A Block-by-Block Method for Volterra Integro-Differential Equations With Weakly-Singular Kernel

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Abstract. The theory of a block-by-block method for solving Volterra integro-differential equations with continuous kernels (see Makroglou [4], [5]) is adapted to Volterra integro-differential equations with weakly-singular kernels, and a rate of convergence is given.

1. Introduction. Consider the nonlinear Volterra integro-differential equation

\[(1.1) \quad y'(x) = G(x, y(x), \int_0^x K(x, t, y(t)) \, dt) \quad (x > 0),\]

given \(y(0)\), written in the form,

\[(1.2) \quad y(x) = \int_0^x G(s, y(s), z(s)) \, ds + y(0) \quad (x > 0),\]

\[(1.3) \quad z(x) = \int_0^x K(x, t, y(t)) \, dt \quad (x > 0),\]

with

\[(1.4) \quad K(x, s, y(s)) = K(x, s)y(s), \quad 0 < \alpha < 1, 0 < s < x < X.\]

For the discretization of the equation (1.3), we shall use a product integration technique in such a way that when the method is used for solving examples with 
\(K(x, s, y(s)) = H(x, s, y(s))/|x - s|^\alpha\) it will not require the evaluation of 
\(H(x, s, y(s))\) for \(s > x\), where it might, for example, not be defined (see Section 2).

Product integration techniques have been used for the solution of weakly-singular integral equations; see for example Linz [3], Weiss [6], de Hoog and Weiss [2], Baker [1].

For the discretization of Eq. (1.2) we shall use Eqs. (2.3) in Makroglou [5] and produce a scheme which we called a generalized block-by-block method after Weiss, scheme GC, though it is a new method for integro-differential equations, see Section 3 below, originated in [4]. ('G' stands for 'Generalized' and 'C' is kept here in agreement with the notation used in [4] where it meant the third of the G schemes GA, GB, GC.)

A rate of convergence of the scheme is given in Section 4.

For use in the discussion to follow, we define \(x_{m,j} = mh + uh, \quad x_{m,j,k} = mh + uh_0 h, \quad j = 0, 1, \ldots, p; \quad m = 0, 1, \ldots, N - 1\), where \(N, p\) integers, \(h > 0\) so that 
\(Nh = X\) and \(0 < u_0 < u_1 < \cdots < u_p = 1\). We also assume the preliminaries and definitions given in Makroglou [5].

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2. Discretization of Eq. (1.3). Consider the equation (1.3) with $K(x, s, y(s))$ as in (1.4), that is the equation,

$$z(x) = \int_0^x K(x, t)y(t) \, dt,$$

where $K(x, t)$ is given by (1.4). Discretizing at the points $x_{m_j}$ we have

$$z(x_{m_j}) = \sum_{i=0}^{m-1} \int_{ih}^{(i+1)h} K(x_{m_j}, s)y(s) \, ds + \int_{m-1}^{x_{m_j}} K(x_{m_j}, s)y(s) \, ds,$$

or

$$z(x_{m_j}) = h \sum_{i=0}^{m-1} \int_0^1 K(x_{m_j}, ih + ht)y(ih + ht) \, dt$$

$$+ hu_j \int_0^1 K(x_{m_j}, mh + u_jht)y(mh + u_jht) \, dt.$$

We now use the approximations

$$y(ih + ht) \approx \sum_{k=0}^p L_k(t)y(x_{i,k}),$$

$$y(mh + u_jrt) \approx \sum_{k=0}^p L_k(t)y(mh + u_ju_kh),$$

where $L_k(t)$ are the Lagrangian coefficients, giving

$$z_{m_j} = hu_j \sum_{r=0}^p \sum_{k=0}^p V^{(m)}(m, j, k)L_r(u_ju_k)y_{m,r}$$

$$+ h \sum_{i=0}^{m-1} \sum_{k=0}^p V^{(m)}(i, j, k)y_{i,k},$$

$m = 0, 1, \ldots, N - 1; j = 0, 1, \ldots, p, (j = 1, 2, \ldots, p, if \ u_0 = 0)$, where we have put

$$V^{(m)}(i, j, k) = \int_0^1 K(x_{m_j}, ih + uht)L_k(t) \, dt,$$

with

$$u = u_j \ \text{if} \ i = m,$$

$$u = 1 \ \text{if} \ i = 0, 1, \ldots, m - 1.$$

2.1. Estimation of the Coefficients $V^{(m)}(i, j, k)$. Using the kernel (1.4) in (2.7), we obtain

$$V^{(m)}(i, j, k) = \int_0^1 \frac{t - u_q}{|l - l^a|} \, dt / (u^n h^n D(k)),$$

where

$$D(k) = \prod_{q=0, q \neq k}^p (u_k - u_q).$$
and

\[ l = m + u_j - i \quad \text{for } i = 0, 1, \ldots, m - 1, \]
\[ l = 1 \quad \text{for } i = m, \]

or

\[ V^{(m)}(i, j, k) = (-1)^{p+1} \int_{t_i}^{t_j} \prod_{q=1}^{p} (t^{1/\alpha} - a_q)^{r^{1/\alpha - 2}} dt / (\alpha u_h^r D(k)), \]

where

\[ a_{q+1} = l - u_q, \quad q = 0, 1, \ldots, k - 1, \]
\[ a_q = l - u_q, \quad q = k + 1, \ldots, p. \]

The product \( \prod_{q=1}^{p} (t^{1/\alpha} - a_q) \) in (2.12) can be written as

\[ \prod_{q=1}^{p} (t^{1/\alpha} - a_q) = c_0(t^{1/\alpha})^p + c_1(t^{1/\alpha})^{p-1} + \cdots + c_p, \]

where, with \( S_m = a_1^m + a_2^m + \cdots + a_p^m \), we have

\[ c_0 = 1, \]
\[ c_1 = -S_1, \]
\[ c_j = -(S_j + c_1 S_{j-1} + c_2 S_{j-2} + \cdots + c_{j-1} S_1)/j, \quad j = 2, 3, \ldots. \]

Substituting (2.14) in (2.12) and integrating, we find

\[ V^{(m)}(i, j, k) = \frac{(-1)^{p+1}}{u^r h^{\alpha} D(k)} \sum_{r=0}^{p} c_{p-r} \left\{ (l - 1)^{r-a+1} - l^{r-a+1} \right\}/(r - \alpha + 1), \]

\( i = 0, 1, \ldots, m; \quad k = 0, 1, \ldots, p; \quad j = 1, \ldots, p \) if \( u_0 = 0, \quad j = 0, 1, \ldots, p \) if \( u_0 \neq 0. \)

3. Statement of the Method. According to the illustration given in the introduction, the approximate equations for scheme GC are

\[ y_{m,j} = h \sum_{k=0}^{p} w_k G(x_{m,k}, y_{m,k}, z_{m,k}) \]
\[ + h \sum_{i=0}^{m-1} \sum_{k=0}^{p} w_k G(x_{i,k}, y_{i,k}, z_{i,k}) + y(0), \]
\[ z_{m,j} = h u_j \sum_{r=0}^{m} \sum_{k=0}^{p} V^{(m)}(m, j, k)L_r(u_j u_k) y_{m,r} \]
\[ + h \sum_{i=0}^{m-1} \sum_{k=0}^{p} V^{(m)}(i, j, k) y_{i,k}, \]

\( m = 0, 1, \ldots, N - 1; j = 0, 1, \ldots, p, \quad (j = 1, 2, \ldots, p \text{ if } u_0 = 0), \) where

\[ w_k^j = \int_{0}^{u_j} L_k(x) \, dx, \]
\[ w_k = \int_{0}^{1} L_k(x) \, dx, \]
(3.5) \[ L_k(x) = \prod_{j=0, j \neq k}^{p} \frac{x - u_j}{u_k - u_j}, \]

and \( V^{(m)}(i, j, k) \) are given by (2.16).

Equations (3.1)–(3.2) constitute a system of \( 2p + 2 \) (or \( 2p \) if \( u_0 = 0 \)) in general nonlinear equations for \( y_{m,0}, y_{m,1}, \ldots, y_{m,p}; z_{m,0}, z_{m,1}, \ldots, z_{m,p} \).

4. Convergence. For the complete convergence proofs we refer to [4]. There, we started by obtaining an asymptotic expansion for the error \( e_m \equiv \max_{0 < x < \rho} |e_m| \), \( e_{m,j} \equiv z(x_{m,j}) - z_{m,j} \) in the approximations (3.2). In doing this, the work in [2] was of great help. Having obtained this expansion, one can then obtain a bound on \( s_m = [e_m, e_m]^T \) along the lines of the convergence proof given in [5]. The convergence result obtained is given as Theorem 1 below.

**Theorem 1.** Let

(i) \( g(x) \in P_v \) (see preliminaries in [5]),

(ii) \( y(x) \) is \( p + 2 \) times continuously differentiable on \( 0 < x < X \),

(iii) \( G(x, y, z) \) be \( p + v + 2 \) times continuously differentiable with respect to \( x, y, z \), respectively, on \( 0 < x < X, \max_{0 < x < \rho} |y(x)| \) and \( \bar{z} = \max_{0 < x < \rho} |z(x)| \). Then, there are constants \( C_1, C_2, C_3, C_4, C_5 \) such that

\[
\|s_m\|_\infty \leq C_5 h^{p+1} \quad \text{if } v = 0, \tag{4.1}
\]

\[
\|s_m\|_\infty \leq \begin{cases} C_1 h^{p+2} & \text{if } v > 0, \\
C_2 h^{p+2-\alpha} & \text{if } v > 0 \end{cases} \tag{4.2}
\]

and the inequalities occur with (1) or (2) according to where the maximum occurs when considering \( \|\cdot\|_\infty \).

Some numerical results obtained by testing scheme GC on a linear and a nonlinear example for both \( u_0 = 0, u_0 \neq 0 \) are displayed in [4] (see [4, Examples 3, 4, p. 97; pp. 152, 153, 157, 158]). Order of convergence at least \( O(h^{p+1}) \) was verified.

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**The result (2) in (4.1) is changed here to \( C_2 h^{p+2-\alpha} \) from \( C_2 h^{p+1} \) in [4]. This because in [4, p. 201, Eq. III-1.108] we have \( \int_0^\rho g(t) P_d(t) \, dt = 0 \) for \( g \in P_{v(>\rho)} \).**
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