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The Exact Degree of Precision of Generalized Gauss-Kronrod Integration Rules

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Abstract. It is shown that the Kronrod extension to the n-point Gauss integration rule, with respect to the weight function $(1 - x^2)^{\mu - 1/2}$, $0 < \mu \leqslant 2$, $\mu \neq 1$, is of exact precision $3n + 1$ for n even and $3n + 2$ for n odd. Similarly, for the $(n+1)$-point Lobatto rule, with $-1/2 < \mu \leqslant 1$, $\mu \neq 0$, the exact precision is $3n$ for n odd and $3n + 1$ for n even.

1. Introduction. In this paper we shall consider the Kronrod extensions (KE) to the Gauss-Gegenbauer integration rules (GGIR) and the Lobatto-Gegenbauer rules (LGIR). The Gegenbauer polynomials, $C_n^\mu(x)$, $\mu > -1/2$, are those polynomials which are orthogonal with respect to the weight function $w(x; \mu) \equiv (1 - x^2)^{\mu - 1/2}$ and have the following normalization [4, p. 174]

$$\int_{-1}^{1} w(x; \mu) C_n^\mu(x) C_m^\mu(x)\, dx = \delta_{nm} h_{n\mu}, \tag{1}$$

where

$$h_{n\mu} = \pi^{1/2}\Gamma(n + 2\mu)\Gamma(\mu + 1/2)/((n + \mu)n!\Gamma(\mu)\Gamma(2\mu)), \tag{2}$$

which implies that $C_n^\mu(x) = k_{n\mu}x^n + \dots$, where

$$k_{n\mu} = 2^n\Gamma(n + \mu)/(n!\Gamma(\mu)). \tag{3}$$

$C_n^\mu(x)$ is even (odd) if n is even (odd). Special cases of $C_n^\mu(x)$, perhaps with a different normalization, are $T_n(x)$, the Chebyshev polynomials of the first kind ($\mu = 0$), $P_n(x)$, the Legendre polynomials ($\mu = 1/2$), and $U_n(x)$, the Chebyshev polynomials of the second kind ($\mu = 1$).

The n-point GGIR is given by

$$If \equiv \int_{-1}^{1} w(x; \mu) f(x)\, dx = \sum_{i=1}^{n} w_i f(x_i) + c_{n\mu} M_{2n}(f), \tag{4}$$

where we have omitted the dependence of w_i and x_i on μ and n, x_i are the zeros of $C_n^\mu(x)$,

$$c_{n\mu} = 2^{2n} h_{n\mu}/k_{n\mu}^2, \tag{5}$$

and $M_j(f)$ is defined to be equal to $f^{(j)}(\xi)/2^j j!$ for some $\xi \in (-1, 1)$. The corresponding LGIR has $n + 1$ points and is given by

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$$If = \sum_{i=1}^{n+1} \bar{w}_i f(\bar{x}_i) + \bar{c}_{n\mu} M_{2n}(f), \tag{6}$$

where the $\bar{x}_i$ are the zeros of $(1 - x^2)C_{n-1}^{\mu+1}(x)$, and

$$\bar{c}_{n\mu} = -\frac{2^{2n} h_{n-1,\mu+1}}{k_{n-1,\mu+1}^2} = -4c_{n-1,\mu+1}. \tag{7}$$

Since the weights of the integration rules considered do not play a part in the discussion, we shall not treat them here except to remark that Monegato [9], [10] has shown that the weights u_i in (8) below are positive for $0 \leqslant \mu \leqslant 1$ and the v_i, for $0 \leqslant \mu \leqslant 2$.

The KEGGIR is given by

$$If = \sum_{i=1}^{n} u_i f(x_i) + \sum_{i=1}^{n+1} v_i f(y_i) + E_{p_n}(f), \tag{8}$$

where $E_s(f) = 0$ if f is a polynomial of degree $< s$ and $p_n = 2[(3n + 3)/2]$. The y_i are the zeros of a certain polynomial $E_{n+1,\mu}(x)$ which we shall study in the next section. For the moment we state a result of Szegö [16] that for $0 \leqslant \mu \leqslant 2$, the y_i are real, lie in $[-1, 1]$, and are separated by the x_i. (For $\mu \neq 0$, the y_i lie in $(-1, 1)$.) The corresponding KELGIR is given by

$$If = \sum_{i=1}^{n+1} \bar{u}_i f(\bar{x}_i) + \sum_{i=1}^{n} \bar{v}_i f(\bar{y}_i) + E_{q_n}(f), \tag{9}$$

where $q_n = 2[(3n + 2)/2]$, and the $\bar{y}_i$ are the zeros of $E_{n,\mu+1}(x)$. Thus, taking into account that $\mu > -\frac{1}{2}$, we see that practical KEGGIR's exist for $0 \leqslant \mu \leqslant 2$ and KELGIR's, for $-\frac{1}{2} < \mu \leqslant 1$.

The first one to discover a KEGGIR was Kronrod [7] who dealt with the case $\mu = \frac{1}{2}$, the Gauss-Legendre or standard Gauss rule. Subsequently, Patterson [13], Piessens and Branders [14], and Monegato [11] improved on Kronrod's original work and extended his results to the usual Lobatto case ($\mu = \frac{1}{2}$). Barrucand [2] was the first to point out the connection between the KE's and the Szegö polynomials $E_{n+1,\mu}(x)$. KE's to other integration rules are discussed by Baratella [1], Kahaner and Monegato [5], Monegato [9], [12], and Ramskiĭ [15].

In the entire literature on this subject, it is stated that the KE's have error terms which vanish for polynomials of degree less than p_n (Gauss) or q_n (Lobatto), and in Kronrod's tables, he gives the error in the integration of x^{p_n} by the KEGGIR with $\mu = \frac{1}{2}$. However, nowhere is it *proved* that these KE's are of exact degree $p_n - 1$ or $q_n - 1$, as the case may be, that is, that there exists a polynomial of degree p_n or q_n for which the corresponding KE is not exact. Indeed, such a statement is not true for all μ. Thus, as Monegato [9] points out, the KE of the n-point GGIR with $\mu = 0$, the first Gauss-Chebyshev rule, is exact for polynomials of degree $\leqslant 4n - 1$ and in fact is identical with the KE of the corresponding $(n + 1)$-point LGIR, being the $(2n + 1)$-point LGIR, the first Lobatto-Chebyshev rule. Furthermore, the KE of the n-point GGIR with $\mu = 1$, the second Gauss-Chebyshev rule, is exact for polynomials of degree

$\leqslant 4n + 1$ and, in fact, is identical with the $(2n + 1)$-point GGIR. In the present work, we shall show that, except for $\mu = 0, 1$ in the GGIR case and $\mu = 0$ in the LGIR case, we have the result that the exact precision of the KEGGIR is $p_n - 1$ while that of the KELGIR is $q_n - 1$. Furthermore, if these rules are of simplex type, i.e., if we can express the error term in the form $K_{n\mu} f^{(p_n)}(\xi)$ or $K_{n\mu} f^{(q_n)}(\xi)$, which we have not been able to prove, then we have the following result:

$$If = \sum_{i=1}^{n} u_i f(x_i) + \sum_{i=1}^{n+1} v_i f(y_i) + d_{n\mu} c_{n\mu} M_{p_n}(f), \tag{10}$$

$$If = \sum_{i=1}^{n+1} \bar{u}_i f(\bar{x}_i) + \sum_{i=1}^{n} \bar{v}_i f(\bar{y}_i) + d_{n-1,\mu+1} \bar{c}_{n\mu} M_{q_n}(f), \tag{11}$$

where $d_{n\mu}$ is easily computable and does not vanish for $0 < \mu \leqslant 2, \mu \neq 1$, and all $n \geqslant 2$. For $\mu = 2$ we have the explicit expression

$$d_{n\mu} = \begin{cases} \dfrac{2}{n+3}\left(\dfrac{n+1}{n+3}\right)^m, & n \text{ even}, \\ 4(n+2)(n+1)^{m-1}/(n+3)^{m+1}, & n \text{ odd}, \end{cases} \tag{12}$$

where $m = [(n + 1)/2]$.

2. The Szegö Polynomials $E_{n+1,\mu}$. We give here the main results of Szegö with some minor modification of his notation and refer to [16] for details. See also Davis and Rabinowitz [3, pp, 82–89] with Monegato [11].

The Gegenbauer function of the second kind, $Q_n^\mu(z)$, defined by

$$\begin{aligned} Q_n^\mu(z) &= \frac{\Gamma(2\mu)}{2\Gamma(\mu + \frac{1}{2})}\int_{-1}^{1} w(t;\mu)\frac{C_n^\mu(t)}{z-t}\,dt \\ &= \frac{\Gamma(2\mu)}{2\Gamma(\mu + \frac{1}{2})} z^{-n-1}\sum_{i=0}^{\infty} \beta_i z^{-2i}, \end{aligned} \tag{13}$$

where

$$\beta_i = \int_{-1}^{1} w(t;\mu) C_n^\mu(t) t^{n+2i}\,dt, \qquad i = 0, 1, \ldots, \tag{14}$$

is analytic in the entire complex plane with a slit on the closed interval $[-1, 1]$. Hence

$$\frac{1}{Q_n^\mu(z)} = z^{n+1}\sum_{i=0}^{\infty} \gamma_i z^{-2i} = E_{n+1,\mu}(z) + \delta_1 z^{-1} + \delta_2 z^{-2} + \ldots, \tag{15}$$

defining the polynomial $E_{n+1,\mu}(z)$ which is even (odd) for n odd (even). Thus,

$$Q_n^\mu(z) E_{n+1,\mu}(z) = 1 + b_1 z^{-n-2} + b_2 z^{-n-3} + \ldots, \tag{16}$$

and by the argument given in [16] or [3]

$$Q_n^\mu(z) E_{n+1,\mu}(z) = 1 + \sum_{i=0}^{n} c_i Q_{n+1+i}^\mu(z), \tag{17}$$

for certain constants $c_0, \ldots, c_n$ depending on μ and n. Since $Q_n^\mu(z)$ is an odd (even) function if n is even (odd), we have that $Q_n^\mu(z)E_{n+1,\mu}(z)$ is always an odd function which implies that $c_0 = 0$ if n is odd.

Now, the functions of the second kind satisfy the following relations:

$$\lim_{\epsilon\to 0}(Q_n^\mu(x + i\epsilon) - Q_n^\mu(x - i\epsilon)) = -i\pi \frac{\Gamma(2\mu)}{\Gamma(\mu + ½)} w(x; \mu)C_n^\mu(x), \tag{18}$$

$$\lim_{\epsilon\to 0}(Q_n^\mu(x + i\epsilon) + Q_n^\mu(x - i\epsilon)) = 2\widetilde{Q}_n^\mu(x), \tag{19}$$

where $\widetilde{Q}_n^\mu(x)$ is defined on the segment $[-1, 1]$. Hence

$$C_n^\mu(x)E_{n+1,\mu}(x) = \sum_{i=0}^{n} c_i C_{n+1+i}^\mu(x), \tag{20}$$

and

$$\widetilde{Q}_n^\mu(x)E_{n+1,\mu}(x) = 1 + \sum_{i=0}^{n} c_i\widetilde{Q}_{n+1+i}^\mu(x). \tag{21}$$

From (20) it follows that

$$\int_{-1}^{1} w(x; \mu)C_n^\mu(x)E_{n+1,\mu}(x)x^k\, dx = 0, \qquad k = 0, 1, 2, \ldots, n, \tag{22}$$

so that, by the theorem in [3, p. 77], an interpolatory integration rule based on the zeros of $C_n^\mu(x)$ and $E_{n+1,\mu}(x)$ is exact for all polynomials of degree $\leqslant 3n + 1$ which forms the basis for KEGGIR's.

Now, it can be shown that

$$\begin{aligned} Q_n^\mu(z) &= \gamma_{n\mu} w^{-n-1} F(1 - \mu, n + 1; n + \mu + 1; w^{-2}) \\ &= \gamma_{n\mu} \sum_{j=0}^{\infty} f_{j\mu} w^{-n-1-2j}, \end{aligned} \tag{23}$$

where $z = ½(w + w^{-1})$, $\gamma_{n\mu} = \sqrt{\pi}\,\Gamma(n + 2\mu)/\Gamma(n + \mu + 1)$, $F(a, b; c; z)$ is the usual hypergeometric function, $f_{0\mu} = 1$,

$$f_{j\mu} = (1 - \mu/j)(1 - \mu/(n + \mu + j))f_{j-1,\mu}, \tag{24}$$

and we have not shown the dependence on n of the $f_{j\mu}$.

Setting $w = e^{-i\theta}$ and $x = \cos\theta$, we get that

$$\widetilde{Q}_n^\mu(x) = \gamma_{n\mu} \sum_{j=0}^{\infty} f_{j\mu} T_{n+1+2j}(x). \tag{25}$$

Since $E_{n+1,\mu}(x)$ contains only even or odd powers of x, we can write $E_{n+1,\mu}(x)$ in the form

$$E_{n+1,\mu}(x) = \sum_{i=0}^{m-1} \lambda_{i\mu} T_{n+1-2i}(x) + \begin{cases} \lambda_{m\mu} T_1(x), & n \text{ even}, \\ \frac{1}{2}\lambda_{m\mu}, & n \text{ odd}, \end{cases} \tag{26}$$

where $m = [(n + 1)/2]$.

To determine the coefficients $\lambda_{i\mu}$, we equate, in view of (21) and (25), the coefficients of $T_k(x)$, $k = 1, \ldots, n + 1$, in the product

$$\tilde{Q}_n^{\mu}(x)E_{n+1,\mu}(x) = \gamma_{n\mu}\left(\sum_{j=0}^{\infty} f_{j\mu}T_{n+1+2j}(x)\right)\left({\sum_{i=0}^{m}}' \lambda_{i\mu}T_{n+1-2i}(x)\right) \tag{27}$$

to zero and the coefficient of $T_0(x)$ to unity. Here the prime means that if n is odd, we replace $\lambda_{m\mu}$ by $\frac{1}{2}\lambda_{m\mu}$. Since $T_r(x)T_s(x) = \frac{1}{2}(T_{r+s}(x) + T_{|r-s|}(x))$, we see that the $\lambda_{i\mu}$ must satisfy the following equations

$$\lambda_{0\mu} = 2\gamma_{n\mu}^{-1}, \qquad \sum_{i=0}^{k} f_{i,\mu}\lambda_{k-i,\mu} = 0, \qquad k = 1, \ldots, m. \tag{28}$$

Following Monegato [11], we define $\alpha_{i\mu} = \lambda_{i\mu}/\lambda_{0\mu}$ so that $\alpha_{0\mu} = 1$, $\alpha_{1\mu} = -f_{1\mu}$, and

$$\alpha_{k\mu} = -f_{k\mu} - \sum_{i=1}^{k-1} f_{i\mu}\alpha_{k-i,\mu}, \qquad k = 2, \ldots, m. \tag{29}$$

From this, we see that the $\alpha_{i\mu}$ are the first $m + 1$ coefficients in the series

$$\sum_{i=0}^{\infty}\alpha_{i\mu}u^i = \left\{\sum_{j=0}^{\infty} f_{j\mu}u^j\right\}^{-1}, \tag{30}$$

so that we can also use (29) for indices $k > m$. Here also we have not indicated the dependence on n of the $\lambda_{i\mu}$ and $\alpha_{i\mu}$.

3. The Exact Degree of Precision of KEGGIR's and KELGIR's. Let us define

$$f_k(x) = C_n^{\mu}(x)E_{n+1,\mu}(x)C_{n+1+k}^{\mu}(x), \qquad k = 0, \ldots, n. \tag{31}$$

Then from (20) it follows that $If_k = c_k h_{n+1+k,\mu}$. Since the KEGGIR applied to $f_k(x)$ vanishes, we have from (8) that $E_{p_n}(f_k) = c_k h_{n+1+k,\mu}$ so that the exact precision of the KEGGIR is determined by the first index k, say k_0, for which $c_{k_0} \neq 0$. Indeed, $p_n = 3n + 1 + k_0$. We now show that for $0 < \mu \leqslant 2$, $\mu \neq 1$, $c_0 \neq 0$ for n even and $c_1 \neq 0$ for n odd.

Consider first the case n even. Substituting (25) and (27) into (21) and equating the coefficients of $T_{n+2}(x)$, we find that

$$\begin{aligned} c_0\gamma_{n+1,\mu} &= \frac{\gamma_{n\mu}}{2}\{\lambda_{m\mu}f_{0\mu} + \lambda_{m\mu}f_{1\mu} + \lambda_{m-1,\mu}f_{2\mu} + \cdots + \lambda_{0\mu}f_{m+1,\mu}\} \\ &= \alpha_{m\mu} + \alpha_{m\mu}f_{1\mu} + \alpha_{m-1,\mu}f_{2\mu} + \cdots + \alpha_{1\mu}f_{m\mu} + f_{m+1,\mu} \\ &= \alpha_{m\mu} - \alpha_{m+1,\mu}. \end{aligned} \tag{32}$$

Thus, it suffices to show that $\alpha_{m\mu} - \alpha_{m+1,\mu}$ does not vanish. In fact, we shall show that the $\alpha_{i\mu}$ are strictly monotonic. For $0 < \mu < 1$, the sequence $\{f_{j\mu}\}$ is completely monotonic, i.e., $(-1)^k\Delta^k f_{j\mu} > 0$ for all j and k [17, p. 137]. Hence, by a theorem of Kaluza [6], the sequence $\{-\alpha_{i+1,\mu}\}$ is also completely monotonic and hence strictly monotonic. For $1 < \mu < 2$, the sequence $\{-f_{j+1,\mu}\}$ is completely monotonic. From this it follows, by some results in [6], that

$$\frac{\alpha_{i-1,\mu}}{\alpha_{i\mu}} > \frac{\alpha_{i\mu}}{\alpha_{i+1,\mu}}, \qquad i = 1, 2, \ldots .$$

Since $\Sigma_{i=0}^{\infty}\alpha_{i\mu}$ converges, and in fact equals $\{F(1-\mu, n+1; n+\mu+1; 1)\}^{-1}$, it follows that the sequence $\{\alpha_{i\mu}\}$ is strictly monotonic. For $\mu = 2$, Szegö [16] gives an explicit expression for the $\lambda_{i\mu}$,

$$\lambda_{i2} = \frac{2}{\sqrt{\pi}} \frac{1}{n+3} \left(\frac{n+1}{n+3}\right)^i, \qquad i = 0, 1, \ldots, \tag{33}$$

which again shows that the α_{i2} are strictly monotonic.

We now consider the case n odd. Proceeding as before, this time equating the coefficients of $T_{n+3}(x)$, we find that

$$\begin{aligned} c_1\gamma_{n+2,\mu} &= \frac{\gamma_{n\mu}}{2}\{\lambda_{m\mu}f_{1\mu} + \lambda_{m-1,\mu}f_{0\mu} + \lambda_{m-1,\mu}f_{2\mu} \\ &\qquad + \lambda_{m-2,\mu}f_{3\mu} + \cdots + \lambda_{0\mu}f_{m+1,\mu}\} \\ &= \alpha_{m-1,\mu} + \alpha_{m\mu}f_{1\mu} + \alpha_{m-1,\mu}f_{2\mu} + \cdots + \alpha_{1\mu}f_{m\mu} + f_{m+1,\mu} \\ &= \alpha_{m-1,\mu} - \alpha_{m+1,\mu}. \end{aligned} \tag{34}$$

Since the $\alpha_{i\mu}$ are strictly monotonic, it follows that $c_1 \neq 0$.

For $\mu = 0, f_{j0} = 1, j = 0, 1, 2, \ldots$, so that $\lambda_{00} = -\lambda_{10} = 2n/\pi^{1/2}$, $\lambda_{i0} = 0$, $i > 1$ and $E_{n+1,0} = (2n/\pi^{1/2})\{T_{n+1}(x) - T_{n-1}(x)\}$, $n \geqslant 2$. Hence

$$\begin{aligned} C_n^0(x)E_{n+1,0}(x) &= k_1T_n\{T_{n+1} - T_{n-1}\} = \frac{k_1}{2}\{T_{2n+1} - T_{2n-1}\} \\ &= k_2(1-x^2)U_{2n-1} = k_3(1-x^2)C_{2n-1}^1(x), \end{aligned} \tag{35}$$

and the zeros of $C_n^0(x)E_{n+1,0}(x)$ are the abscissas of the $(2n+1)$-point LGIR for the weight $w(x; 0)$ which is of exact precision $4n-1$, as can also be seen from the fact that c_{n-2} is the first c_k which does not vanish.

For $\mu = 1, f_{01} = 1, f_{j1} = 0, j > 0$ so that, $\lambda_{01} = 2/\sqrt{\pi}$, $\lambda_{i1} = 0$, $i > 0$, and $E_{n+1,1}(x) = (2/\sqrt{\pi})T_{n+1}(x)$. Hence

$$C_n^1(x)E_{n+1,1}(x) = k_1'U_n(x)T_{n+1}(x) = k_2'C_{2n+1}^1(x), \tag{36}$$

and the zeros of $C_n^1(x)E_{n+1,1}(x)$ are the abscissas of the $(2n+1)$-point GGIR for the weight $w(x; 1)$ which is of exact precision $4n+1$ and which also follows from the fact that c_n is the first c_k which does not vanish.

In the case of the KELGIR, we define

$$\bar{f}_k(x) = (1-x^2)C_{n-1}^{\mu+1}(x)E_{n,\mu+1}(x)C_{n+k}^{\mu+1}(x), \qquad k = 0, 1, \ldots, n-1, \tag{37}$$

so that $I\bar{f}_k = c_kh_{n+k,\mu+1}$. Hence, since $c_0 = c_0(n-1, \mu+1) \neq 0$ for $n-1$ even, i.e., for n odd, while $c_1 \neq 0$ for $n-1$ odd, we have that the $(2n+1)$-point KELGIR is of exact precision $3n+1$, for n even, and $3n$, for n odd, provided that $\mu \neq 0$. For $\mu = 0$, we have as before that $E_{n1}(x) = (2/\pi^{1/2})T_n(x)$, so that

$$(1-x^2)C_{n-1}^1(x)E_{n1}(x) = \hat{k}_1(1-x^2)C_{2n-1}^1(x), \tag{38}$$

whose zeros are again the abscissas of the $(2n+1)$-point LGIR for the weight $w(x; 0)$.

If we now define

$$d_{n\mu} = \begin{cases} \alpha_{m\mu} - \alpha_{m+1,\mu}, & n \text{ even}, \\ \alpha_{m-1,\mu} - \alpha_{m+1,\mu}, & n \text{ odd}, m = [(n+1)/2], \end{cases} \tag{39}$$

we have that for the Gauss case

$$d_{n\mu} = \begin{cases} c_0\gamma_{n+1,\mu}, & n \text{ even}, \\ c_1\gamma_{n+2,\mu}, & n \text{ odd}, \end{cases} \tag{40}$$

while for the Lobatto case

$$d_{n-1,\mu+1} = \begin{cases} c_0\gamma_{n,\mu+1}, & n \text{ even}, \\ c_1\gamma_{n+1,\mu+1}, & n \text{ odd}, \end{cases}$$

where we have suppressed the dependence of c_0 and c_1 on n and μ. This leads us immediately to formulas (10) and (11). For example, applying (8) with n even to $f_0(x)$, we have that

$$c_0 h_{n+1,\mu} = K_{n\mu}k_{n\mu}2\gamma_{n\mu}^{-1}2^n k_{n+1,\mu}(3n+2)!, \tag{41}$$

so that

$$K_{n\mu} = \frac{d_{n\mu}}{\gamma_{n+1,\mu}} \frac{h_{n+1,\mu}\gamma_{n\mu}}{2^{n+1}k_{n\mu}k_{n+1,\mu}(3n+2)!} = \frac{d_{n\mu}c_{n\mu}}{2^{p_n}p_n!}. \tag{42}$$

For n odd, we consider $f_1(x)$ while in the Lobatto case we work with $\bar{f}_0(x)$ and $\bar{f}_1(x)$.

4. Remarks. a. Monegato [11] gives an error bound for KEGGIR's with $0 < \mu < 1$. We shall show how to improve this bound slightly and extend it to the case $1 < \mu < 2$, as well as to KELGIR's with $-\frac{1}{2} < \mu \leqslant 1$, $\mu \neq 0$.

For n even, Monegato writes the error $E_{p_n}(f)$ for $f \in C^{3n+2}[-1, 1]$ in the form

$$E_{p_n}(f) = \frac{2^{-2n}}{k_{n\mu}(3n+2)!} \int_{-1}^{1} w(x;\mu)C_n^\mu(x)(\bar{E}_{n+1,\mu}(x))^2 f^{(3n+2)}(\xi_x)\,dx, \tag{43}$$

where

$$\bar{E}_{n+1,\mu}(x) = E_{n+1,\mu}(x)/\lambda_{0\mu} = \sum_{i=0}^{m}{}' \alpha_{i\mu}T_{n+1-2i}(x). \tag{44}$$

Hence

$$|E_{p_n}(f)| \leqslant \frac{\pi\Gamma(n+2\mu)B_{n+1,\mu}^2}{2^{3n+2\mu-1}p_n!\Gamma(\mu+1)\Gamma(n+\mu)} M_{p_n}, \tag{45}$$

where

$$M_s = \max_{-1 \leqslant x \leqslant 1} |f^{(s)}(x)| \quad \text{and} \quad B_{n+1,\mu} = \max_{-1 \leqslant x \leqslant 1} |\bar{E}_{n+1,\mu}(x)|.$$

For $0 < \mu < 1$, Monegato states that $B_{n+1,\mu} < 2$ and replaces $B_{n+1,\mu}$ by 2 in (45). Now, while this bound is the best available for $0 < \mu \leqslant 1/2$, we can improve on it for $1/2 < \mu < 1$. In addition, a bound on $B_{n+1,\mu}$ is also available for $1 < \mu \leqslant 2$. This follows from our observation above that

$$\begin{aligned}\sum_{i=0}^{\infty} \alpha_{i\mu} &= \{F(1-\mu, n+1; n+\mu+1, 1)\}^{-1} \equiv T_{n\mu} \\ &= \frac{\Gamma(\mu)\Gamma(n+2\mu)}{\Gamma(n+\mu+1)\Gamma(2\mu-1)}, \qquad \mu > 1/2, \mu \neq 1, 2.\end{aligned} \tag{46}$$

Now for $1/2 < \mu < 1$, $\alpha_{0\mu} = 1$, $\alpha_{i\mu} < 0$, $i > 0$. Since

$$B_{n+1,\mu} \leqslant \sum_{i=0}^{m} |\alpha_{i\mu}| = 1 - \sum_{i=1}^{m} \alpha_{i\mu} < 1 - \sum_{i=1}^{\infty} \alpha_{i\mu},$$

it follows that $B_{n+1,\mu} < 2 - T_{n\mu} < 2$. For $1 < \mu < 2$, we have that $\alpha_{i\mu} > 0$, for all i. Hence $B_{n+1,\mu} \leqslant \Sigma_{i=0}^{m}\alpha_{i\mu} < T_{n\mu}$. For $\mu = 2$,

$$\sum_{i=0}^{\infty} \alpha_{i2} = \left(1 - \frac{n+1}{n+3}\right)^{-1} = \frac{n+3}{2} > B_{n+1,2}.$$

For n odd, using classical arguments, we have the same bound.

In the Lobatto case, we have, similarly for n odd, that

$$E_{q_n}(x) = \frac{2^{2-2n}}{k_{n-1,\mu+1}(3n+1)!} \int_{-1}^{1} w(x; \mu+1) C_{n-1}^{\mu+1}(x)(\bar{E}_{n,\mu+1}(x))^2 f^{(3n+1)}(\bar{\xi}_x)\, dx, \tag{47}$$

whence

$$|E_{q_n}(f)| \leqslant \frac{\pi\Gamma(n+2\mu+1)B_{n,\mu+1}^2}{2^{3n+2\mu-2}q_n!\Gamma(n+\mu)\Gamma(\mu+2)} M_{q_n}, \tag{48}$$

where for $-1/2 < \mu < 0$, $B_{n,\mu+1} < 2 - T_{n-1,\mu+1}$ and for $0 < \mu < 1$, $B_{n,\mu+1} < T_{n-1,\mu+1}$. For $\mu = 1$, $B_{n2} < (n+2)/2$. As before, the same bound holds for n even.

b. The Fourier-Gegenbauer coefficients of a function $f(x)$ are defined by

$$\mathrm{FG}_{n\mu}(f) = h_{n\mu}^{-1} \int_{-1}^{1} w(x; \mu) C_n^{\mu}(x) f(x)\, dx, \qquad n = 0, 1, \ldots . \tag{49}$$

As Barrucand [2] points out, the integral is most efficiently evaluated by a $(2n+1)$-point KEGGIR applied to the function $C_n^{\mu}(x)f(x)$ which reduces to the $(n+1)$-point formula

$$\mathrm{FG}_{n\mu}(f) \simeq h_{n\mu}^{-1} \sum_{i=1}^{n+1} v_i C_n^{\mu}(y_i) f(y_i) = \sum_{i=1}^{n+1} \tilde{v}_i f(y_i). \tag{50}$$

For $\mu \neq 0, 1$, we get a rule which is exact for polynomials of degree $< p_n - n$, which is the best possible. For assume that there existed an $(n+1)$-point rule, say

$$\mathrm{FG}_n(f) \simeq \sum_{i=1}^{n+1} \hat{v}_i f(\hat{y}_i), \tag{51}$$

exact for polynomials of degree $p_n - n$, n even. This would imply that

$$\int_{-1}^{1} w(x;\mu)C_n^\mu(x)E_{n+1,\mu}(x)\prod_{i=1}^{n+1}(x-\hat{y}_i)\,dx = 0, \tag{52}$$

which contradicts our results above. Similarly for n odd.

For $\mu = 0$, the rule (50) is exact for polynomials of degree $\leqslant 3n - 1$, a result which has already been reported in [8]. For $\mu = 1$, (50) is exact for polynomials of degree $\leqslant 3n + 1$ which is the best possible result, so that the highest precision is achieved for Fourier-Chebyshev coefficients of the second kind. However, we should warn the user that the weights $\tilde{v}_i$ in (50) alternate in sign inasmuch as the v_i are positive and the zeros of $C_n^\mu(x)$ separate those of $E_{n+1,\mu}(x)$, so that the $C_n^\mu(y_i)$ alternate in sign.

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