On the BN Stability of the Runge-Kutta Methods

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Abstract. In this note sufficient conditions that let Runge-Kutta $s$ stages methods of at least order $s$ be BN stable are given.

1. Introduction. When a numerical method is applied to solve a system of stiff differential equations,

\begin{equation}
y' = f(t, y),
\end{equation}

it is necessary to analyze the properties of stability of the method. Usually the property of $A$-stability is required [6]. This property is related to the test equation, which is scalar, in which

\[ f(t, y) = \lambda y, \quad \lambda \in \mathbb{C}, \quad R_e(\lambda) < 0. \]

Recently Burrage and Butcher [1] have taken into account the following, more general, test equation:

\begin{equation}
y' = f(t, y), \quad f: \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N,
\end{equation}

with

\begin{equation}
\langle f(t, y) - f(t, z), y - z \rangle < 0 \quad \forall y, z \in \mathbb{R}^N, t \in \mathbb{R},
\end{equation}

where $\langle \cdot, \cdot \rangle$ is a scalar product in $\mathbb{R}^N$ with $\| \cdot \|$ as a corresponding norm and they have defined a criterion of stability called BN stability for this particular test equation.

Burrage [4] has constructed a class of high-order BN stable Runge-Kutta methods, but, as he has pointed out, the construction of low-order BN stable methods is not as simple. In this note the sufficient conditions that let a Runge-Kutta $s$ stages method of at least order $s$ be stable are given.

A result that has already been demonstrated in another way [5] about the BN stability of implicit Runge-Kutta methods of maximum order has been obtained as a corollary.

2. Review of Known Results. Before presenting the result of this study I would like to recall some known definitions and results [2], [3].

Consider a Runge-Kutta $s$ stages method which is defined by the following matrix form:
We shall denote the approximation to \( y(t_n) \) with \( y_n \), where \( y(t) \) is the solution to (1.1) and \( t_n = t_{n-1} + h, h > 0, n = 1, 2, \ldots \).

**Definition 1.** The method (2.1) is BN stable if applied to the test equation (1.2), (1.3) it is such that for each pair of solutions \( \ldots y_{n-1}, y_n, \ldots \) and \( \ldots z_{n-1}, z_n, \ldots \), the result will be

\[
\|y_n - z_n\| \leq \|y_{n-1} - z_{n-1}\|.
\]

**Definition 2.**

- \( C(p) : \sum_{j=1}^{s} a_{ij} c_{j}^{k-1} = c_{i}^{k}/k, \quad i = 1, 2, \ldots, s, k \leq p. \)
- \( D(p) : \sum_{i=1}^{j} b_{i} c_{i}^{k} a_{ij} = b_{j}(1 - c_{i}^{k}), \quad j = 1, 2, \ldots, s, k \leq p. \)
- \( B(p) : \sum_{i=1}^{j} b_{i} c_{i}^{k-1} = \frac{1}{k}, \quad k \leq p. \)
- \( L(s) : c_{i}, i = 1, 2, \ldots s, \) are the zeros of the polynomial \( P_{s}(2c - 1) \),

where \( P_{s} \) denotes the \( s \) degree Legendre polynomial.

**Theorem 1.** If (2.1) is such that \( b_{i} > 0, \ i = 1, 2, \ldots s, \) and the matrix \( BB + A^{T}B - bb^{T} \) is not negatively defined (\( B = \text{diag}(b_{1}, b_{2}, \ldots, b_{s}) \)), then (2.1) is BN stable.

**Lemma 1.** If \( C(\eta) \land D(\xi) \land B(p) \), where \( p < \xi + \eta + 1, p < 2\eta + 2, \) then (2.1) is of the order \( p \) at least.

**Theorem 2.** \( C(s) \land D(s) \land B(s) \land L(s) \) if and only if (2.1) is of the order \( 2s \).

### 3. Sufficient Conditions for the BN Stability of Runge-Kutta Methods of Order \( s \) at Least.

We define the following matrices and vectors:

- \( D = \text{diag}(1, \frac{1}{2}, \ldots, \frac{1}{s}) \), \( e_{s}^{T}(1, 1, \ldots, 1), \)
- \( C = \text{diag}(c_{1}, c_{2}, \ldots, c_{s}), \ B = \text{diag}(b_{1}, b_{2}, \ldots, b_{s}), \)
- \( E = \begin{bmatrix}
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1
\end{bmatrix} \) matrix \( s \times s \),
- \( V_{s} = \begin{bmatrix}
1 & c_{1} & \ldots & c_{1}^{s-1} \\
1 & c_{2} & \ldots & c_{2}^{s-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & c_{s} & \ldots & c_{s}^{s-1}
\end{bmatrix} \).
Note. From Lemma 1 if \( C(s) \land D(s) \land B(s) \), then (2.1) is of order \( s \) at least.

Using the above defined matrices, \( C(s) \), \( D(s) \), \( B(s) \) will become respectively:

\[
C(s): AV_s = CV_sD, \\
D(s): V_s^TBA = D(E - V_s^TC)B, \\
B(s): (Be)^T V_s = (De)^T.
\]

**Theorem 3.** The class of Runge-Kutta \( s \) stages methods satisfy the properties \( C(s) \), \( D(s) \), \( B(s) \) and for which \( c_i, i = 1, 2, \ldots, s, \) are distinct and \( b_i > 0, i = 1, 2, \ldots, s, \) are BN stable and have an order \( s \) at least.

**Proof.** Using the property \( D(s) \) and \( C(s) \),

\[
V_s^TBA = DEB - DV_s^T CB = DEB - V_s^TA^TB
\]

from which

\[
BA + A^TB = V_s^{-T}DEB = BEDV_s^{-1} = B \begin{bmatrix}
\text{e}^T \\
\text{e}^T \\
\vdots \\
\text{e}^T
\end{bmatrix} D V_s^{-1} = B \begin{bmatrix}
\text{e}^T D V_s^{-1} \\
\text{e}^T D V_s^{-1} \\
\vdots \\
\text{e}^T D V_s^{-1}
\end{bmatrix};
\]

from \( B(s) \)

\[
V_s^T Be = De \iff Be = V_s^{-T}De \iff e^TDV_s^{-1}.
\]

Therefore it follows that

\[
BA + A^TB = B \begin{bmatrix}
\text{e}^T B \\
\text{e}^T B \\
\vdots \\
\text{e}^T B
\end{bmatrix} \iff BA + A^TB - bb^T = 0.
\]

At this point we would like to recall the fact that there is only one Runge-Kutta \( s \) stages method of order \( 2s \) [2] and that according to Theorem 2 it belongs to the class introduced in this note. Having observed that for that method \( b_i > 0, i = 1, 2, \ldots, s \) [2] and \( \det V_s \neq 0 \), it follows that

**Corollary.** The Runge-Kutta \( s \) stages method of order \( 2s \) is BN stable.