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Galerkin Methods for Singular Integral Equations

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Abstract. The approximate solution of a singular integral equation by Galerkin's method is studied. We discuss the theoretical aspects of such problems and give error bounds for the approximate solution.

1. Introduction. In this paper we will discuss Galerkin's method for the approximate solution of the singular integral equation

$$u(s) - \frac{1}{\pi}\int_{-1}^{+1} \frac{k(s, t)u(t)}{t - s}\, dt = f(s), \tag{1.1}$$

in which $k(s, t)$ is a real-valued kernel (assumed smooth), $f(s)$ is a given function, and $u(s)$ is the unknown function. The integral is to be interpreted as a Cauchy principal value throughout the paper. It is well known that the solution of (1.1) is not unique unless one restricts the space of functions in which the solution is sought in some manner. In this paper we restrict u to lie in $L^2[-1, 1]$.

In Section 2 we outline the theory for singular integral equations. This is mainly based on the treatment in [12]. We will show the uniqueness of the L^2-solution of (1.1). Also, from the theoretical treatment, one obtains the asymptotic behavior of the solution at ± 1. The solution, in fact, nearly always has endpoint singularities.

In Section 3 we give the error analysis for Galerkin's method. The main result is that

$$\|u - u_n\|_{L^2} \leqslant C(1 + o(1))\|(I - P_n)u\|, \tag{1.2}$$

where u_n is the approximate solution, C a constant, and P_n is the projection operator from $L^2[-1, 1]$ into the space of trial functions.

Section 4 deals with the use of spline functions as a basis for Galerkin's method. The endpoint singularities mean that one must use splines on nonuniform partitions and singular functions as the trial space. We show how to calculate a partition that gives an asymptotic rate of convergence of $O(N^{-k})$ where $k - 1$ is the degree of the spline. The paper is concluded by a numerical example.

The basis for this paper is the section on singular integral equations in the book by Cherruault [4]. Different treatments for singular integral equations include [6], [7], [8], [9], [10].

2. Theoretical Treatment of Singular Integral Equations. The theoretical analysis of singular integral equations using complex variable theory dates from Carleman [3] and may be found in numerous text books (e.g. Tricomi [15], Muskhelishvili [12]). We follow the treatment of [12, Chapters 10, 14] and show that (1.1) has a

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unique solution on $L^2[-1, 1]$. The reader is left to fill in the details.

[12] shows that, in order to solve a singular integral equation such as (1.1) on an open interval, one must specify the class $\mathcal{C}$ of functions in which the solution is sought. This is often accomplished by imposing additional boundary conditions on the solution. The questions of existence and uniqueness of solutions are answered by computing the index κ (an integer) of the integral equation with respect to the class of functions $\mathcal{C}$. (When considering singular integral equations on a contour in the complex plane the index does not depend on additional boundary conditions.)

The integral equation (1.1) is rewritten in the form

$$Au - Ku = f, \tag{2.1}$$

where

$$Au(s) = u(s) - \frac{k(s, s)}{\pi} \int_{-1}^{+1} \frac{u(t)}{t - s}\, dt \tag{2.2}$$

and

$$Ku(s) = \frac{1}{\pi} \int_{-1}^{+1} \frac{k(s, t) - k(s, s)}{t - s} u(t)\, dt. \tag{2.3}$$

The operator A is called the dominant part of the integral equation. K is compact because of the smoothness (Hölder-continuity) of $k(s, t)$.

The first stage of the theoretical analysis is to solve the equation

$$Av = g, \tag{2.4}$$

where $v, g \in L^2[-1, 1]$.

[12, Chapter 14] is used to determine the analytic solution of (2.4). We define the function

$$\theta(s) = \frac{1}{\pi} \underset{(-\pi/2, \pi/2)}{\arctan}\; k(s, s), \tag{2.5}$$

integers n_1, n_2 such that

$$-1 < \theta(1) + n_1 < 1, \qquad -1 < -\theta(-1) + n_2 < 1, \tag{2.6}$$

and the function

$$\Omega(s) = (1 - s)^{n_1}(1 + s)^{n_2} \exp\left[\int_{-1}^{+1} \frac{\theta(t)}{t - s}\, dt\right].$$

The index of (2.4) is then given by

$$\kappa = -(n_1 + n_2), \tag{2.7}$$

and the analytic solution of (2.4) by

$$\begin{aligned} v &= K^* g \\ &= \frac{g(s)}{1 + k^2(s, s)} + \frac{k(s, s)\Omega(s)}{\pi\sqrt{1 + k^2(s, s)}} \int_{-1}^{+1} \frac{g(t)}{\sqrt{(1 + k^2(t, t))}\Omega(t)(t - s)}\, dt \\ &\quad + \frac{k(s, s)\Omega(s)P_{\kappa - 1}(s)}{\sqrt{1 + k^2(s, s)}}, \end{aligned} \tag{2.8}$$

where $P_{\kappa-1}(s)$ is an arbitrary polynomial of degree $\kappa - 1$, ($P_{\kappa-1}(s) \equiv 0$ if $\kappa < 0$).

The analytic solution gives useful information about the asymptotic behavior near the ends $s = \pm 1$. By applying the results in [12, Chapter 4] we can show that

(2.9) $$v(s) \sim a_1(1-s)^{\theta(1)+n_1}, \quad \text{near } s = 1,$$

(2.10) $$v(s) \sim a_2(1+s)^{-\theta(-1)+n_2}, \quad \text{near } s = -1,$$

provided $\theta(1) \neq 0$ and $\theta(-1) \neq 0$.

If $\theta(s)$ has a zero of order r at $s = \pm 1$, then

(2.11) $$v(s) \sim f(s) + b_1(1 \pm s)^r \log(1 \pm s), \quad \text{near } s = \pm 1.$$

The conditions (2.6) indicate that n_1, n_2 are only determined to within ± 1. Only by imposing the condition $v \in \mathcal{C}$ do we determine n_1, n_2. By choosing n_1, n_2 to minimize the value $n_1 + n_2$ can take without violating the condition $v \in \mathcal{C}$, (2.8) will give the most general solution in the class $\mathcal{C}$ and (2.7) the index.

If we require that $v \in L^2[-1, 1]$, we should replace (2.6) by

(2.12) $$-\frac{1}{2} < \theta(1) + n_1 < 1, \quad -\frac{1}{2} < -\theta(-1) + n_2 < 1$$

to ensure that our solution is square-integrable. We can manipulate the first of these inequalities to get

$$-1 \leqslant -\frac{1}{2} - \theta(1) < n_1 < 1 - \theta(1) \leqslant \frac{3}{2}$$

since $-\frac{1}{2} < \theta(s) < \frac{1}{2}$. Hence, n_1 can only take the values 0 or 1. Similarly n_2 can only take the values 0 or 1. Hence, the minimum value of $n_1 + n_2$, subject to $v \in L^2[-1, 1]$, is zero, i.e. $\kappa = 0$.

This establishes the uniqueness of the solution of the dominant equation.

Having solved the dominant part of the singular equation, the full equation can be reduced to the Fredholm equation,

$$u - K^*Ku = K^*f.$$

Since the operator K^*K is compact, the usual theory of the Fredholm alternative is applicable. This means that either the integral equation will have a unique solution in $L^2[-1, 1]$ or unity will be an eigenvalue of K^*K. We will assume for the remainder of the paper the existence of a unique L^2 solution.

Moreover, we have assumed that $f(s)$ and $k(s, t)$ are smooth functions, so that $g = Ku + f$ defines a smooth function g. Since $u = K^*g$, $u(s)$ will have similar asymptotic behavior as $v(s)$ near $s = \pm 1$, governed by the equations (2.9)–(2.11).

3. The Error Analysis of Galerkin's Method. To give an error analysis of Galerkin's method, we rewrite (1.1) in the form

(3.1) $$u - K_1u - K_2u = f,$$

where

(3.2) $$K_1u(s) = \frac{1}{2\pi}\int_{-1}^{+1} \frac{k(s,t) + k(t,s)}{t-s} u(t)\, dt$$

and

(3.3) $$K_2u(s) = \frac{1}{2\pi}\int_{-1}^{+1} \frac{k(s,t) - k(t,s)}{t-s} u(t)\, dt.$$

The operator K_1 is skew-hermitian and the operator K_2 is symmetric and compact. We first give the analysis of Galerkin's method for the problem

(3.4) $$Bu = f,$$

where $B = I - K_1$. We define the bilinear functional $a(u, v) = (Bu, v)$ for $u, v \in L^2[-1, 1]$.

LEMMA 3.1. (i) Re $a(u, u) = \|u\|^2$, $u \in L^2[-1, 1]$.
(ii) $|a(u, v)| \leqslant \|B\| \, \|u\| \, \|v\|$, $u, v \in L^2[-1, 1]$.
(iii) B^{-1} *exists and* $\|B^{-1}\| \leqslant 1$.

Proof. (i) $\mathrm{Re}(K_1 u, u) = \frac{1}{2}[(K_1 u, u) + \overline{(K_1 u, u)}] = \frac{1}{2}((K_1 + K_1^*)u, u) = 0$ since $K_1^* = -K_1$. Hence, Re $a(u, u) = (u, u) - \mathrm{Re}(K_1 u, u) = \|u\|^2$.

(ii) This follows directly from the Schwarz inequality.

(iii) From (i) and the Schwarz inequality, one can show $\|u\|^2 \leqslant \|Bu\| \, \|u\|$. Hence, $\|u\| \leqslant \|Bu\|$. (If $\|u\| \neq 0$, the division is permissible, if $u = 0$, the result is trivial.) This establishes the existence of B^{-1}, and $\|B^{-1}\| \leqslant 1$ is established from a standard result in functional analysis.

We let $S_n \subset L^2[-1, 1]$ be the finite-dimensional subspace (the dimension of S_n is n) in which we find the Galerkin approximation u_n. We let P_n be the projection operator onto the subspace S_n. The Galerkin approximation will satisfy

$$a(u_n, v_n) = (f, v_n) \quad \text{for } v_n \in S_n,$$

and the error in method is given by the following.

THEOREM 3.2. *The error in Galerkin's method for the problem $Bu = f$ is given by*

(3.5) $$\|u - u_n\| \leqslant \|B\| \, \|(I - P_n)u\|.$$

Proof. It is readily verified that $a(u - u_n, w_n) = 0$ for all $w_n \in S_n$. Hence,

$$\begin{aligned} \|u - u_n\|^2 &= \mathrm{Re}\, a(u - u_n, u - u_n) \leqslant |a(u - u_n, u - u_n)| \\ &\leqslant |a(u - u_n, u - P_n u)| + |a(u - u_n, P_n u - u_n)| \\ &= |a(u - u_n, (I - P_n)u)| \leqslant \|B\| \, \|u - u_n\| \, \|(I - P_n)u\| \end{aligned}$$

from Lemma 3.1. Hence, we may divide through by $\|u - u_n\|$ and get the result.

For the full equation (3.1) we put

(3.6) $$Bu = v.$$

Then (3.1) is equivalent to

(3.7) $$v - K_2 B^{-1} v = f.$$

Since B^{-1} is continuous and K_2 compact, $K_2 B^{-1}$ forms a compact operator. We can use the theory of prolongation and restriction operators developed in [11], [14] to produce error bounds. We define operators q_n and s_n as follows. $\phi_i(t)$, $i = 1, \ldots, n$, is a set of linearly independent functions that span S_n. Define

$$q_n \colon E_n \to L^2[-1, 1]$$

by

(3.8) $$q_n \mathbf{v}_n = \sum_{j=1}^{n} v_j \phi_j$$

and

$$s_n \colon L^2[-1, 1] \to E_n$$

by

(3.9) $$s_n f = \{(f, \phi_i)\}.$$

It is readily confirmed that

(3.10) $$s_n q_n = G_n,$$

the Gramm matrix of the basis.

We put

(3.11) $$B_n = s_n B q_n.$$

Then Galerkin's method for the problem (3.4) produces the linear equations

(3.12) $$B_n \mathbf{u}_n = s_n f$$

and $q_n \mathbf{u}_n$ is the Galerkin approximation. However, [14, Eq. 4] is not satisfied by q_n, s_n, so we define prolongation and restriction operators p_n, r_n by

(3.13) $$p_n = Bq_n, \quad r_n = B_n^{-1} s_n,$$

and now $r_n p_n = I_n$.

We define our norm in E_n by

(3.14) $$\|\mathbf{v}_n\|_{E_n} = \|q_n \mathbf{v}_n\|_{L^2}.$$

This norm is related to the Euclidean vector norm by the following

LEMMA 3.3. (i) $(q_n \mathbf{f}_n, g) = \langle \mathbf{f}_n, s_n g\rangle$, $\mathbf{f}_n \in E_n$, $g \in L^2[-1, 1]$ (*i.e.* $s_n^* = q_n$, $q_n^* = s_n$).

(ii) $\|\mathbf{v}_n\|_{E_n} = \|G_n^{1/2}\mathbf{v}_n\|_2$, *where* $\langle \mathbf{f}_n, g_n\rangle = \sum f_i \bar{g}_i$, *denotes the inner-product in* E_n *and* $\| \ \|_2$ *is the usual Euclidean vector norm*

$$\|\mathbf{v}_n\|_2^2 = \langle \mathbf{v}_n, \mathbf{v}_n\rangle.$$

Proof.

(i) $$(q_n \mathbf{f}_n, g) = \left(\sum_{i=1}^{n} f_i \phi_i, g\right) = \sum_{i=1}^{n} f_i \left(\overline{g, \phi_i}\right) = \langle \mathbf{f}_n, s_n g\rangle.$$

(ii) $$\begin{aligned}\|\mathbf{v}_n\|_{E_n}^2 &= \|q_n \mathbf{v}_n\|_{L^2}^2 = (q_n \mathbf{v}_n, q_n \mathbf{v}_n) = \langle \mathbf{v}_n, s_n q_n \mathbf{v}_n\rangle \\ &= \langle \mathbf{v}_n, G_n \mathbf{v}_n\rangle = \langle G_n^{1/2}\mathbf{v}_n, G_n^{1/2}\mathbf{v}_n\rangle = \|G_n^{1/2}\mathbf{v}_n\|^2.\end{aligned}$$

Lemma 3.3 is useful in establishing the stability of the prolongation and restriction operators.

THEOREM 3.4. *If the norm in Euclidean space is defined by* (3.14), *then* (i) $\|p_n\| \leqslant \|B\|$, (ii) $\|r_n\| \leqslant 1$.

Proof. (i) If $\mathbf{u}_n \in E_n$ with $\|\mathbf{u}_n\|_{E_n} = 1$,

$$\|p_n \mathbf{u}_n\|_{L^2} = \|Bq_n \mathbf{u}_n\|_{L^2} \leqslant \|B\| \ \|q_n u_n\|_{L^2} = \|B\|.$$

(ii) Let $f \in L^2[-1, 1]$ with $\|f\| = 1$. Then

$$\|r_n f\|_{E_n} = \|G_n^{1/2} r_n f\|_2 = \|G_n^{1/2} B_n^{-1} s_n f\|_2.$$

Now

$$B_n = G_n - s_n K_1 q_n = G_n^{1/2}(I - G_n^{-1/2} s_n K_1 q_n G_n^{-1/2}) G_n^{1/2},$$

$$\therefore \quad B_n^{-1} = G_n^{-1/2}(I - G_n^{-1/2} s_n K_1 q_n G_n^{-1/2})^{-1} G_n^{-1/2}.$$

So

$$\|r_n f\|_{E_n} \leq \|(I - G_n^{-1/2} s_n K_1 q_n G_n^{-1/2})^{-1}\|_2 \cdot \|G_n^{-1/2} s_n f\|_2 \leq \|G_n^{-1/2} s_n f\|_2,$$

since $G_n^{-1/2} s_n K_1 q_n G_n^{-1/2}$ is skew-symmetric (Lemma 3.3(i)) and thus

$$\|(I - G_n^{-1/2} s_n K_1 q_n G_n^{-1/2})^{-1}\|_2 \leq 1.$$

Now

$$\|G_n^{-1/2} s_n f\|_2^2 = \langle s_n f, G_n^{-1} s_n f\rangle = (f, q_n G_n^{-1} s_n f).$$

$q_n G_n^{-1} s_n$ is simply the projection operator P_n from $L^2[-1, 1]$ into S_n and thus $\|q_n G_n^{-1} s_n\| \leq 1$. Hence, $(f, q_n G_n^{-1} S_n f) \leq \|f\|^2$

$$\therefore \quad \|r_n f\|_{E_n} \leq \|f\|$$

establishing the result.

LEMMA 3.5. *For $g \in L^2[-1, 1]$ we have*

$$\|(1 - p_n r_n)g\| \leq \|B\|^2 \|(I - P_n)B^{-1}g\|. \tag{3.15}$$

Proof.

$$\begin{aligned}\|(1 - p_n r_n)g\| &= \|B(B^{-1} - q_n B_n^{-1} s_n)g\| \\ &\leq \|B\| \, \|(B^{-1} - q_n B_n^{-1} s_n)g\| \leq \|B\|^2 \|(I - P_n)B^{-1}g\|,\end{aligned}$$

since the second term is merely the error in Galerkin's method applied to the problem $BW = g$ and we may use Theorem 3.2.

Following the method described in [14, Section 6], (3.4) is to be approximated by

$$(I_n - r_n K_2 B^{-1} p_n)\mathbf{v}_n = r_n f,$$

which is equivalent to

$$(G_n - s_n(K_1 + K_2))\mathbf{v}_n = s_n f,$$

the Galerkin approximation of the full equation. For brevity we put

$$K_n = r_n K_2 B^{-1} p_n.$$

[14, Theorem 3.2] can now be used to establish a bound on $\|(I_n - K_n)^{-1}\|_{E_n}$. We recall the assumption of a unique L^2-solution which implies the existence of $(I - K_2 B^{-1})^{-1}$.

THEOREM 3.6. *If $(I - K_2 B^{-1})^{-1}$ exists and*

$$\delta_n = \|(I - K_2 B^{-1})^{-1}\| \, \|B\|^3 \, \|(I - P_n)B^{-1}K_2\| < 1, \tag{3.16}$$

then $(I_n - K_n)$ is nonsingular and satisfies

$$\|(I_n - K_n)^{-1}\|_{E_n} \leq \frac{\|B\| \, \|(I - K_2 B^{-1})^{-1}\|}{1 - \delta_n}.$$

Proof. The term

$$\begin{aligned}\|(1 - p_n r_n)K_2 B^{-1}\| &\leq \|B\|^2 \|(I - P_n)B^{-1}K_2 B^{-1}\| \\ &\leq \|B\|^2 \|(I - P_n)B^{-1}K_2\|,\end{aligned}$$

since $\|B^{-1}\| \leq 1$ by Lemma 3.1(iii).

The theorem follows directly from [14, Theorem 3.2]. We can now calculate the error $\|u - q_n \mathrm{v}_n\|$.

THEOREM 3.7. *The error in Galerkin's method for the equation* (3.1) *satisfies*

$$\|u - q_n \mathrm{v}_n\| \leqslant \|B\|(1 + C(n))\|(I - P_n)u\|,$$

where

$$C(n) = \frac{\|B\|^2 \|(I - K_2 B^{-1})^{-1}\| \, \|(I - Q_n^*) B^{*-1} K_2^*\|}{1 - \delta_n}, \tag{3.17}$$

$Q_n = p_n r_n$, $\delta_n < 1$ *is in Eq.* (3.16), *and* * *denotes the adjoint.*

Proof. Since $Bu = v$, we have

$$\begin{aligned} \|u - q_n \mathrm{v}_n\| &\leqslant \|u - q_n B_n^{-1} s_n B u\| + \|q_n(B_n^{-1} s_n v - \mathrm{v}_n)\| \\ &= \|u - q_n B_n^{-1} s_n B u\| + \|r_n v - \mathrm{v}_n\|_{E_n}. \end{aligned}$$

The first term is simply the error in Galerkin's method for the problem $Bu = Bu$, and

$$\|u - q_n B_n^{-1} s_n B u\| \leqslant \|B\| \, \|(I - P_n)u\|.$$

For the second term we use [14, Eq. 18] to get

$$\|r_n v - \mathrm{v}_n\|_{E_n} \leqslant \|(I - K_n)^{-1}\|_{E_n} \|r_n\| \, \|K_2 B^{-1}(1 - p_n r_n) B u\|.$$

A bound for $\|(I - K_n)^{-1}\|$ is given in Theorem 3.6 and

$$\begin{aligned} &\|K_2 B^{-1}(I - p_n r_n) B u\| \\ &\quad = \|K_2 B^{-1}(I - Q_n)^2 B u\| \leqslant \|K_2 B^{-1}(I - Q_n)\| \, \|(I - Q_n) B u\| \\ &\quad \leqslant \|(I - Q_n^*) B^{*-1} K_2^*\| \, \|B\|^2 \, \|(I - P_n)u\|. \end{aligned}$$

Hence,

$$\begin{aligned} \|u - q_n \mathrm{v}_n\| &\leqslant \|B\| \left(1 + \frac{\|B\|^2 \|(I - K_2 B^{-1})\| \, \|(I - Q_n^*) B^{*-1} K_2^*\|}{1 - \delta_n} \right) \|(I - P_n)u\| \\ &= \|B\|(1 + C(n))\|(I - P_n)u\| \quad \text{as required.} \end{aligned}$$

To discuss convergence as $n \to \infty$, we assume that $P_n g \to g$ as $n \to \infty$ for all $g \in L^2[-1, 1]$. We can then produce

LEMMA 3.8. *Suppose $P_n g \to g$ as $n \to \infty$ for all $g \in L^2[-1, 1]$. Then*
(i) *$p_n r_n g \to g$ for all $g \in L^2[-1, 1]$,*
(ii) *$(p_n r_n)^* g \to g$ for all $g \in L^2[-1, 1]$.*
(iii) *$c(n) \to 0$ in* (3.17).

Proof. (i) $B^{-1} g \in L^2[-1, 1]$, so we can use Lemma 3.5.

(ii) Since $(p_n r_n)^* g = q_n (B^*)^{-1} s_n B^* g$, the convergence of $(p_n r_n)^* g$ to g depends on the convergence of Galerkin's method for the problem $B^* w = B^* g$. It is readily verified that B may be replaced by B^* in the analysis prior to Theorem 3.2, and the desired convergence is readily established.

(iii) Since $(B^*)^{-1}$ is continuous and K_2 compact, the composition $(B^*)^{-1}K_2$ is compact and $\|(I - Q_n^*)(B^*)^{-1}K_2\| \to 0$ from a standard theorem in functional analysis; cf. [14]. Thus $c(n) \to 0$.

This lemma gives the interesting result

$$\|(I - P_n)u\| \leqslant \|u - q_n\mathrm{v}_n\| \leqslant \|B\| \, \|(I - P_n)u\| + o(1),$$

which is of the form (1.2).

4. Spline Functions as a Basis for Galerkin's Method. The spline functions form a convenient basis for Galerkin's method. One can often calculate the resulting "stiffness" matrix analytically. However, we saw from Section 2 that the solution of the singular integral equations under discussion invariably possesses endpoint singularities. Hence there is a need to use splines defined on nonuniform meshes and to possibly incorporate suitable singular functions into the basis.

The principal contributions to the theory of splines on nonuniform meshes have been Rice [13], de Boor [1], Burchard [2], and Dodson [5]. All of these authors have shown that with careful knot selection one can achieve asymptotically optimal rates of convergence ($O(N^{-k})$ for splines of degree less than k). The difficulty in applying the theory is that the function we wish to approximate is unknown. In this case we may use the known asymptotic behavior at the endpoints to calculate a good set of knots.

First we will define our notation. π will denote a partition

$$\pi\colon a = t_0 < t_1 < \cdots < t_N = b$$

of the interval $[a, b]$. P_π^k denotes the set of piecewise polynomials of degree less than k and having breakpoints at the t_i, $i = 1, \ldots, N - 1$. We will find approximations in the class

$$S_\pi^k = \begin{cases} P_\pi^k, & k = 1, \\ P_\pi^k \cap C[a, b], & k \geqslant 2, \end{cases}$$

and determine a partition to give $O(N^{-k})$ rates of convergence in the space $L^2[a, b]$. (One can easily adapt the method to give estimates in more general spaces $L^p[a, b]$.)

The first lemma is adapted from [2, Lemma 1].

LEMMA 4.1. *Let* $f \in C^{(k)}[a, b]$. *Then there exist* $s \in S_k^\pi$ *and a constant* A_k, *independent of* f, *for which*

(a)

$$\max_{t_i \leqslant t \leqslant t_{i+1}} |f(t) - s(t)| \leqslant A_k h_i^k \max_{t_i \leqslant t \leqslant t_{i+1}} |f^{(k)}|,$$

for $i = 0, \ldots, N - 1$, *and* $h_i = t_{i+1} - t_i$;

(b) *if* $k > 1$,

$$s(t_i) = f(t_i), \qquad i = 0, \ldots, N.$$

Proof. One can construct s by equidistant Lagrange interpolation of degree $k - 1$ in the interval $[t_i, t_{i+1}]$. Then $A_k = 1/k!$. One takes $A_1 = \frac{1}{2}$.

Remark. By a more judicious choice of interpolation points in $[t_i, t_{i+1}]$, one could obtain smaller values for A_k when $k > 2$. We can now produce our main theorem.

THEOREM 4.2. *Let $f \in C^{(k)}[a, b]$ and $g(t)$ be a continuous function that satisfies*
(i) $g(t) \geqslant \beta > 0$, *for* $t \in [a, b]$,
(ii) $|f^{(k)}(t)|^\sigma \leqslant g(t)$, *for* $t \in [a, b]$ *and* $\sigma = 2/(2k+1)$.
For an integer N, define the partition $\pi^* = (t_i)_{i=0}^N$ *of* $[a, b]$, *where* $t_0 = a$ *and*

$$\int_{t_i}^{t_{i+1}} g(t)\,dt = \frac{1}{N}\int_a^b g(t)\,dt. \tag{4.1}$$

Then there exists an integer N_0 such that, for $N \geqslant N_0$, there exists $s^* \in S_{\pi^*}^k$ such that

$$\|f - s^*\| \leqslant \frac{C_k}{N^k}\left\{\int_a^b g(t)\,dt\right\}^{k+1/2},$$

where C_k is independent of N and f.

Proof (*cf.* [2]). $\log g(t)$ is uniformly continuous, so there exists $\delta > 0$ such that

$$\max_{x \leqslant t \leqslant x+\delta} g(t) \leqslant 2 \min_{x \leqslant t \leqslant x+\delta} g(t).$$

Also, for t_i defined by (4.1),

$$h_i = t_{i+1} - t_i \leqslant \frac{1}{\beta}\int_{t_i}^{t_{i+1}} g(t)\,dt = \frac{1}{N\beta}\int_a^b g(t)\,dt.$$

We choose N_0 such that $h_i < \delta$, $i = 0, \ldots, N-1$, if $N \geqslant N_0$. Then, for $N \geqslant N_0$ and $t \in [t_i, t_{i+1}]$,

$$\begin{aligned}|f^{(k)}(t)|^\sigma &\leqslant \max_{t_i \leqslant t \leqslant t_{i+1}} g(t) \leqslant 2 \min_{t_i \leqslant t \leqslant t_{i+1}} g(t)\\ &\leqslant \frac{2}{h_i}\int_{t_i}^{t_{i+1}} g(t)\,dt = \frac{2}{h_i N}\int_a^b g(t)\,dt.\end{aligned}$$

Hence, using the fact $\sigma = 2/(2k+1)$,

$$|f^{(k)}(t)|^2 \leqslant \left(\frac{2}{h_i N}\int_a^b g(t)\,dt\right)^{2k+1} \quad \text{for } t \in [t_i, t_{i+1}].$$

Choose $s^* \in S_{\pi^*}^k$ from Lemma 1. Then

$$\begin{aligned}\|f - s^*\|^2 &\leqslant \sum_{i=0}^{N-1} A_k^2 h_i^{2k+1} \max_{t_i \leqslant t \leqslant t_{i+1}} |f^{(k)}(t)|^2\\ &\leqslant \sum_{i=0}^{N-1} A_k^2 h_i^{2k+1}\left(\frac{2}{h_i N}\int_a^b g(t)\,dt\right)^{2k+1} = \frac{A_k^2 2^{2k+1}}{N^{2k}}\left(\int_a^b g(t)\,dt\right)^{2k+1}.\end{aligned}$$

Hence,

$$\|f - s^*\| \leqslant \frac{C_k}{N^k}\left\{\int_a^b g(t)\,dt\right\}^{k+1/2},$$

where $C_k = 2^{k+1/2}A_k$.

The partition π^* will produce a fine mesh in regions where $g(t)$ is large. However, the theorem is not directly applicable to functions with endpoint singularities. Dodson [5] has shown that a similar result will hold for functions whose kth derivative has a finite number of singularities and is monotonic in a neighborhood of such singularities. The key property is the integrability of the function $g(t)$. It is instructive to consider the function $f(t) = t^\alpha$ ($\alpha > -\frac{1}{2}$) over the integral [0, 1].

Here we would take

$$g(t) = P_{\alpha,k} t^{2(\alpha-k)/(2k+1)},$$

where $P_{\alpha,k}$ is a constant. $g(t)$ is clearly integrable, and from (4.1) we recover the partition due to Rice [13]

$$t_i = \left(\frac{i}{N}\right)^q, \qquad i = 0, \ldots, N,$$
$$q = \frac{2k+1}{2\alpha+1}.$$

This result is useful since it gives insight into how the knots should be placed near the endpoints. It also points out a practical difficulty. If α is close to $-\frac{1}{2}$, then q will get very large, so, although in theory we can get $O(N^{-k})$ rates of convergence, the distance between knots becomes so small that computing with them is well-nigh impossible. Also, one can calculate

$$\left(\int_0^1 g(t)\,dt\right)^{1/\sigma} = \left(P_{\alpha,k}\frac{2k+1}{2\alpha+1}\right)^{k+1/2}$$

and see that the size of this term will be large when $\alpha \sim -\frac{1}{2}$, and the accuracy obtained with this partition may only be modest.

The introduction of singular functions into the basis reduces, to a large extent, the problem of small intervals. We know that the solution of (1.1) has the form

$$u(s) = (1-s)^\alpha (1+s)^\beta w(s),$$

where $w(s)$ is smooth and α, β are known. For a given integer N, we construct a solution as follows. In an interval $[-1, -1 + \delta_1(N)]$, where $\delta_1(N)$ is a function of N, we find an approximation of the form

$$U_N(s) = C_1(1+s)^\beta, \tag{4.2}$$

and in an interval $[1 - \delta_2(N), 1]$ ($\delta_2(N)$ is a function of N) we find an approximation of the form

$$U_N(s) = C_2(1-s)^\alpha. \tag{4.3}$$

C_1, C_2 are constants to be determined. $\delta_1(N)$ and $\delta_2(N)$ are chosen so that

$$\int_{-1}^{-1+\delta_1(N)} (u(s) - u_N(s))^2\,ds = O\left(\frac{1}{N^{2k}}\right),$$
$$\int_{1-\delta_2(N)}^{1} (u(s) - u_N(s))^2\,ds = O\left(\frac{1}{N^{2k}}\right).$$

In the interval $[-1 + \delta_1(N), 1 - \delta_2(N)]$, $u(s)$ is to be approximated by a spline $u_N(s)$ of degree k and with N knots chosen according to Theorem 4.2.

If π denotes a partition of $[-1 + \delta_1, 1 - \delta_2]$, we let $\tilde{S}_\pi^k$ be the space formed by augmenting the spline space with the singular functions. We can construct a function $\tilde{s} \in \tilde{S}_\pi^k$ such that $\|\tilde{s} - u\| = O(N^{-k})$, as follows.

In the interval $[-1, -1 + \delta_1]$, $u(s)$ has the form

$$u(s) = (1+s)^\beta v(s) = (1+s)^\beta \{v(-1) + (1+s)v'(\xi)\},$$

where $v(s)$ is smooth and $\xi \in (-1, -1 + \delta_1)$. Hence, if $u(s)$ is approximated by $v(-1)(1+s)^\beta$ in $[-1, -1 + \delta_1]$, the L^2 error will be

$$\left[\int_{-1}^{-1+\delta_1}(1+s)^{2\beta+2}|v'(\xi)|^2\,ds\right]^{1/2} \leq \frac{\delta_1^{\beta+1.5}}{\sqrt{2\beta+3}} \max_{-1\leq s\leq -1+\delta_1} |v'(s)|.$$

Hence, by taking

$$\delta_1^{\beta+1.5}/\sqrt{2\beta+3} = N^{-k},$$

i.e.

$$\delta_1 = (2\beta+3)^{1/(2\beta+3)}N^{-k/(\beta+1.5)}, \tag{4.4}$$

we will achieve the desired rate of convergence. Similarly, we take

$$\delta_2 = (2\alpha+3)^{1/(2\alpha+3)}N^{-k/(\beta+1.5)} \tag{4.5}$$

to achieve the desired rate of convergence in $[1-\delta_2, 1]$.

To calculate a partition of $[-1+\delta_1, 1-\delta_2]$, we must find a suitable function $g(s)$ for use in Theorem 4.2. One can show that

$$u^{(k)}(s) = (1-s)^{\alpha-k}(1+s)^{\beta-k}W_k(s),$$

where $W_k(s)$ is bounded, and we put

$$g(s) = \begin{cases} D(1-s)^{(\alpha-k)\sigma}, & 0 \leq s \leq 1, \\ D(1+s)^{(\beta-k)\sigma}, & -1 \leq s \leq 0, \end{cases}$$

where D is a constant chosen such that $g(s) \geq [u^{(k)}(s)]^{\sigma}$. It is then easy to show that

$$\int_{-1+\delta_1}^{1-\delta_2} g(t)\,dt = D\left\{\frac{1}{\gamma_1}(1-\delta_1^{\gamma_1}) + \frac{1}{\gamma_2}(1-\delta_2^{\gamma_2})\right\},$$

with

$$\gamma_1 = (\beta-k)\sigma+1, \quad \gamma_2 = (\alpha-k)\sigma+1.$$

The desired partition is calculated from (4.1) and is

$$t_i = -1 + \left\{\delta_1^{\gamma_1} + \frac{i}{N}\left((1-\delta_1^{\gamma_1}) + \frac{\gamma_1}{\gamma_2}(1-\delta_2^{\gamma_2})\right)\right\}^{1/\gamma_1},$$

for $i = 0, 1, \ldots, N^*$, and

$$t_i = 1 - \left\{\delta_2^{\gamma_2} + \frac{(N-i)}{N}\left[(1-\delta_2^{\gamma_2}) + \frac{\gamma_2}{\gamma_1}(1-\delta_1^{\gamma_1})\right]\right\}^{1/\gamma_2},$$
$$i = N^*+1, \ldots, N,$$

where

$$\frac{N^*}{N} \leq \frac{1-\delta_1^{\gamma_1}}{(1-\delta_1^{\gamma_1}) + \dfrac{\gamma_1}{\gamma_2}(1-\delta_2^{\gamma_2})}.$$

The integer N^* is needed because of the different forms of $g(s)$ for s negative and positive. The constant D cancels out in the computation.

Hence we may show the existence of a function $\tilde{s} \in \tilde{S}_{\pi^*}^k$ with the desired rate of convergence. From Theorem 3.7 it will then follow that the rate of convergence of the Galerkin approximation $U_N(s)$, when our trial space is $\tilde{S}_{\pi^*}^k$, satisfies

$$\|u - U_N\| = O(N^{-k}).$$

The numerical solution will suffer a mild discontinuity at the points $-1 + \delta_1$, $1 - \delta_2$, but numerical experience has shown that this is not a serious defect.

5. Numerical Example. Galerkin's method, using a basis of piecewise linear functions and the singular functions of Section 4, was implemented and applied to the example

$$u(s) - \frac{1}{\pi}\int_{-1}^{+1} \frac{u(t)}{t-s}\,dt = 1,$$

which has the exact solution

$$u(s) = \frac{1}{\sqrt{2}}\left(\frac{1-s}{1+s}\right)^{1/4}.$$

For $\lambda = 1$, the following results were obtained. The first table gives the value of the constants C_1, C_2 of (4.2) and (4.3), together with δ_1 and δ_2 from (4.4) and (4.5).

N	C_1	δ_1	C_2	δ_2
16	0.8405	0.0155	0.5971	0.0561
32	0.8407	0.0054	0.5958	0.0263

The value of C_1 should tend to $2^{-1/4} = 0.8409$ as $N \to \infty$, and C_2 should tend to $2^{-3/4} = 0.5946$ as $N \to \infty$. We will now compare the Galerkin approximation with the true solution at interior points

s	$N = 16$	$N = 32$	true
−0.8	1.2256	1.2243	1.2247
−0.4	0.8747	0.8738	0.8739
0.0	0.7068	0.7073	0.7071
0.4	0.5720	0.5721	0.5721
0.8	0.4082	0.4082	0.4082

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