Spectral and Pseudo Spectral Methods for Advection Equations*

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Abstract. Spectral and pseudo spectral methods for advection equations are investigated. A basic framework is given which allows the application of techniques used in finite element analysis to spectral methods with trigonometric polynomials. Error estimates for semidiscrete spectral and pseudo spectral as well as fully discrete explicit pseudo spectral methods are given. The approximation schemes are shown to converge with infinite order.

1. Introduction. Spectral and pseudo spectral methods have become popular in approximating solutions of advection equations arising in many sciences. Christensen and Prahm have developed spectral models for dispersion of atmospheric pollutants [4]. Gazdag proposes spectral methods for advection equations and Burger’s equation [6]. Gottlieb and Orzag present many spectral applications in [7]. Numerical evidence supporting spectral type approximations is abundant in the literature, see for example [4], [5], [6], [7], [11]. Numerical tests indicate that spectral methods outperform finite difference methods for many hyperbolic problems [7]. The general consensus among users is that spectral methods work well whenever they are stable.

Recently there have been many theoretical advances in the understanding of these methods. Results on stability of semidiscrete spectral type methods have been given in [5], [9], [10]. Investigations into mollifying the method for nonsmooth initial data have appeared in [9], [10]. Error estimates are implied in the above literature, however, the explicit dependence of convergence on smoothness of initial data is not always given. Also, the compatibility requirements on the initial data at the boundary necessary for convergence are not stated.

In this paper a basic framework is given which allows the application of techniques used in finite element theory to spectral and pseudo spectral methods with trigonometric polynomials. The spectral method is just a Galerkin projection and usual finite element analysis leads to error estimates. Unfortunately, the matrices corresponding to the spectral methods are not sparse and any implementation is costly. Pseudo spectral
methods involve collocation projections instead of $L^2$ projections. Using a result given in [9], the finite element analysis carries over for pseudo spectral methods. Furthermore, by use of the fast Fourier transform (FFT) algorithms, implementation of explicit pseudo spectral schemes can be accomplished economically.

For simplicity, the analysis in this paper only deals with advection equations with coefficients which are constant in time. Extensions of the methods and analysis to problems with coefficients that vary with time or to nonlinear problems are possible.

It is illustrated in this paper that spectral techniques can lead to rapidly convergent approximations to evolution equations. A necessary condition for rapid global convergence is that the solution of the equation can be approximated accurately in the finite-dimensional subspace of trigonometric polynomials. The domain under consideration will always be rectangular and periodic boundary conditions will be imposed. In addition, assumptions on smoothness and compatibility of initial data and coefficients of the advection equation shall be made.

The rest of the paper is broken into four parts. In Section 2, the advection equation is defined and "a priori" regularity estimates for its solutions are proven. In Sections 3 and 4 spectral and pseudo spectral approximations are introduced, and error estimates are established for the corresponding semidiscrete schemes. The convergence for the semidiscrete spectral and pseudo spectral approximation is of order $s$ for compatible initial data in the Sobolev space of order $s + 1$. In Section 5, conditions for stability and convergence of fully discrete explicit pseudo spectral schemes are given, and error estimates for the explicit pseudo spectral method are proven.

2. Regularity for the Advection Equation. We shall consider advection equations on rectangles in $n$-dimensional Euclidean space $\mathbb{R}^n$ with periodic boundary conditions. By changing variables, we may assume, without loss of generality, that the rectangle is the unit rectangle

$$\Omega = [0, 1]^n = [0, 1] \times [0, 1] \times \cdots \times [0, 1].$$

A function $f$ defined on $\mathbb{R}^n$ is periodic if $f(x + z) = f(x)$ for every point $x$ in $\Omega$ and every multi-integer $z$ in $\mathbb{Z}^n$. The space $C_p^\infty(\Omega)$ is the set of infinitely differentiable periodic functions defined on $\mathbb{R}^n$. Let $\| \|$ denote the $L^2$ norm on $\Omega$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-integer with nonnegative entries. Denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and define

$$D_\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

For positive integers $s$, the Sobolev norms on $C_p^\infty(\Omega)$ are given by

$$\|u\|_s = \left( \sum_{|\alpha| \leq s} \|D_\alpha u\|^2 \right)^{1/2}.$$

We also denote the Sobolev seminorms

$$|u|_j = \left( \sum_{|\alpha| = j} \|D_\alpha u\|^2 \right)^{1/2}.$$
Let $H^s_p(\Omega)$ be the completion of $C^\infty_p(\Omega)$ under the norm $\| \cdot \|_s$. The space $H^s_p(\Omega)$ is a Hilbert space under the obvious inner product. For $s \geq 0$, define the space $H^s_p(\Omega)$ by interpolation; see [8].

Define the unbounded operator $L$ on $L^2(\Omega)$ by

$$LU = \sum_{i=1}^{n} V_i \frac{\partial}{\partial x_i} U + \frac{\partial}{\partial x_i} (V_i U),$$

with domain $\mathcal{D}(L) = H^1_p(\Omega)$. We assume for convenience that $V(x)$ is in $C^\infty_p(\Omega)$. It follows that multiplication by $V_i$ is a bounded operator from $H^k_p(\Omega)$ into $H^k_p(\Omega)$. Hence $L$ is a bounded operator from $H^k_p(\Omega)$ into $H^{k-1}_p(\Omega)$.

The advection problem is defined by

$$D_t u + Lu = 0, \quad \Omega \times [0, \infty),$$

$$u(x, 0) = u_0(x), \quad x \in \Omega,$$

$$u \text{ periodic on } \partial \Omega.$$

Specifically, the boundary condition $u$ periodic on $\partial \Omega$ means for smooth $u$ that $u(\cdot, t)$ is in $H^r_p(\Omega)$ for every $t$. The following theorem characterizes the solutions of (2.1).

**Theorem 1.** Let $r \geq 1$ and $u_0$ be in $H^r_p(\Omega)$. There exists a unique function $u(x, t)$ in $C([0, T_0], H^r_p(\Omega))$ satisfying (2.1) and a constant $C$ not depending on $u_0$ or $t$ such that

$$\|u(\cdot, t)\|_r \leq C\|u_0\|_r \quad \text{for } t \in [0, T_0].$$

Before proving Theorem 1, we shall give an alternative characterization for the spaces $H^k_p(\Omega)$. Let $M = (M_1, \ldots, M_n)$ be a multi-integer and define

$$\varphi_M(X) = \text{Exp}(2\pi i M \cdot X) \quad \text{for } X \in \mathbb{R}^n.$$

Let

$$\lambda_M = 1 + 4\pi^2 \sum_{j=1}^{n} M_j^2.$$

$\varphi_M$ is an eigenfunction with eigenvalue $\lambda_M$ for the elliptic problem

$$W - \Delta W = f \quad \text{in } \Omega, \quad W \text{ periodic on } \partial \Omega,$$

where $\Delta$ is the Laplace operator $\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. For $r \geq 0$ consider the sum

$$\sum_{M \in \mathbb{Z}^n} \lambda_M^r |\beta_M|^2,$$

where $\beta_M = (u, \varphi_M)$ and $(\cdot, \cdot)$ denotes the $L^2$ inner product on $\Omega$ given by

$$(u, v) = \int_{\Omega} u \overline{v} \, dx.$$
space $\hat{H}^r$ has the natural norm

\begin{equation}
\|u\|_r = \left( \sum_{M \in \mathbb{Z}^n} \lambda_M^r |\beta_M|^2 \right)^{1/2}.
\end{equation}

**Lemma 1.** For $r \geq 0$, the spaces $H^r_p(\Omega)$ and $\hat{H}^r$ coincide and their norms are equivalent.

**Proof.** Let $k$ be an even integer and $u$ be in $\hat{H}^k$. Then $u$ is obviously in $H^k(\Omega)$ and the norm given by (2.3) is equivalent to the usual Sobolev norm. The partial sums of the series $u = \sum_{M \in \mathbb{Z}^n} \beta_M \varphi_M$ exhibit $u$ as an $H^k(\Omega)$ limit of functions in $C_0^\infty(\Omega)$. Hence $\hat{H}^k$ is contained in $H^k_p(\Omega)$.

Let $u$ be in $H^k_p(\Omega)$ and $g = (I - \Delta)^{k/2}u$. Then $g$ is in $L^2(\Omega)$ and the function

\begin{equation}
w = \sum_{M \in \mathbb{Z}^n} \lambda_M^{-k/2}(g, \varphi_M)\varphi_M
\end{equation}

is in $\hat{H}^k$ and hence $H^k_p(\Omega)$. Furthermore $(I - \Delta)^{k/2}w = g$. The operator $(I - \Delta)$ is an injective map of $H^k_p(\Omega)$ into $H^{k-2}_p(\Omega)$ so $w$ and $u$ are identical. Hence $H^k_p(\Omega)$ is contained in $\hat{H}^k$ which proves the lemma for even $k$. Since $\hat{H}^r$ and $H^r_p(\Omega)$ are Hilbert scales, the lemma follows by interpolation [8].

**Proof of Theorem 1.** The proof of Theorem 1 is essentially a proof given by Taylor in [12]. We shall indicate the changes in Taylor's proof due to our boundary conditions.

Define the mollifier $J_\varepsilon$ by

$$J_\varepsilon U = \sum_{M \in \mathbb{Z}^n} e^{-\varepsilon |M|^2} (U, \varphi_M)\varphi_M.$$ 

By Lemma 1, it is evident that $J_\varepsilon$ is a bounded map of $H^k_p(\Omega)$ into $H^{k+r}_p(\Omega)$ for any real $r$. Furthermore, for $u$ in $H^k_p(\Omega)$,

$$\|J_\varepsilon u\|_k \leq \|u\|_k.$$ 

The proof of Theorem 1 continues by exhibiting the solution of (2.1) as a limit of solutions of the problems

$$\frac{\partial u_\varepsilon}{\partial t} + J_\varepsilon L J_\varepsilon u_\varepsilon = 0, \quad \Omega \times [0, T_0],$$

$$u_\varepsilon(x, 0) = u_0(x), \quad x \in \Omega,$$

$$u_\varepsilon \text{ periodic on } \partial \Omega.$$ 

From here on the proof proceeds exactly as in Taylor [12] and hence shall be omitted.

**3. The Semidiscrete Spectral Approximation.** Let $M$ be a multi-integer, $M = (M_1, \ldots, M_n) \in \mathbb{Z}^n$, and define $\varphi_M(X) = \text{Exp}(2\pi i M \cdot X)$ for $X$ in $\mathbb{R}^n$. Let

$$\Omega_N = \{(M_1, \ldots, M_n)|-N + 1 \leq M_j \leq N \text{ for } j = 1, \ldots, n\}.$$
The approximation spaces $S_N$ are defined to be the span of $\varphi_M$ as $M$ varies over $\Omega_N$. We note the approximation properties of $S_N$ given in [3]: Let $m \geq j \geq 0$. There exists a constant $C$, not depending on $w$ in $H^m_p(\Omega)$ or $N$, such that

\begin{equation}
\|w - P_N w\|_j \leq CN^{j-m} |w|_m,
\end{equation}

where $P_N$ is the $L^2$ projection onto $S_N$. The subspaces $S_N$ also have the inverse properties, for $u$ in $S_N$,

\begin{equation}
\|u\|_{k+j} \leq CN^j \|u\|_k.
\end{equation}

The spectral approximation $L_N$ is the $L^2$ projection of $L$ into the subspace $S_N$, that is

\begin{equation}
L_N f = P_N L f.
\end{equation}

The operator $L_N$ is obviously skew symmetric on $S_N$. Since $S_N$ is finite dimensional, $L_N$ generates a unique unitary semigroup. Let $U_N$ be the solution to

\begin{equation}
D_t U_N + L_N U_N = 0, \quad U_N(0) = P_N u_0.
\end{equation}

Error analysis for the semidiscrete approximation $U_N$ could proceed by the usual finite element approach. We could first prove convergence estimates for $(I + L_N)^{-1}$ as an approximate to $(I + L)^{-1}$. Then the usual techniques (see [2]) would give rise to semidiscrete error estimates. In general, $(I + L)^{-1}$ is not a smoothing operator and standard techniques in finite element theory only give that $(I + L_N)^{-1}$ is a suboptimal approximation to $(I + L)^{-1}$. Thus, any analysis requiring approximation of $(I + L)^{-1}$ may lead to inferior convergence estimates. In the analysis in the rest of this paper we shall always approximate $L$.

**Theorem 2.** Let $u_0$ be in $H^{s+1}_p(\Omega)$ and $u$ be the solution of the advection equation (2.1). There is a constant $C$ independent of $u_0$ and $N$ satisfying

\begin{equation}
\|u(t) - U_N(t)\|_0 \leq CN^{-s} \|u_0\|_{s+1}, \quad t \in [0, T_0].
\end{equation}

**Proof.** Let $X(t) = P_N u(t)$ and $v(t) = u(t) - X(t)$. Then for $\theta$ in $S_N$

\begin{equation}
(D_t X, \theta) + (L X, \theta) = - (L v, \theta).
\end{equation}

Using the definition of $L_N$, we have for $\theta$ in $S_N$

\begin{equation}
(D_t U_N, \theta) + (L U_N, \theta) = 0.
\end{equation}

Let $e = U_N - W$. Subtracting (3.3) and (3.4) and setting $\theta = e$ gives $(D_t e, e) + (L e, e) = (L v, e)$. A similar argument shows $(e, D_t e) + (e, L e) = (e, L v)$, and since $L$ is skew symmetric,

\begin{align*}
2\|e\|D_t\|e\| = D_t\|e\|^2 &= 2 \Re(e, L v) \\
&\leq 2\|e\|\|L v\|.
\end{align*}

Integrating the above equation gives

\begin{equation}
\|e\| \leq C \sup_{t \in [0, T_0]} \|L v(t)\|.
\end{equation}
By the approximation properties of $S_N$ and Theorem 1

\[
\|Lu\| \leq C\|u - P_Nu\|_1 \leq CN^{-s}\|u_0\|_{s+1},
\]

Combining (3.5) and (3.6) and again using (3.1),

\[
\|u(t) - U_N(t)\| \leq \|e(t)\| + \|u(t)\| \leq CN^{-s}\|u_0\|_{s+1},
\]

which completes the proof of the theorem.

**Remark.** I suspect that Theorem 2 is not sharp. Indeed, for $V_j$ constant, one can prove the stronger result that $N^{-s}$ convergence is achieved with initial data in $H^s_p(\Omega)$. The proof does not generalize to nonconstant $V_j$ due to the lack of commutativity between $P_h$ and multiplication by $V_j$. Numerical tests on a few variable coefficient problems suggest that the stronger result holds. The numerical results, however, could be misleading since the cases computed were by no means extensive.

**4. The Semidiscrete Pseudo Spectral Approximation.** The spectral approximation $L_N$, defined by (3.2), is given by the alternative formula

\[
L_N f = \sum_{j=1}^{n} P_N\left(V_j \frac{\partial}{\partial x_j} f\right) + \frac{\partial}{\partial x_j} P_N(V_j f).
\]

That $\partial/\partial x_j$ and $P_N$ commute is readily seen by expansion in the basis of trigonometric polynomials. The pseudo spectral approximation to $L$ will be defined by replacing the $L^2$ projections in (4.1) by collocation projections.

Let $h = 1/2N$ and set $x_j = jh$. Let $\Omega_x$ be the collection of points

\[
\Omega_x = \{(x_{i_1}, \ldots, x_{i_n})|0 \leq i_k < 2N\}.
\]

For any continuous function $f$ on $\overline{\Omega}$, define $P_c f$ to be the function in $S_N$ which interpolates $f$ on the grid points of $\Omega_x$. To see that $P_c$ is well defined, we shall introduce discrete Fourier transforms. Let $C_x$ be the space of complex-valued functions on $\Omega_x$ and define the inner product

\[
(f, g)_x = h^n \sum_{y \in \Omega_x} f(y)\overline{g}(y) \quad \text{for } f, g \in C_x.
\]

Let $C_N$ be the space of complex-valued functions on $\Omega_N$ and define the inner product

\[
(f, g)_N = \sum_{I \in \Omega_N} f(I)\overline{g}(I) \quad \text{for } f, g \in C_N.
\]

The discrete Fourier transform $F_N$ maps $C_x$ onto $C_N$ and is defined by

\[
F_N f(I) = h^n \sum_{y \in \Omega_x} f(y) \Exp(-2\pi i y \cdot I).
\]

$F_N$ is a unitary transformation from $C_x$ onto $C_N$. The inverse of $F_N$ is given by

\[
F_N^{-1} f(y) = \sum_{I \in \Omega_N} f(I) \Exp(2\pi i y \cdot I).
\]
We shall see that $P_c f$ is the function $g$ given by

\begin{equation}
(4.2) \quad g(y) = \sum_{I \in \Omega_N} \mathcal{F}_N f(I) \exp(2\pi iy \cdot I).
\end{equation}

Indeed, the inversion formula $F^{-1}_{\Omega} F_N = I$ implies that $g$ is a function in $S_N$ which equals $f$ at each point of $\Omega_x$. An easy exercise in linear algebra shows that $g$ is the unique function in $S_N$ assuming the values of $f$ on $\Omega_x$. Thus $P_c$ is well defined. We note that (4.2) gives an algorithm for finding the coefficients of $P_c f$ in the basis of trigonometric polynomials given the nodal values of $f$ on $\Omega_x$. Also note that the inner product $(\cdot, \cdot)_x$ is defined so that $P_c$ is a unitary transformation of $C_x$ onto $S_N$ with $L^2$ inner product. Thus for functions in $S_N$, the $(\cdot, \cdot)_x$ inner product and $L^2$ inner product and hence their respective norms are interchangeable.

The next theorem is essentially an $n$-dimensional version of Theorem 3.3 of [9] and demonstrates that $P_c$ has approximation properties similar to those given by (3.1) for $P_N$. We shall give a new proof of Theorem 3 using arguments which are similar to those used to derive approximation properties for finite element interpolation.

**Theorem 3.** Let $0 \leq j \leq m$ and $m > n/2$. There exists a constant $C$, independent of $w$ in $H^m_p(\Omega)$ and $N$, such that

$$||w - P_c w||_j \leq CN^{-m+j}||w||_m.$$ 

**Proof.** First we introduce the notation for the proof. Let $\hat{\Omega}$ be the rectangle $[0, 2N]^n$. For a function $f$ defined on $\Omega$, let $\hat{f}$ be the function defined on $\hat{\Omega}$ by $\hat{f}(x) = f(x/2N)$. Let $\hat{S}_N$ be the image of $S_N$ under the above dilation. We also denote the seminorms on $\hat{\Omega}$,

$$|f|_{j, \hat{\Omega}} = \left( \sum_{|\alpha| = j} ||D_\alpha f||_L^2(\hat{\Omega}) \right)^{1/2},$$

and the corresponding Sobolev norms

$$||f||_{j, \hat{\Omega}} = \left( \sum_{k=0}^j |f|_{k, \hat{\Omega}}^2 \right)^{1/2}.$$ 

We make the following observations:

(i) $\hat{S}_N$ is a space of trigonometric polynomials and the interpolation projection $\hat{P}_c$ satisfying $\hat{P}_c \hat{f}(2Nx) = \hat{f}(2Nx)$, for all $x$ in $\Omega_x$, is well defined.

(ii) $P_c f = \hat{P}_c \hat{f}$ for all $f$ in $C_x$.

(iii) $(\hat{P}_c - I)f = 0$ for $f$ in $\hat{S}_N$.

(iv) For $w$ in $H^m_p(\Omega)$,

$$||\hat{w} - \hat{P}_N \hat{w}||_{m, \hat{\Omega}} \leq C||\hat{w}||_{m, \hat{\Omega}}.$$ 

Observations (i), (ii), and (iii) follow easily from the definitions of $S_N$, $P_c$, $\hat{P}_c$ and earlier arguments. For (iv) we first note that a change of variables implies
(4.3) \[ |f|_k = (2N)^{k-n/2} |\hat{f}|_{k, \Omega}. \]

Using (3.1) and (4.3) gives
\[
\|\hat{w} - \hat{P}_N w\|_{m, \Omega}^2 = \sum_{k=0}^{j} (2N)^{n-2k} |w - P_N w|_k^2
\leq C(2N)^{-2m + n} |w|_m^2 = C|\hat{w}|_{m, \Omega}^2.
\]

which proves (iv).

Using (ii) and (iii), we can compute
\[
\|w - P_c w\|_j \leq (2N)^{j-n/2} \|\hat{w} - \hat{P}_c \hat{w}\|_{j, \Omega}
\leq (2N)^{j-n/2} \|(I - \hat{P}_c)(\hat{w} - \hat{P}_N w)\|_{j, \Omega}.
\]

Using (iv) and (4.3) gives
\[
\|w - P_c w\|_j \leq C N^{-n/2} \|I - \hat{P}_c\|_{L(H^m(\Omega), H^n(\Omega))} \|\hat{w}\|_{m, \Omega}
\leq C N^{-m} \|I - \hat{P}_c\|_{L(H^m(\Omega), H^n(\Omega))} \|w\|_m.
\]

Thus, the theorem will follow if we can bound the operator norm of \(\hat{P}_c\) in
\(L(H^m(\Omega), H^n(\Omega))\) independently of \(N\).

Let \(\hat{f}\) be in \(H^m(\Omega)\), then Sobolev inequalities on the domain \(\Omega\) imply
\[
(4.4) \sum_{x \in \Omega} |\hat{f}(2Nx)|^2 \leq C \sum_{x \in \Omega} \|\hat{f}(\cdot + 2Nx)\|^2_m \leq C\|\hat{f}\|^2_{m, \Omega}.
\]

By (4.2)
\[
(4.5) \hat{P}_c \hat{f}(X) = \sum_{I \in \Omega_N} F_N f(I) \exp\left(\frac{\pi i}{N} X \cdot I\right).
\]

Using the fact that \(F_N\) is an isometry from \(C_x\) onto \(C_N\) and (4.4) gives
\[
\|\hat{P}_c \hat{f}\|_{0, \Omega} = \left(\sum_{x \in \Omega} |\hat{f}(2Nx)|^2\right)^{1/2} \leq C\|\hat{f}\|_{m, \Omega}.
\]

Finally (4.5) implies
\[
\|\hat{P}_c \hat{f}\|_{j, \Omega} \leq C\|\hat{P}_c \hat{f}\|_{0, \Omega} \leq C\|\hat{f}\|_{m, \Omega},
\]

which completes the proof of the theorem.

We can now define the pseudo spectral approximation to \(L\) by replacing \(P_N\) in
(4.1) by \(P_c\):
\[
L_c f = \sum_{j=1}^{n} P_c \left(V_j \frac{\partial}{\partial x_j} f + \frac{\partial}{\partial x_j} P_c (V_j f)\right).
\]

The operator \(L_c\) is skew symmetric on \(S_N\), indeed, for \(f, g\) in \(S_N\)
As in Section 3, \( L_c \) generates a unitary semigroup on \( S_N \). Let \( U \) be the solution to the problem

\[
(D_t U + L_c U = 0, \quad U(0) = P_c u_0).
\]

Theorem 4. Let \( u_0 \) be in \( H^{s+1}_p(\Omega) \) for \( s > n/2 \) and let \( u \) be the solution to the advection equation (2.1). There exists a constant \( C \) independent of \( u_0 \) satisfying

\[
\| u(t) - U(t) \| \leq C N^{-s} \| u_0 \|_{s+1}, \quad t \in [0, T_0].
\]

Theorem 4 can also be viewed as an extension to a theorem given by B. Fornberg. In [5], Fornberg derives estimates for semidiscrete pseudo spectral methods which bound the errors by norms of certain remainder terms. Fornberg gives heuristic arguments to show that these bounding norms are small. The norms in Fornberg's theorem can be estimated using Theorem 1 and Theorem 3. We give a proof which is similar to familiar finite element proofs.

Proof of Theorem 4. Let \( X \) and \( v \) be defined as in the proof of Theorem 2 and set \( e = U - X \). Using the definition of \( U \), we have

\[
(D_t U, \theta) + (L_c U, \theta) = 0,
\]

for all \( \theta \) in \( S_N \).

Subtracting (3.3) and setting \( \theta = e \) gives

\[
(D_t e, e) + (L_c e, e) = (L v, e) + ((L - L_c)X, e).
\]

As in the proof of Theorem 2, it follows that

\[
\| e(0) \| \leq \| e(0) \| + T_0 \sup_{t \in [0, T_0]} \{ \| (L - L_c)X(t) \| + \| L_v(t) \| \}.
\]

From the proof of Theorem 2,

\[
\| L_v(t) \| \leq C N^{-s} \| u_0 \|_{s+1}.
\]

Also the triangle inequality and Theorem 3 imply

\[
\| (L - L_c)X \| \leq \sum_{j=1}^{n} \left\| \frac{\partial}{\partial x_j} (I - P_c) V_j X \right\| + \left\| (I - P_c) V_i \frac{\partial}{\partial x_j} X \right\|
\]

\[
\leq C N^{-s} \| X \|_{s+1} \leq C N^{-s} \| u_0 \|_{s+1}.
\]

The last inequality made use of (3.1) and Theorem 1. Theorem 3 implies

\[
\| e(0) \| \leq C N^{-s} \| u_0 \|_{s}.
\]

Combining the above estimates proves Theorem 4.
5. Fully Discrete Pseudo Spectral Approximation. In this section we shall describe fully discrete explicit pseudo spectral approximations to (2.1). By taking advantage of the fast Fourier transform package, the pseudo spectral operator $L_c$ can be evaluated economically. Thus fully discrete explicit timestepping procedures will run efficiently. To get error estimates, arguments proceed along the lines given by Baker, Bramble, and Thomée in [1]. We prove “smooth” data estimates (Lemma 3) and then expand the discretization error and apply the smooth estimates. In [1], Baker et al. expand the discretization error in terms of differences of $T$ and $T_h$. For advection equations, expanding the errors in terms of $L_N - L_c$ gives rise to better error estimates because the operator $(I + L)^{-1}$, corresponding to $T$ in [1], is not a smoothing operator.

Let $P_j(t)$ be the truncated Taylor series

$$P_j(t) = \sum_{j=0}^{J} \frac{t^j}{j!}.$$  

We shall only consider $J$ such that there exists $\delta > 0$ satisfying

$$|P_j(\tau)| < 1 \quad \text{for real } \tau \text{ with } |\tau| < \delta. \quad (5.1)$$

By expanding $|P_j(\tau)|$ as a function of $\tau$, it is easily checked that the values $J = 3, 4, 7, 8$ satisfy the above assumption. Inequality (5.1), however, fails to hold for $J = 1, 2, 5$ or 6 for all choices of $\delta$.

We approximate the solution of (2.1) by the sequence

$$W^0 = P_c u_0, \quad W^{j+1} = P_j(-L_c k)W^j \quad \text{for } j = 0, 1, 2, \ldots. \quad (5.2)$$

Then $W^j$ approximates $u(t_j)$ for $t_j = kj$. Note that the evaluation of (5.2) requires only $J$ evaluations of $L_c$ per timestep. Evaluation of $L_c$ only requires multiplications and discrete Fourier transforms. Each timestep of (5.2) involves work of order $m \log(m)$ where $m$ is the number of points in $\Omega_x$.

Let the maximum norm over the grid points of $\Omega_x$ be denoted

$$\|V\|_{x, \infty} = \max_{y \in \Omega_x} |V(y)|.$$  

For stability of (5.2) we use the following lemma:

**Lemma 2.** Let $\theta$ be an element of $S_N$. The following estimate holds:

$$\|L_c \theta\| \leq C_1 N \|\theta\|, \quad \text{where } C_1 = 4\pi n \max_{j=1,\ldots,n} \|V_j\|_{x, \infty}. \quad$$

**Proof.** We clearly have

$$\|V\theta\|_x \leq \|V\|_{x, \infty} \|\theta\|_x.$$ 

Also, for $\theta$ in $S_N$,

$$\left\| \frac{\partial \theta}{\partial x_i} \right\|_x = \left\| \frac{\partial \theta}{\partial x_i} \right\|_x \leq 2\pi N \|\theta\| = 2\pi N \|\theta\|_x.$$ 

The lemma follows immediately from the above estimates and the definition of $L_c$. 


As a consequence of (5.1) and Lemma 2, an obvious eigenfunction analysis gives that (5.2) is stable in $L^2$ whenever $kC_1N \leq \delta$. The following theorem is the main result of this section.

**Theorem 5.** Let $u$ be the solution of (2.1) and $W$ be the solution of (5.2) with $kC_1N \leq \delta$. Let $u_0$ be in $H^J(\Omega)$ where $\tau = \max(s + 1, J + 1)$. For $\tau > n/2 + 1$ there exists a constant $C$ independent of $u_0$ satisfying

$$
||W^n - u(t_n)|| \leq C \{N^{-s} + k^J\} ||u_0||_r \text{ for } t_n \leq T_0.
$$

As a result of Theorem 4, to prove Theorem 5, it is sufficient to analyze the error between (5.2) and (4.6). To accomplish this we shall break the error into pieces and analyze the pieces. Introduce the error function $E_j(g)$ for $g$ in $S_N$ defined by $E_j(g) = W^j - U(t_j)$ where $W^j$ satisfies (5.2) with initial data $W^0 = g$ and $U$ satisfies (4.6) with data $U(0) = g$. Note that $E_j(g)$ is a linear function of $g$ and that to prove Theorem 5 we need to bound $E_j(P_c u_0)$.

Since $L_c$ is skew symmetric on a finite-dimensional space, the operator $T_c = (I + LC)^{-1}P_N$ is well defined. We prove the following approximation result:

**Lemma 3.** For $kj \leq T_0$ and $kC_1N \leq \delta$ the following estimates hold:

(i) $||E_j(g)|| \leq C||g||$,

(ii) $||E_j(T^m c g)|| \leq Ck^{m-1}||g||$ for $2 \leq m \leq J + 1$.

**Proof.** (i) is just stability in $L^2$ of (5.2) and (4.6). For (ii) we note the following recursion

(5.3) $E_{j+1}(f) = P_j(-L_c k)E_j(f) + [P_j(-L_c k) - \text{Exp}(-L_c k)] U(t_j),$

where $U$ is defined by (4.6) with initial data $f$. By (5.1)

$$
||P_j(-L_c k)E_j(f)|| \leq ||E_j(f)||.
$$

Now for $f = T^m c g$, $U(t) = T^m c \tilde{U}(t)$ where $\tilde{U}$ satisfies (4.6) with initial data $g$. Expanding the second term in (5.3) in the eigenfunctions of $L_c$ implies that

$$
||[P_j(-L_c k) - \text{Exp}(-L_c k)] T^m c \tilde{U}(t)|| \leq Ck^m ||\tilde{U}(t)||.
$$

Using the stability of (4.6) and combining the above results gives

$$
||E_{j+1}(T^m c g)|| \leq ||E_j(T^m c g)|| + Ck^m ||g||,
$$

and summing proves Lemma 3.

**Proof of Theorem 5.** We note the following identity: For $\theta$ in $S_N,$

(5.4) $\theta = \sum_{j=1}^m T^j_c(L_c - L_N)(I + L_N)^{-1} \theta + T^m_c(I + L_N)^m \theta.$

The identity is obvious for $m = 1$, and validity for all $m$ follows from an easy induction argument.
Using (5.4) and the linearity of $E_m$ gives

$$E_m(P_c u_0) = \sum_{j=1}^{J+1} E_m(T_c^j(L_c - L_N)(I + L_N)^{j-1}P_N u_0)$$

$$+ E_m(T_c^{J+1}(I + L_N)^{J+1}P_N u_0) + E_m(P_c u_0 - P_N u_0).$$

As a consequence of (3.1), for $f$ in $H^m_p(\Omega)$,

$$(5.5) \quad \|I + L_N\|^j_{m-j} \leq C\|f\|_m$$

Lemma 3, the triangle inequality, and (3.1) give

$$\|E_m(P_c u_0)\| \leq \sum_{j=2}^{J+1} Ck^{j-1}\|(L_c - L_N)(I + L_N)^{j-1}P_N u_0\|$$

$$+ C\|(L_c - L_N)P_N u_0\| + Ck^j\|u_0\|_{j+1} + CN^{-s}\|u_0\|_s.$$

By estimates similar to those given on (4.7), the inverse properties of $S_N$, and (5.5)

$$(5.7) \quad \|(L_c - L_N)(I + L_N)^{j-1}P_N u_0\| \leq CN^{-s}\|P_N u_0\|_{s+j} \leq CN^{-s+j-1}\|u_0\|_{s+1}.$$

Combining (5.6) and (5.7) with the inequality $kNC_1 \leq \delta$ proves the theorem.

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