A Block-by-Block Method for Volterra Integro-Differential Equations With Weakly-Singular Kernel

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Abstract. The theory of a block-by-block method for solving Volterra integro-differential equations with continuous kernels (see Makroglou [4], [5]) is adapted to Volterra integro-differential equations with weakly-singular kernels, and a rate of convergence is given.

1. Introduction. Consider the nonlinear Volterra integro-differential equation

\[ y'(x) = G(x, y(x), \int_0^x K(x, t, y(t)) \, dt) \quad (x > 0), \]

given \( y(0) \), written in the form,

\[ y(x) = \int_0^x G(s, y(s), z(s)) \, ds + y(0) \quad (x > 0), \]

\[ z(x) = \int_0^x K(x, t, y(t)) \, dt \quad (x > 0), \]

with

\[ K(x, s, y(s)) = K(x, s)y(s), \]

\[ K(x, s) = 1/|x - s|^\alpha, \quad 0 < \alpha < 1, \quad 0 < s < x < X. \]

For the discretization of the equation (1.3), we shall use a product integration technique in such a way that when the method is used for solving examples with \( K(x, s, y(s)) = H(x, s, y(s))/|x - s|^\alpha \) it will not require the evaluation of \( H(x, s, y(s)) \) for \( s > x \), where it might, for example, not be defined (see Section 2). Product integration techniques have been used for the solution of weakly-singular integral equations; see for example Linz [3], Weiss [6], de Hoog and Weiss [2], Baker [1].

For the discretization of Eq. (1.2) we shall use Eqs. (2.3) in Makroglou [5] and produce a scheme which we called a generalized block-by-block method after Weiss, scheme GC, though it is a new method for integro-differential equations, see Section 3 below, originated in [4]. ('G' stands for 'Generalized' and 'C' is kept here in agreement with the notation used in [4] where it meant the third of the G schemes GA, GB, GC.)

A rate of convergence of the scheme is given in Section 4.

For use in the discussion to follow, we define \( x_{m,j} = mh + u_jh, \quad x_{m,j,k} = mh + u_ju_kh, \quad j = 0, 1, \ldots, p; \quad m = 0, 1, \ldots, N - 1, \) where \( N, p \) integers, \( h > 0 \) so that \( Nh = X \) and \( 0 < u_0 < u_1 < \cdots < u_p = 1 \). We also assume the preliminaries and definitions given in Makroglou [5].
2. Discretization of Eq. (1.3). Consider the equation (1.3) with $K(x, s, y(s))$ as in (1.4), that is the equation,

\begin{equation}
z(x) = \int_0^x K(x, t)y(t) \, dt,
\end{equation}

where $K(x, t)$ is given by (1.4). Discretizing at the points $x_{m,j}$ we have

\begin{equation}
z(x_{m,j}) = \sum_{i=0}^{m-1} \int_{ih}^{(i+1)h} K(x_{m,j}, s)y(s) \, ds + \int_{mh}^{x_{m,j}} K(x_{m,j}, s)y(s) \, ds,
\end{equation}

or

\begin{equation}
z(x_{m,j}) = h \sum_{i=0}^{m-1} \int_0^1 K(x_{m,j}, ih + ht)y(ih + ht) \, dt
+ \sum_{j=0}^{m-1} K(x_{m,j}, mh + u_jht)y(mh + u_jht) \, dt.
\end{equation}

We now use the approximations

\begin{equation}
y(ih + ht) \simeq \sum_{k=0}^p L_k(t)y(x_{i,k}),
\end{equation}

\begin{equation}
y(mh + u_jt) \simeq \sum_{k=0}^p L_k(t)y(mh + u_ju_kh)
= \sum_{k=0}^p L_k(t) \sum_{r=0}^p L_r(u_ju_k)y(x_{m,r}),
\end{equation}

where $L_k(t)$ are the Lagrangian coefficients, giving

\begin{equation}
z_{m,j} = h u_j \sum_{r=0}^p \sum_{k=0}^p V^{(m)}(m, j, k)L_r(u_ju_k)y_{m,r}
+ h \sum_{i=0}^{m-1} \int_0^1 K(x_{m,j}, ih + uht)L_k(t) \, dt,
\end{equation}

where

\begin{equation}
V^{(m)}(i, j, k) = \int_0^1 K(x_{m,j}, ih + uht)L_k(t) \, dt,
\end{equation}

with

\begin{equation}
u = u_j \quad \text{if } i = m,
\end{equation}

\begin{equation}
u = 1 \quad \text{if } i = 0, 1, \ldots, m - 1.
\end{equation}

2.1. Estimation of the Coefficients $V^{(m)}(i, j, k)$. Using the kernel (1.4) in (2.7), we obtain

\begin{equation}
V^{(m)}(i, j, k) = \prod_{q=0; q \neq k}^{p} \frac{(t - u_q)}{|l - l|^{m}} \, dt / (u^a h^a D(k)),
\end{equation}

where

\begin{equation}
D(k) = \prod_{q=0; q \neq k}^{p} (u_k - u_q),
\end{equation}

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D(k) = \prod_{q=0; q \neq k}^{p} (u_k - u_q),
\end{equation}
and

\[ l = m + u_j - i \quad \text{for } i = 0, 1, \ldots, m - 1, \]
\[ l = 1 \quad \text{for } i = m, \]

or

\[ V^{(m)}(i, j, k) = (-1)^{p+1} \int_{l^{(i-1)}}^{(l^{i-1})^*} \prod_{q=1}^{p} \frac{(l^{1/\alpha} - a_q)l^{1/\alpha-2} dt}{(\alpha u_h^p D(k)),} \]

where

\[ a_{q+1} = l - u_q, \quad q = 0, 1, \ldots, k - 1, \]
\[ a_q = l - u_q, \quad q = k + 1, \ldots, p. \]

The product \( \prod_{q=1}^{p} (l^{1/\alpha} - a_q) \) in (2.12) can be written as

\[ \prod_{q=1}^{p} (l^{1/\alpha} - a_q) = c_0(l^{1/\alpha})^p + c_1(l^{1/\alpha})^{p-1} + \cdots + c_p, \]

where, with \( S_m = a_1^m + a_2^m + \cdots + a_p^m \), we have

\[ c_0 = 1, \]
\[ c_1 = -S_1, \]
\[ c_j = -(S_j + c_1S_{j-1} + c_2S_{j-2} + \cdots + c_{j-1}S_1)/j, \quad j = 2, 3, \ldots. \]

Substituting (2.14) in (2.12) and integrating, we find

\[ V^{(m)}(i, j, k) = \frac{(-1)^{p+1}}{u_h^p D(k)} \sum_{r=0}^{p} \frac{e_{p-r} \left( (l - 1)^{r-\alpha+1} - l^{r-\alpha+1} \right)}{r - \alpha + 1}, \]

\[ i = 0, 1, \ldots, m; \quad k = 0, 1, \ldots, p; \quad j = 1, \ldots, p \quad \text{if } u_0 = 0, \quad j = 0, 1, \ldots, p \quad \text{if } u_0 \neq 0. \]

3. Statement of the Method. According to the illustration given in the introduction, the approximate equations for scheme GC are

\[ y_{m,j} = h \sum_{k=0}^{p} w_k G(x_{m,k}, y_{m,k}, z_{m,k}) \]
\[ + h \sum_{i=0}^{m-1} \sum_{k=0}^{p} w_k G(x_{i,k}, y_{i,k}, z_{i,k}) + y(0), \]
\[ z_{m,j} = hu_j \sum_{r=0}^{m-1} \sum_{k=0}^{p} V^{(m)}(m, j, k) L_r(u_j u_k y_{m,r}) \]
\[ + h \sum_{i=0}^{m-1} \sum_{k=0}^{p} V^{(m)}(i, j, k) y_{i,k}, \]

\[ m = 0, 1, \ldots, N - 1; \quad j = 0, 1, \ldots, p; \quad (j = 1, 2, \ldots, p \quad \text{if } u_0 = 0), \]

where

\[ w_k^j = \int_0^{u_j} L_k(x) \, dx, \]
\[ w_k = w_k^p = \int_0^1 L_k(x) \, dx, \]
\( L_k(x) = \prod_{j=0, j \neq k}^{p} (x - u_j)/(u_k - u_j), \)

and \( V^{(m)}(i, j, k) \) are given by (2.16).

Equations (3.1)–(3.2) constitute a system of \( 2p + 2 \) (\( 2p \) if \( u_0 = 0 \)) in general nonlinear equations for \( y_0, y_1, \ldots, y_p; z_0, z_1, \ldots, z_p \).

4. Convergence. For the complete convergence proofs we refer to [4]. There, we started by obtaining an asymptotic expansion for the error \( e_m \equiv \max_{0 < l < p} |e_m| \), \( e_{m,j} \equiv z(x_{m,j}) - z_m \) in the approximations (3.2). In doing this, the work in [2] was of great help. Having obtained this expansion, one can then obtain a bound on \( s_m = [e_m, e_m] \) along the lines of the convergence proof given in [5]. The convergence result obtained is given as Theorem 1 below.

**Theorem 1.** Let

(i) \( g(x) \in P_v \) (see preliminaries in [5]),

(ii) \( y(x) \) is \( p + 2 \) times continuously differentiable on \( 0 < x < X \),

(iii) \( G(x, y, z) \) be \( p + v + 2 \) times continuously differentiable with respect to \( x, y, z \), respectively, on \( 0 < x < X, |y| < y, |z| < z \) where \( y = \max_{0 < x < X} |y(x)| \) and \( z = \max_{0 < x < X} |z(x)| \). Then, there are constants \( C_1, C_2, C_3, C_4, C_5 \) such that

\[
\|s_m\|_\infty < C_5 h^{p+1} \quad \text{if } v = 0,
\]

\[
\|s_m\|_\infty < \begin{cases} \frac{C_1 h^{p+2}}{1} & \text{if } v > 0, \\ \frac{C_2 h^{p+2-\alpha}}{2} & \end{cases}
\]

for \( m = 1, 2, \ldots, N - 1 \), and

\[
\|s_0\|_\infty < \begin{cases} \frac{C_3 h^{p+2}}{1} & \text{if } v = 0, \\ \frac{C_4 h^{p+2-\alpha}}{2} & \end{cases}
\]

and the inequalities occur with (1) or (2) according to where the maximum occurs when considering \( \|\cdot\|_\infty \).

Some numerical results obtained by testing scheme GC on a linear and a nonlinear example for both \( u_0 = 0, u_0 \neq 0 \) are displayed in [4] (see [4, Examples 3, 4, p. 97; pp. 152, 153, 157, 158]). Order of convergence at least \( O(h^{p+1}) \) was verified.

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**The result (2) in (4.1) is changed here to \( C_2 h^{p+2-\alpha} \) from \( C_2 h^{p+1} \) in [4]. This because in [4, p. 201, Eq. III.1.108] we have \( \int_0^\infty g(t)P_d(t) \, dt = 0 \) for \( g \in P_{v \geq 0} \).**