On Determination of Best-Possible Constants in Integral Inequalities Involving Derivatives

By Beny Neta

Abstract. This paper is concerned with the numerical approximation of the best possible constants $\gamma_{n,k}$ in the inequality

$$\|F^{(k)}\|^2 \leq \gamma_{n,k}^{-1} \{\|F\|^2 + \|F^{(n)}\|^2\},$$

where

$$\|F\|^2 = \int_{0}^{\infty} |F(x)|^2 \, dx.$$

A list of all constants $\gamma_{n,k}$ for $n \leq 10$ is given.

1. Introduction. This paper utilizes the algorithm given in [1] to numerically approximate the best possible constants $\gamma_{n,k}$, $1 \leq k < n$, for $n \leq 10$ in the inequality:

$$\|F^{(k)}\|^2 \leq \gamma_{n,k}^{-1} \{\|F\|^2 + \|F^{(n)}\|^2\},$$

where $\| \cdot \|$ denotes the $L_2[0, \infty)$ norm. The function $F$ has a locally absolutely continuous $(n-1)$st derivative. The inequality (1) is equivalent to

$$\|F^{(k)}\| \leq M_{n,k} \|F\|^{(n-k)/n} \|F^{(n)}\|^{k/n},$$

where

$$M_{n,k}^2 = \gamma_{n,k}^{-1} \left(\frac{n-k}{k}\right)^{k/n} + \left(\frac{k}{n-k}\right)^{(n-k)/n};$$

see [1].

Interest in inequalities (1) and (2) increased because of their close connection with problems of best approximation of the differentiation operator by bounded operators; see [2], [3], [4], [5], and with the problem of best approximation of one class of functions by another; see [4], [6], [7].

In the next section we shall give lower and upper bounds for the best possible constants $\gamma_{n,k}$ and $M_{n,k}$ for $n \leq 10$.

2. Numerical Results. In this section the best possible constants $\gamma_{n,k}$ and $M_{n,k}$ are listed.

$$\gamma_{21} = 1, \text{ see [1].}$$

$$\gamma_{31} = \gamma_{32} = \sqrt[3]{3} - 2\sqrt{2} = .555669, \text{ see [1].}$$
In [1], \( \gamma_{41} \) is characterized as the smallest positive zero of the polynomial \( Z^8 - 6Z^4 - 8Z^2 + 1 \), and \( \gamma_{42} \) is the smallest positive zero of the polynomial \( Z^4 - 2Z^2 - 4Z + 1 \). Using Müller’s method [8], we obtain \( \gamma_{41} = \gamma_{43} = .339246, \gamma_{42} = .225270 \).

Remark. It is known, see [1], that

\[
\gamma_{n,n-k} = \gamma_{n,k} \quad \text{for all } n, k.
\]

Using the algorithm in [1], one has the following table of lower and upper bounds on \( \gamma_{n,k} \) for \( 2 \leq n \leq 10 \) and \( 1 \leq k \leq \lfloor n/2 \rfloor \). For other values of \( k \), use (4).

Table 1

<table>
<thead>
<tr>
<th>( n )</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
<th>( k = 3 )</th>
<th>( k = 4 )</th>
<th>( k = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(.555669)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(.339246)</td>
<td>(.225271)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(.225837, .2258375)</td>
<td>(.102266, .102268)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(.160328, .160338)</td>
<td>(.051986, .05199)</td>
<td>(.0361167, .0361177)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>(.11936, .11943)</td>
<td>(.028924, .02895)</td>
<td>(.014698, .0147)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>(.09128, .09129)</td>
<td>(.0172, .01723)</td>
<td>(.0068112, .00681124)</td>
<td>(.005014, .0050145)</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>(.07593, .07594)</td>
<td>(.010795, .0108)</td>
<td>(.00345, .0036)</td>
<td>(.00193, .001939)</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>(.0479, .048)</td>
<td>(.0068, .007)</td>
<td>(.0016163, .0014165)</td>
<td>(.000681505, .00068151)</td>
<td>(.00064256, .00064257)</td>
</tr>
</tbody>
</table>

Using (3) and the values listed in Table 1, one has the following table of lower and upper bounds on \( M_{n,k} \) for \( 2 \leq n \leq 10 \) and \( 1 \leq k \leq \lfloor n/2 \rfloor \). For other values of \( k \), use \( M_{n,n-k} = M_{n,k} \) for all \( n, k \).

Table 2

<table>
<thead>
<tr>
<th>( n )</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
<th>( k = 3 )</th>
<th>( k = 4 )</th>
<th>( k = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.41421</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2.07005</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2.72732</td>
<td>2.97963</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(2.70248, 2.70249)</td>
<td>(4.37797, 4.37801)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(3.12838, 3.12848)</td>
<td>(6.02917, 6.02940)</td>
<td>(7.44141, 7.44151)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>(3.55221, 3.55325)</td>
<td>(7.92662, 7.93019)</td>
<td>(11.60467, 11.60564)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>(3.99579, 3.99601)</td>
<td>(10.09176, 10.10056)</td>
<td>(16.86722, 16.86727)</td>
<td>(19.97108, 19.97206)</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>(4.32029, 4.32057)</td>
<td>(12.54043, 12.54333)</td>
<td>(23.0795, 23.23717)</td>
<td>(32.02543, 32.09137)</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>(5.36995, 5.37555)</td>
<td>(15.35033, 15.57423)</td>
<td>(36.06112, 36.06367)</td>
<td>(53.62986, 53.63004)</td>
<td>(55.78980, 55.79001)</td>
</tr>
</tbody>
</table>

Remarks. 1. The lower and upper bounds for each \( n \) and \( k \) are given in parentheses and separated by a comma, for example, \(.11936 < \gamma_{7,1} < .11943\).

2. The number \( M_{4,2} \) in Table 2 agrees with that obtained by Bradley and Everitt [7].

3. The number \( M_{6,3} \) in this table agrees with a result of Dawson and Everitt [9].

Conjecture. For fixed \( k \) the \( \gamma_{n,k} \) are decreasing functions of \( n \). For fixed \( n \) the \( \gamma_{n,k} \) are decreasing functions of \( k \) up to \( k = \lfloor n/2 \rfloor \).
Thus the initial value of $\gamma_{n,k}$ may be taken in the interval

$$I_{n,k}^* = (0, \gamma_{n-1,k})$$

for $n > 2$

rather than the interval suggested by Kupcov, namely

$$I_{n,k} = (0, g_{n,k}),$$

where

$$g_{n,k} = \frac{n}{k^k/n(n-k)(n-k)/n}.$$

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Department of Mathematical Sciences
Northern Illinois University
DeKalb, Illinois 60115