On Odd Perfect, Quasiperfect, and Odd Almost Perfect Numbers

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Abstract. We establish upper bounds for the six smallest prime factors of odd perfect, quasiperfect, and odd almost perfect numbers.

1. Suppose \( N = \prod_{i=1}^{r} p_i^{a_i} \) is an odd perfect (OP) number, i.e. \( \sigma(N) = 2N \), where \( p_i \)'s are odd primes, \( p_1 < \cdots < p_r \), and \( a_i \)'s are positive integers. Grun [1] proved that
\[
p_1 < 2 + 2r/3,
\]
and Pomerance [5] proved that
\[
(1) \quad p_i < (4r)^{2r(1+1)/2} \quad \text{for} \quad 1 < i < r.
\]
In [3] we showed that if \( N \) is an odd integer and the number \( \omega(N) \) of distinct prime factors of \( N \) is 5, then
\[
(2) \quad |2 - \sigma(N)/N| > 10^{-14}.
\]
From this it follows immediately that if \( M \) is an odd integer, \( \sigma(M) = 2M + L \), and if \( |L/M| < 10^{-14} \), then \( \omega(M) > 6 \). OP, quasiperfect (QP) numbers, i.e. \( \sigma(N) = 2N + 1 \), and odd almost perfect (OAP) numbers, i.e. \( \sigma(N) = 2N - 1 \), are such examples.

Also, it can be proved from (2) that if \( M = \prod_{i=1}^{r} p_i^{a_i} \) is OP,
\[
p_6 < 2 \cdot 10^{14}(r - 5).
\]
However, if we consider only those \( N = \prod_{i=1}^{r} p_i^{a_i} \) in (2) for which \( \prod_{i=1}^{r} p_i^{a_i} \) is OP, then exponents \( a_i \) are restricted, and hence we have a better lower bound in (2). Consequently we have a better upper bound for \( p_6 \).

In this paper we prove

THEOREM. Suppose \( M = \prod_{i=1}^{r} p_i^{a_i} \). If \( M \) is OP or QP,
\[
p_i < 2^{2^i - 1}(r - i + 1) \quad \text{for} \quad 2 < i < 6.
\]
If \( M \) is OAP,
\[
p_i < 2^{2^i - 1}(r - i + 1) \quad \text{for} \quad 2 < i < 5, \quad \text{and}
\]
\[
p_6 < 23775427335(r - 5).
\]
Although our Theorem gives upper bounds for \( p_i \) only for \( 2 < i < 6 \), they are better than (1). For example, if \( M \) is OP, then \( p_5 < 65536(r - 4) \) by our Theorem.

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and \( p_r > 100110 \) by Hargis and McDaniel [2]. Hence, we have another proof that \( \omega(M) > 6 \).

2. In order to prove our Theorem, we need three lemmas.

Definition. \( S(N) = \sigma(N)/N \).

**Lemma 1.** Suppose \( M = \prod_{i=1}^{r} p_i^{a_i} \) is OP. Then

\[
S\left( \prod_{i=1}^{5} p_i^{a_i} \right) < \frac{3}{2} \frac{5}{4} \frac{17}{16} \frac{257}{256} \frac{65537}{65536} = \alpha \approx 2 - 4/10^{10}.
\]

**Proof.** Since \( M \) is OP, by Euler,

(3) \( \text{if } p_i \equiv 1 \ (4), \ a_i \equiv 0, 1, 2 \ (4), \ \text{and if } p_i \equiv 3 \ (4), \ a_i \equiv 0 \ (2), \) and if \( q \) is an odd prime factor of \( \sigma(p_i^{a_i}) \) for some \( i \), then \( q \mid M \). Suppose

(4)

\[ \alpha < S\left( \prod_{i=1}^{5} p_i^{a_i} \right) < 2, \]

and \( q \neq p_i \) for \( 1 < i < 5 \). If \( q < 10^9 \), then

\[
\log 2 = \log S(M) > \log S\left( \prod_{i=1}^{5} p_i^{a_i} \right) + \sum_{i=6}^{r} \log S(p_i^{a_i})
\]

\[ > \log \alpha + \log(q + 1)/q > \log \alpha + \log(10^9 + 1)/10^9 > \log 2, \]

a contradiction. Hence,

(5) \( \text{If } q \) is an odd prime factor of \( \sigma(p_i^{a_i}) \) for some \( i \) and \( q \neq p_j \) for \( 1 < j < 5 \), then \( q > 10^9 \).

As in [3], we used a computer (PDP11 at the University of Toledo) to find odd integers \( \prod_{i=1}^{r} p_i^{a_i} \) satisfying (3) and (4). There were infinitely many such \( \prod_{i=1}^{5} p_i^{a_i} \). (However, there were finitely many (just over one hundred) \( \prod_{i=1}^{5} p_i^{a_i} \) if \( a_i < a(p_i) \) where

\[ a(p_i) = \min\{ a_i | a_i \text{ satisfies (3) and } p_i^{a_i+1} > 10^{11}\}. \]

See [3].) In every case such \( \prod_{i=1}^{5} p_i^{a_i} \) had a component \( p_i^{a_i} \) such that \( a_i < a(p_i) \), \( q \) is an odd prime factor of \( \sigma(p_i^{a_i}) \), \( q \neq p_j \) for \( 1 < j < 5 \) and \( q < 10^9 \), contradicting (5).

Q.E.D.

**Lemma 2.** Suppose \( M = \prod_{i=1}^{r} p_i^{a_i} \) is QP. Then

\[
S\left( \prod_{i=1}^{5} p_i^{a_i} \right) < \frac{3}{2} \frac{5}{4} \frac{17}{16} \frac{257}{256} \frac{65537}{65536} = \alpha \approx 2 - 4/10^{10}.
\]

**Proof.** Since \( M \) is QP, by [3], \( r > 6 \), \( S(\prod_{i=1}^{5} p_i^{a_i}) < 2, \) and

\[ a_i \equiv 0 \ (2) \text{ for any } i, \]

(6)

if \( p_i = 3, a_i = 4, 12 \text{ or } > 24, \)

if \( p_i = 5, a_i = 6 \text{ or } > 16, \)

if \( p_i = 17, a_i = 2 \text{ or } > 8. \)
We used the computer to find odd integers $\prod_{i=1}^{5} p_i^{a_i}$ satisfying (6) and

$$\alpha < S\left(\prod_{i=1}^{5} p_i^{a_i}\right) < 2,$$

but there were none. Q.E.D.

**Lemma 3.** Suppose $M = \prod_{i=1}^{5} p_i^{a_i}$ is OAP. Then

$$S\left(\prod_{i=1}^{5} p_i^{a_i}\right) < S\left(3^{12}\right) \frac{5}{4} S(17^{6}) \frac{257}{256} \frac{62939}{62938} = \beta \approx 2 - 8/10^{11}.$$  

**Proof.** Since $M$ is OAP, by [3], $r > 6$ and

$$a_i \equiv 0 \pmod{2} \text{ for all } i,$$

if $p_i = 3$, $a_i = 12, 16$ or $> 24$,

if $p_i = 5$, $a_i = 2, 10$ or $> 16$,

if $p_i = 257$, $a_i > 16$.

We used the computer to find odd integers $\prod_{i=1}^{5} p_i^{a_i}$ satisfying (7) and

$$\alpha < S\left(\prod_{i=1}^{5} p_i^{a_i}\right) < 2,$$

and the results were

$3^{a_1} 5^{a_2} 17^{a_3} 257^{a_4} 65449^{a_5}$, where $a_1 > 24, a_3 > 8, a_4 > 16, a_5 > 2$, and

$3^{12} 5^{a_2} 17^{6} 257^{a_4} 62939^{a_5}$, where $a_2 > 16, a_4 > 16, a_5 > 2$.

Since

$$\frac{3}{2} S(5^{10}) \frac{17}{16} \frac{257}{256} \frac{65449}{65448} < S\left(3^{12}\right) \frac{5}{4} S(17^{6}) \frac{257}{256} \frac{62939}{62938} = \beta,$$

Lemma 3 follows. Q.E.D.

**Proof of Theorem.** We prove only the case $i = 5$. Suppose $M = \prod_{i=1}^{5} p_i^{a_i}$ is OP or QP, $N = \prod_{i=1}^{5} p_i^{a_i}$, and

$$\frac{2}{2 - \alpha} (r - 5) + 1 < p_6 < \cdots < p_r.$$  

Since $\log(1 + x) < x$ and $\log(1 - x) < -x$ if $0 < x < 1$, we have, by Lemmas 1 and 2,

$$\log 2 < \log S(M) = \log S(N) + \sum_{i=6}^{r} \log S(p_i^{a_i})$$

$$< \log \alpha + (r - 5) \log S(p_6^{a_6})$$

$$< \log 2 + \log \alpha/2 + (r - 5) \log p_6/ (p_6 - 1)$$

$$= \log 2 + \log(1 - (2 - \alpha)/2) + (r - 5) \log(1 + 1/ (p_6 - 1))$$

$$< \log 2 - (2 - \alpha)/2 + (r - 5)/ (p_6 - 1)$$

$$< \log 2 - (2 - \alpha)/2 + (2 - \alpha)/2 = \log 2,$$

a contradiction. Hence,

$$p_6 < \frac{2}{2 - \alpha} (r - 5) + 1 = 2^2(r - 5) + 1.$$  

Since $p_6$ is a prime, $p_6 < 2^2(r - 5)$.  

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Suppose \( M = \prod_{i=1}^{r} p_i^a \) is OAP, \( N = \prod_{i=1}^{s} p_i^a \), and
\[
\frac{2}{2 - \beta} (r - 5) + 1 < p_6 < \cdots < p_r.
\]
Since \( M > 10^{30} \) by [4] and \( \log(1 - x) < -x - x^2/2 \) if \( 0 < x < 1 \), we have, by Lemma 3,
\[
\log 2 - \frac{1}{2} \cdot 10^{30} \approx \log 2 + \log\left(1 - \frac{1}{2} \cdot 10^{30}\right)
= \log(2 - 1/10^{30}) < \log(2 - 1/M) = \log(S(M)/M)
= \log S(N) + \sum_{i=6}^{r} \log p_i^a < \log \beta + (r - 5) \log p_6 / (p_6 - 1)
< \log 2 + \log(1 - (2 - \beta)/2) + (r - 5)/(p_6 - 1)
< \log 2 - (2 - \beta)/2 - (2 - \beta)^2/8 + (2 - \beta)/2
= \log 2 - (2 - \beta)^2/8 \approx \log 2 - 9 \cdot 10^{-22},
\]
a contradiction. Hence
\[
p_6 < \frac{2}{2 - \beta} (r - 5) + 1 < 23775427335(r - 5) + 1.
\]
Since \( p_6 \) is a prime, \( p_6 < 23775427335(r - 5) \). Q.E.D.

Finally, we (re)state the following

**Theorem.** Suppose \( N = \prod_{i=1}^{r} p_i^a \) is an integer.

(a) If \( r = 5 \), \( |2 - S(N)| > 2 - S(3756172233) \cdot 36550429/36550428 > 10^{-14} \).

(b) If \( r = 4 \), \( |2 - S(N)| > 2 - S(3756172233) > 5/10^8 \).

(c) If \( r = 3 \), \( |2 - S(N)| > S(355213) - 2 > 3/10^4 \).

(d) If \( r = 2 \), \( |2 - S(N)| > 2 - \frac{3}{2} \cdot \frac{5}{4} = 0.125 \).

(e) If \( r = 1 \), \( |2 - S(N)| > 2 - \frac{3}{2} = 0.5 \).

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4. M. Kishore, *The Number of Distinct Prime Factors of \( N \) for Which \( \sigma(N) = 2N, \sigma(N) = 2N \pm 1, \) and \( \phi(N)/N - 1 \),* Doctoral dissertation, Princeton University, Princeton, N. J., 1977.