On the $BN$ Stability of the Runge-Kutta Methods

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Abstract. In this note sufficient conditions that let Runge-Kutta $s$ stages methods of at least order $s$ be $BN$ stable are given.

1. Introduction. When a numerical method is applied to solve a system of stiff differential equations,

$$y' = f(t, y),$$

it is necessary to analyze the properties of stability of the method. Usually the property of $A$-stability is required [6]. This property is related to the test equation, which is scalar, in which

$$f(t, y) = \lambda y, \quad \lambda \in \mathbb{C}, \quad R_e(\lambda) < 0.$$

Recently Burrage and Butcher [1] have taken into account the following, more general, test equation:

$$y' = f(t, y), \quad f: \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N,$$

with

$$\langle f(t, y) - f(t, z), y - z \rangle < 0 \quad \forall y, z \in \mathbb{R}^N, t \in \mathbb{R},$$

where $\langle \cdot, \cdot \rangle$ is a scalar product in $\mathbb{R}^N$ with $\| \cdot \|$ as a corresponding norm and they have defined a criterion of stability called $BN$ stability for this particular test equation.

Burrage [4] has constructed a class of high-order $BN$ stable Runge-Kutta methods, but, as he has pointed out, the construction of low-order $BN$ stable methods is not as simple. In this note the sufficient conditions that let a Runge-Kutta $s$ stages method of at least order $s$ be stable are given.

A result that has already been demonstrated in another way [5] about the $BN$ stability of implicit Runge-Kutta methods of maximum order has been obtained as a corollary.

2. Review of Known Results. Before presenting the result of this study I would like to recall some known definitions and results [2], [3].

Consider a Runge-Kutta $s$ stages method which is defined by the following matrix form:
We shall denote the approximation to \( y(t_n) \), with \( y_n \), where \( y(t) \) is the solution to (1.1) and \( t_n = t_{n-1} + h, h > 0, n = 1, 2, \ldots \).

**Definition 1.** The method (2.1) is BN stable if applied to the test equation (1.2), (1.3) it is such that for each pair of solution \( \ldots y_{n-1}, y_n, \ldots \) and \( \ldots z_{n-1}, z_n, \ldots \), the result will be

\[
\|y_n - z_n\| \leq \|y_{n-1} - z_{n-1}\|.
\]

**Definition 2.**

\[
C(p): \quad c_k^{j-1} = c_k^j/k, \quad j = 1, 2, \ldots, s, k < p.
\]

\[
D(p): \quad d_k^{j-1} = d_j(1 - c_k^j), \quad j = 1, 2, \ldots, s, k < p.
\]

\[
B(p): \quad b_k^{j-1} = \frac{1}{k}, \quad k < p.
\]

\[
L(s): \quad c_i, i = 1, 2, \ldots s, \text{ are the zeros of the polynomial } P_s(2c - 1),
\]

where \( P_s \) denotes the s degree Legendre polynomial.

**Theorem 1.** If (2.1) is such that \( b_j > 0, i = 1, 2, \ldots s, \text{ and the matrix } BA + A^TB - bb^T \text{ is not negatively defined } (B = \text{diag}(b_1, b_2, \ldots, b_s)), \text{ then (2.1) is BN stable.}

**Lemma 1.** If \( C(\eta) \land D(\xi) \land B(p), \text{ where } p < \xi + \eta + 1, p < 2\eta + 2, \text{ then (2.1) is of the order } p \text{ at least.}

**Theorem 2.** \( C(s) \land D(s) \land B(s) \land L(s) \text{ if and only if (2.1) is of the order } 2s. \)

3. **Sufficient Conditions for the BN Stability of Runge-Kutta Methods of Order s at Least.** We define the following matrices and vectors:

\[
D = \text{diag}\left(1, \frac{1}{2}, \ldots, \frac{1}{s}\right), \quad e_{1 \times s}^T(1, 1, \ldots, 1),
\]

\[
C = \text{diag}(c_1, c_2, \ldots, c_s), \quad B = \text{diag}(b_1, b_2, \ldots, b_s),
\]

\[
E = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \text{ matrix } s \times s,
\]

\[
V_s = \begin{bmatrix} 1 & c_1 & \cdots & c_1^{s-1} \\ 1 & c_2 & \cdots & c_2^{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_s & \cdots & c_s^{s-1} \end{bmatrix}.
\]
Note. From Lemma 1 if \( C(s) \land D(s) \land B(s) \), then (2.1) is of order \( s \) at least.

Using the above defined matrices, \( C(s), D(s), B(s) \) will become respectively:

\[
C(s): AV_s = CV_s D,
\]

\[
D(s): V_s^T BA = D(E - V_s^T C) B,
\]

\[
B(s): (B e)^T V_s = (De)^T.
\]

**Theorem 3.** The class of Runge-Kutta \( s \) stages methods satisfy the properties \( C(s), D(s), B(s) \) and for which \( c_i, i = 1, 2, \ldots, s, \) are distinct and \( b_i > 0, i = 1, 2, \ldots, s, \) are BN stable and have an order \( s \) at least.

**Proof.** Using the property \( D(s) \) and \( C(s) \),

\[
V_s^T BA = DEB - DV_s^T CB = DEB - V_s^T A^T B
\]

from which

\[
BA + A^T B = V_s^-T DEB = BEDV_s^-1 = B \begin{bmatrix} e^T \\ e^T \\ \vdots \\ e^T \end{bmatrix} DV_s^-1 = B \begin{bmatrix} e^T V_s^-1 \\ e^T V_s^-1 \\ \vdots \\ e^T V_s^-1 \end{bmatrix};
\]

from \( B(s) \)

\[
V_s^T Be = De \iff Be = V_s^-T De \iff e^T B = e^T DV_s^-1.
\]

Therefore it follows that

\[
BA + A^T B = B \begin{bmatrix} e^T B \\ e^T B \\ \vdots \\ e^T B \end{bmatrix} \iff BA + A^T B - bb^T = 0.
\]

At this point we would like to recall the fact that there is only one Runge-Kutta \( s \) stages method of order \( 2s \) [2] and that according to Theorem 2 it belongs to the class introduced in this note. Having observed that for that method \( b_i > 0, i = 1, 2, \ldots, s \) [2] and \( \det V_s \neq 0, \) it follows that

**Corollary.** The Runge-Kutta \( s \) stages method of order \( 2s \) is BN stable.

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