On Odd Perfect, Quasiperfect, and Odd Almost Perfect Numbers

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Abstract. We establish upper bounds for the six smallest prime factors of odd perfect, quasiperfect, and odd almost perfect numbers.

1. Suppose \( N = \prod_{i=1}^{r} p_i^{a_i} \) is an odd perfect (OP) number, i.e. \( \sigma(N) = 2N \), where \( p_i \)'s are odd primes, \( p_1 < \cdots < p_r \), and \( a_i \)'s are positive integers. Grun [1] proved that

\[ p_1 < 2 + 2r/3, \]

and Pomerance [5] proved that

\[ p_i < (4r)^{2(r+1)/2} \quad \text{for} \quad 1 < i < r. \]  

In [3] we showed that if \( N \) is an odd integer and the number \( \omega(N) \) of distinct prime factors of \( N \) is 5, then

\[ |2 - \sigma(N)/N| > 10^{-14}. \]

From this it follows immediately that if \( M \) is an odd integer, \( \sigma(M) = 2M + L \), and if \( |L/M| < 10^{-14} \), then \( \omega(M) > 6 \). OP, quasiperfect (QP) numbers, i.e. \( \sigma(N) = 2N + 1 \), and odd almost perfect (OAP) numbers, i.e. \( \sigma(N) = 2N - 1 \), are such examples.

Also, it can be proved from (2) that if \( M = \prod_{i=1}^{r} p_i^{a_i} \) is OP,

\[ p_6 < 2 \cdot 10^{14}(r - 5). \]

However, if we consider only those \( N = \prod_{i=1}^{r} p_i^{a_i} \) in (2) for which \( \prod_{i=1}^{r} p_i^{a_i} \) is OP, then exponents \( a_i \) are restricted, and hence we have a better lower bound in (2). Consequently we have a better upper bound for \( p_6 \).

In this paper we prove

**Theorem.** Suppose \( M = \prod_{i=1}^{r} p_i^{a_i} \). If \( M \) is OP or QP,

\[ p_i < 2^{2^{i-1}}(r - i + 1) \quad \text{for} \quad 2 < i < 6. \]

If \( M \) is OAP,

\[ p_i < 2^{2^{i-1}}(r - i + 1) \quad \text{for} \quad 2 < i < 5, \quad \text{and} \]

\[ p_6 < 23775427335(r - 5). \]

Although our Theorem gives upper bounds for \( p_i \) only for \( 2 < i < 6 \), they are better than (1). For example, if \( M \) is OP, then \( p_5 < 65536(r - 4) \) by our Theorem.
and \( p_r > 100110 \) by Hargis and McDaniel [2]. Hence, we have another proof that \( \omega(M) > 6 \).

2. In order to prove our Theorem, we need three lemmas.

**Definition.** \( S(N) = \sigma(N)/N \).

**Lemma 1.** Suppose \( M = \prod_{i=1}^{r} p_i^{a_i} \) is OP. Then
\[
S\left( \prod_{i=1}^{5} p_i^{a_i} \right) < \frac{3}{2} \frac{5}{4} \frac{17}{16} 257 \frac{65537}{256} \frac{65536}{65536} = \alpha \approx 2 - 4/10^6.
\]

**Proof.** Since \( M \) is OP, by Euler,
\[
(3) \quad \text{if } p_i \equiv 1 \pmod{4}, \quad a_i \equiv 0, 1, 2 \pmod{4}, \quad \text{and if } \quad p_i \equiv 3 \pmod{4}, \quad a_i \equiv 0 \pmod{2},
\]
and if \( q \) is an odd prime factor of \( \sigma(p_i^{a_i}) \) for some \( i \), then \( q \mid M \). Suppose
\[
(4) \quad \alpha < S\left( \prod_{i=1}^{5} p_i^{a_i} \right) < 2,
\]
and \( q \neq p_i \) for \( 1 < i < 5 \). If \( q < 10^9 \), then
\[
\log 2 = \log S(M) > \log S\left( \prod_{i=1}^{5} p_i^{a_i} \right) + \sum_{i=6}^{r} \log S(p_i^{a_i}) > \log \alpha + \log(q + 1)/q > \log \alpha + \log(10^9 + 1)/10^9 > \log 2,
\]
a contradiction. Hence,
\[
(5) \quad \text{If } q \text{ is an odd prime factor of } \sigma(p_i^{a_i}) \text{ for some } i \text{ and } q \neq p_j \text{ for } 1 < j < 5, \text{ then } q > 10^9.
\]

As in [3], we used a computer (PDP11 at the University of Toledo) to find odd integers \( \prod_{i=1}^{5} p_i^{a_i} \) satisfying (3) and (4). There were infinitely many such \( \prod_{i=1}^{5} p_i^{a_i} \). (However, there were finitely many (just over one hundred) \( \prod_{i=1}^{5} p_i^{a_i} \) if \( a_i < a(p_i) \) where
\[
a(p_i) = \min\{a_i \mid a_i \text{ satisfies (3) and } p_i^{a_{i+1}} > 10^{11}\}.
\]
See [3].) In every case such \( \prod_{i=1}^{5} p_i^{a_i} \) had a component \( p_i^{a_i} \) such that \( a_i < a(p_i) \), \( q \) is an odd prime factor of \( \sigma(p_i^{a_i}) \), \( q \neq p_j \) for \( 1 < j < 5 \) and \( q < 10^9 \), contradicting (5). Q.E.D.

**Lemma 2.** Suppose \( M = \prod_{i=1}^{r} p_i^{a_i} \) is QP. Then
\[
S\left( \prod_{i=1}^{5} p_i^{a_i} \right) < \frac{3}{2} \frac{5}{4} \frac{17}{16} 257 \frac{65537}{256} \frac{65536}{65536} = \alpha \approx 2 - 4/10^6.
\]

**Proof.** Since \( M \) is QP, by [3], \( r > 6 \), \( S(\prod_{i=1}^{5} p_i^{a_i}) < 2 \), and
\[
(6) \quad a_i \equiv 0 \pmod{2} \text{ for any } i,
\]
if \( p_i = 3, a_i = 4, 12 \text{ or } > 24, \)
if \( p_i = 5, a_i = 6 \text{ or } > 16, \)
if \( p_i = 17, a_i = 2 \text{ or } > 8. \)
We used the computer to find odd integers \( \prod_{i=1}^{5} p_i^{a_i} \) satisfying (6) and
\[
\alpha < \frac{S}{\prod_{i=1}^{5} p_i^{a_i}} < 2,
\]
but there were none. Q.E.D.

**Lemma 3.** Suppose \( M = \prod_{i=1}^{5} p_i^{a_i} \) is OAP. Then
\[
S\left( \prod_{i=1}^{5} p_i^{a_i} \right) < S(3^{12}) \frac{5}{4} S(17^6) \frac{257}{256} \frac{62939}{62938} = \beta < 2 - 8/10^{11}.
\]

**Proof.** Since \( M \) is OAP, by [3], \( r > 6 \) and
\[
\begin{align*}
\alpha_i & \equiv 0 (2) \text{ for all } i, \\
\text{if } p_i = 3, \alpha_i & = 12, 16 \text{ or } > 24, \\
\text{if } p_i = 5, \alpha_i & = 2, 10 \text{ or } > 16, \\
\text{if } p_i = 257, \alpha_i & > 16.
\end{align*}
\]
We used the computer to find odd integers \( \prod_{i=1}^{5} p_i^{a_i} \) satisfying (7) and
\[
\alpha < \frac{S}{\prod_{i=1}^{5} p_i^{a_i}} < 2,
\]
and the results were
\[
3^a 5^{10} 17^a 257^a 65449^a, \quad \text{where } a_1 > 24, a_3 > 8, a_4 > 16, a_5 > 2, \text{ and}
\]
\[
3^{12} 5^a 17^6 257^a 62939^a, \quad \text{where } a_2 > 16, a_4 > 16, a_5 > 2.
\]
Since
\[
\frac{3}{2} S(5^{10}) \frac{17}{16} \frac{257}{256} \frac{65449}{65448} < S(3^{12}) \frac{5}{4} S(17^6) \frac{257}{256} \frac{62939}{62938} = \beta,
\]
Lemma 3 follows. Q.E.D.

**Proof of Theorem.** We prove only the case \( i = 5 \). Suppose \( M = \prod_{i=1}^{5} p_i^{a_i} \) is OP or QP, \( N = \prod_{i=1}^{5} p_i^{a_i} \), and
\[
\frac{2}{2 - \alpha} (r - 5) + 1 < p_6 < \cdots < p_r.
\]
Since \( \log(1 + x) < x \) and \( \log(1 - x) < -x \) if \( 0 < x < 1 \), we have, by Lemmas 1 and 2,
\[
\log 2 < \log S(M) = \log S(N) + \sum_{i=6}^{r} \log S(p_i^{a_i})
\]
\[
< \log \alpha + (r - 5) \log S(p_6^{a_6})
\]
\[
< \log 2 + \log \alpha/2 + (r - 5) \log p_6/(p_6 - 1)
\]
\[
= \log 2 + \log(1 - (2 - \alpha)/2) + (r - 5) \log(1 + 1/(p_6 - 1))
\]
\[
< \log 2 - (2 - \alpha)/2 + (r - 5)/(p_6 - 1)
\]
\[
< \log 2 - (2 - \alpha)/2 + (2 - \alpha)/2 = \log 2,
\]
a contradiction. Hence,
\[
p_6 < \frac{2}{2 - \alpha} (r - 5) + 1 = 2^3 (r - 5) + 1.
\]
Since \( p_6 \) is a prime, \( p_6 < 2^3 (r - 5) \).
Suppose $M = \prod_{i=1}^{r} p_i^{a_i}$ is OAP, $N = \prod_{i=1}^{5} p_i^{a_i}$, and
\[
\frac{2}{2 - \beta} (r - 5) + 1 < p_6 < \cdots < p_r.
\]
Since $M > 10^{30}$ by [4] and $\log(1 - x) < -x - x^2/2$ if $0 < x < 1$, we have, by Lemma 3,
\[
\log 2 - \frac{1}{2} \cdot 10^{30} \approx \log 2 + \log \left(1 - \frac{1}{2} \cdot 10^{30}\right)
\]
\[
= \log(2 - 1/10^{30}) < \log(2 - 1/M) = \log(S(M)/M)
\]
\[
= \log S(N) + \sum_{i=6}^{r} \log(p_i^{a_i}) < \log \beta + (r - 5)\log p_6 / (p_6 - 1)
\]
\[
< \log 2 + \log(1 - (2 - \beta)/2) + (r - 5) / (p_6 - 1)
\]
\[
< \log 2 - (2 - \beta)/2 - (2 - \beta)^2/8 + (2 - \beta)/2
\]
\[
= \log 2 - (2 - \beta)^2/8 \approx \log 2 - 9 \cdot 10^{-22},
\]
a contradiction. Hence
\[
p_6 < \frac{2}{2 - \beta} (r - 5) + 1 < 23775427335(r - 5) + 1.
\]
Since $p_6$ is a prime, $p_6 < 23775427335(r - 5)$. Q.E.D.

Finally, we (re)state the following

**Theorem.** Suppose $N = \prod_{i=1}^{r} p_i^{a_i}$ is an integer.

- (a) If $r = 5$, $\left|2 - S(N)\right| > 2 - S(3756172233) \cdot 36550429/36550428 > 10^{-14}$.
- (b) If $r = 4$, $\left|2 - S(N)\right| > 2 - S(3756172233) > 5/10^8$.
- (c) If $r = 3$, $\left|2 - S(N)\right| > S(3^{5}5^{2}13) - 2 > 3/10^4$.
- (d) If $r = 2$, $\left|2 - S(N)\right| > 2 - \frac{3}{2} \cdot \frac{5}{4} = 0.125$.
- (e) If $r = 1$, $\left|2 - S(N)\right| > 2 - \frac{3}{2} = 0.5$.

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