On Determination of Best-Possible Constants in Integral Inequalities Involving Derivatives

By Beny Neta

Abstract. This paper is concerned with the numerical approximation of the best possible constants \( \gamma_{n,k} \) in the inequality

\[
\|F^{(k)}\|^2 \leq \gamma_{n,k}^{-1} \{\|F\|^2 + \|F^{(n)}\|^2\},
\]

where

\[
\|F\|^2 = \int_0^\infty |F(x)|^2 \, dx.
\]

A list of all constants \( \gamma_{n,k} \) for \( n \leq 10 \) is given.

1. Introduction. This paper utilizes the algorithm given in [1] to numerically approximate the best possible constants \( \gamma_{n,k} \), \( 1 \leq k < n \), for \( n \leq 10 \) in the inequality:

\[
(\ast) \quad \|F^{(k)}\|^2 \leq \gamma_{n,k}^{-1} \{\|F\|^2 + \|F^{(n)}\|^2\},
\]

where \( \| \cdot \| \) denotes the \( L_2 [0, \infty) \) norm. The function \( F \) has a locally absolutely continuous \((n - 1)\)st derivative. The inequality (1) is equivalent to

\[
\|F^{(k)}\| \leq M_{n,k} \|F^{(n-k)}\|^n \|F^{(n)}\|^{k/n},
\]

where

\[
M_{n,k}^2 = \gamma_{n,k}^{-1} \left( \frac{n - k}{k} \right)^{k/n} + \left( \frac{k}{n - k} \right)^{(n-k)/n};
\]

see [1].

Interest in inequalities (1) and (2) increased because of their close connection with problems of best approximation of the differentiation operator by bounded operators; see [2], [3], [4], [5], and with the problem of best approximation of one class of functions by another; see [4], [6], [7].

In the next section we shall give lower and upper bounds for the best possible constants \( \gamma_{n,k} \) and \( M_{n,k} \) for \( n \leq 10 \).

2. Numerical Results. In this section the best possible constants \( \gamma_{n,k} \) and \( M_{n,k} \) are listed.

\[
\gamma_{21} = 1, \quad \text{see [1].}
\]

\[
\gamma_{31} = \gamma_{32} = \frac{\sqrt{3} - 2\sqrt{2}}{\sqrt{3} - 2\sqrt{2}} = .555669, \quad \text{see [1].}
\]
In [1], \( \gamma_{41} \) is characterized as the smallest positive zero of the polynomial \( Z^8 - 6Z^4 - 8Z^2 + 1 \), and \( \gamma_{42} \) is the smallest positive zero of the polynomial \( Z^4 - 2Z^2 - 4Z + 1 \). Using Müller's method [8], we obtain \( \gamma_{41} = \gamma_{43} = .339246, \gamma_{42} = .225270 \).

Remark. It is known, see [1], that

\[
\gamma_{n,n-k} = \gamma_{n,k} \quad \text{for all } n, k.
\]

Using (4) and the values listed in Table 1, one has the following table of lower and upper bounds on \( \gamma_{n,k} \) for \( 2 \leq n \leq 10 \) and \( 1 \leq k \leq \lfloor n/2 \rfloor \). For other values of \( k \), use (4).

<table>
<thead>
<tr>
<th>( n ) ( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</thead>
<tbody>
<tr>
<td>2</td>
<td>.1192</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>3</td>
<td>.555669</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>.339246</td>
<td>.225271</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>5</td>
<td>(.225837, .2258375) (.102266, .102268)</td>
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</tr>
<tr>
<td>6</td>
<td>(.160328, .160338) (.051986, .05199) (.0361167, .0361177)</td>
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<tr>
<td>7</td>
<td>(.11936, .11943) (.028924, .02895) (.014698, .0147)</td>
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<tr>
<td>8</td>
<td>(.09128, .09129) (.0172, .01732) (.0068112, .00681124) (.005014, .005015)</td>
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<tr>
<td>9</td>
<td>(.07593, .07594) (.010795, .0108) (.00345, .0036) (.00193, .001938)</td>
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<tr>
<td>10</td>
<td>(.0479, .048) (.0068, .007) (.001616, .0014165) (.000681505, .00068151) (.000642565, .00064257)</td>
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</tbody>
</table>

Using (3) and the values listed in Table 1, one has the following table of lower and upper bounds on \( M_{n,k} \) for \( 2 \leq n \leq 10 \) and \( 1 \leq k \leq \lfloor n/2 \rfloor \). For other values of \( k \), use \( M_{n,n-k} = M_{n,k} \) for all \( n, k \).

<table>
<thead>
<tr>
<th>( n ) ( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.41421</td>
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</tr>
<tr>
<td>3</td>
<td>2.07005</td>
<td></td>
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</tr>
<tr>
<td>4</td>
<td>2.27432</td>
<td>2.9763</td>
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<tr>
<td>5</td>
<td>(2.070248, 2.070249) (4.37797, 4.37801)</td>
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<tr>
<td>6</td>
<td>(3.12838, 3.12848) (6.02917, 6.02940) (7.44141, 7.44151)</td>
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</tr>
<tr>
<td>7</td>
<td>(3.55221, 3.55325) (7.92662, 7.93019) (11.60467, 11.60566)</td>
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<tr>
<td>9</td>
<td>(4.32029, 4.32057) (12.54043, 12.54333) (23.07295, 23.23717) (32.02543, 32.09173)</td>
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<tr>
<td>10</td>
<td>(5.36995, 5.37555) (15.35033, 15.57423) (36.06112, 36.06367) (53.62986, 53.63004) (55.78980, 55.79001)</td>
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</tbody>
</table>

Remarks. 1. The lower and upper bounds for each \( n \) and \( k \) are given in parentheses and separated by a comma, for example, \( .11936 \leq \gamma_{7,1} \leq .11943 \).

2. The number \( M_{4,2} \) in Table 2 agrees with that obtained by Bradley and Everitt [7].

3. The number \( M_{6,3} \) in this table agrees with a result of Dawson and Everitt [9].

Conjecture. For fixed \( k \) the \( \gamma_{n,k} \) are decreasing functions of \( n \). For fixed \( n \) the \( \gamma_{n,k} \) are decreasing functions of \( k \) up to \( k = \lfloor n/2 \rfloor \).
Thus the initial value of $y_{n,k}$ may be taken in the interval

$$\mathcal{I}_{n,k}^* = (0, y_{n-1,k}) \quad \text{for } n > 2$$

rather than the interval suggested by Kupcov, namely

$$\mathcal{I}_{n,k} = (0, g_{n,k}),$$

where

$$g_{n,k} = \frac{n}{k(n-k)(n-k)n}.$$

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