Primes Differing by a Fixed Integer

By W. G. Leavitt and Albert A. Mullin

Abstract. It is shown that the equation \((n - 1)^2 - \sigma(n)\phi(n) = m^2\) is always solvable by \(n = p_1p_2\) where \(p_1, p_2\) are primes differing by the integer \(m\). This is called the "Standard" solution of \((*)\) and an \(m\) for which this is the only solution is called a "*-number". While there are an infinite number of non *-numbers there are many (almost certainly infinitely many) *-numbers, including \(m = 2\) (the twin prime case). A procedure for calculating all non *-numbers less than a given bound \(L\) is devised and a table is given for \(L = 1000\).

The prime numbers \(p_1, p_2\) are said to form a pair of "twin primes" if \(p_1 - p_2 = 2\). Using \(\sigma(n)\), the sum of the divisors of \(n\) (including \(n\) itself), and \(\phi(n)\), the number of numbers less than \(n\) and relatively prime to \(n\), S. A. Sergusov [1] has recently announced two criteria for an integer to be the product of twin primes. They are: \(n\) is the product of twin primes if and only if either \(\sigma(n) = n + 1 + 2\sqrt{n} + 1\) or \(\phi(n) = n + 1 - 2\sqrt{n} + 1\). Combining these two results gives the sufficiency for:

**Theorem 1.** The integer \(n\) is the product of twin primes if and only if

\[(n - 1)^2 - \sigma(n)\phi(n) = 4.\]

**Proof of the Necessity.** For primes \(p_1 < p_2 < \cdots < p_k\), suppose (1) is satisfied when \(n = \prod^k_1 p_i^{n_i}\). Then (1) can be written

\[2 \prod^k_1 p_i^{n_i} + 3 = \prod^k_1 p_i^{2n_i} - \prod^k_1 (p_i^{2n_i} - p_i^{n_i} - 1).\]

Since (2) would reduce for \(k = 1\) to \(2p^n + 3 = p^{n-1}\), it is clear that \(k > 2\). Then note that if \(p_1 = 2\), the left side of (2) is odd whereas the right side is even, and so \(p_1 > 3\). Also from (2) it follows that if \(p_1 = 3\), then \(n_1 = 2\) or 1, and in all other cases \(n_1 = 1\).

Now if \(k > 3\), it is easy to show that the right-hand side of (2) is greater than \(p_2^3 \prod^k_1 p_i^{n_i}\) and so exceeds the left-hand side, and if \(k = 2\) with \(p_1 = 3\) and \(n_1 = 2\), the right side is \(3p_2^2 + 78\) which again is always greater than the left-hand side.

In the only remaining case \(k = 2\) and \(n_1 = n_2 = 1\), so (2) reduces to \(2p_1p_2 + 3 = p_1^2 + p_2^2 - 1\), that is \((p_1 - p_2)^2 = 4\), and we conclude that \(n = p_1p_2\) with \(p_1 - p_2 = 2\).

We now generalize (1) to

\[(*) \quad (n - 1)^2 - \sigma(n)\phi(n) = m^2\]

for any integer \(m\). It is easy to check that

**Theorem 2.** If \(n = p_1p_2\) with \(p_1, p_2\) primes such that \(p_1 - p_2 = m\), then \(n\) satisfies (*).

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We will call the \( n \) of Theorem 2 the standard solution of (\( \ast \)), and we will say that \( m \) is a \(*\)-number if (\( \ast \)) has only the standard solution, that is if (\( \ast \)) characterizes those \( n \) which are products of two primes differing by the fixed integer \( m \). Thus Theorem 1 states that 2 is a \(*\)-number.

**Theorem 3.** For a given prime \( p \), if \( 2p - 1 \) is also prime, then \( n = p^k(2p - 1) \) satisfies (\( \ast \)) for \( m = p^k - 1 \), so \( m = p^k - 1 \) is not a \(*\)-number for all \( k > 2 \). Similarly (\( \ast \)) has a solution \( n = p^k(2p + 1) \) for \( m = p^k + 1 \) whenever \( p \) and \( 2p + 1 \) are prime.

**Proof.** If \( 2p \pm 1 \) is prime, then for \( n = p^k(2p \pm 1) \) the left-hand side of (\( \ast \)) becomes

\[
(p^k(2p \pm 1) - 1)^2 - (p^{2k} - p^{k-1})(4p^2 \pm 4p) = (p^k \pm 1)^2.
\]

**Corollary.** There are an infinite number of odd non \(*\)-numbers and an infinite number of even non \(*\)-numbers.

**Proof.** This is clear since we have as non \(*\)-numbers \( 2^k - 1 \) and \( 2^k + 1 \), and also \( 3^k - 1 \) and \( 3^k + 1 \) for all \( k > 2 \). Note: There are many other sequences of non \(*\)-numbers such as \( 7^k - 1 \) or \( 11^k + 1 \). Also note that except for 2 and 3 it is impossible for both \( 2p - 1 \) and \( 2p + 1 \) to be prime.

For primes \( p_1 < p_2 < \cdots < p_k \) let

\[
f = \left( \prod_{i=1}^{k} p_i^{n_i} - 1 \right)^2 - \prod_{i=1}^{k} (p_i^{2n_i} - p_i^{n_i-1}),
\]

so that \( n = \prod_{i=1}^{k} p_i^{n_i} \) is a solution of (\( \ast \)) if and only if \( \sqrt{f} = m \) is an integer.

The next two propositions gave some limitations on the type of solutions that (\( \ast \)) may have.

**Proposition 1.** If \( p \) is a prime such that \( p \mid m \) then the Mersenne number \( M_p = 2^p - 1 \) is not a solution of (\( \ast \)).

**Proof.** Let \( n = M_p \) be a solution of (\( \ast \)). For a prime \( q \mid M_p \), we have \( 2^p \equiv 1 \) (mod \( q \)) so \( q \equiv 1 \) (mod \( p \)). But then any \( q^{2r} - q^{r-1} \equiv 0 \) (mod \( p \)) and also \( M_p - 1 \equiv 0 \) (mod \( p \)). Thus from (3) we have the contradiction \( p^2 \mid f \).

**Proposition 2.** If \( p < q \) are primes, then \( n = pq' \) is not a solution of (\( \ast \)) for any \( r > 2 \) and any \( m \).

**Proof.** If \( n = pq' \) is a solution of (\( \ast \)), then since \( r > 2 \) we have \( (q, m) = 1 \). Thus we can write \( m = q^h \pm \alpha \) for either \( h = 0 \) or \( (h, q) = 1 \) with some \( t < 1 \), and some \( 0 < \alpha < (q - 1)/2 \). Then \( \alpha^2 \equiv 1 \) (mod \( q \)), so \( \alpha^2 = 1 \) and (3) becomes

\[
q^{2r} - 2pq' + (p^2 - 1)q^{r-1} = q'h(q'h \pm 2).
\]

**Case 1.** \( q = 2, q = 3 \). Then, since \( p^2 - 1 = 3 \), it follows from (4) that \( t = r + 1 \). Thus (4) reduces to

\[
3^{r+1}h^2 \pm 2h - 3^{r-1} + 1 = 0.
\]

But the left side of this equation is positive for all \( h > 1 \) and is nonzero for \( h = 0 \). Thus no integral value of \( h \) satisfies (4), so \( m \) an integer is impossible.
Case 2. In all other cases, since \( q > p \), we have \( q \nmid (p^2 - 1) \) and so \( t = r - 1 \). Thus (4) becomes

\[
q^{t-1}h^2 + 2p - q^{t+1} + 2pq + 1 - p^2 = 0.
\]

Writing the left side of this equation \( F(h) \) we have, \( F(0) \neq 0 \), and clearly \( F(h) \) is an increasing function for all \( h > 1 \). Since \( q > p \), it is evident that \( F(q) > 0 \). But also

\[
F(q - 1) < q^{t-1}(q - 1)^2 + 2(q - 1) - q^{t+1} + 2pq + 1 - p^2
\]

\[
< q^{t-1}(3 - 2q) + p(2q - p) - 1
\]

\[
< q^{t-1}(3 - 2q + 2q - p) - 1 = q^{t-1}(3 - p) - 1 < 0.
\]

Thus \( F(h) \) has no integral zeros, so again \( m \) an integer is impossible.

Remark. The method of Theorem 1 can be used to show that, for certain values of \( m \), (\( \ast \)) has only the standard solution, so that \( m \) is a \( \ast \)-number. However, with increasing \( m \) the method rapidly becomes more complicated and must in any case be done one \( m \) at a time. The following propositions yield a much simpler method, namely that for any chosen limit \( L \) there is a systematic procedure by which all nonstandard solutions of (\( \ast \)) can be calculated for all \( m < L \). Eliminating all such \( m \) then leaves those \( \ast \)-numbers that are \( < L \).

The following are clear from (3).

**Proposition 3.** If \( k = 1 \), then \( f < 0 \) so (\( \ast \)) is impossible.

**Proposition 4.** If \( k > 2 \), then \( f \) is odd if and only if \( n \) is even.

**Proposition 5.** In all cases \( f \) is an increasing function of \( n_j \) for all \( j \).

**Proof.** We take the partial of \( f \) with respect to \( n_j \) and check directly in the case \( j = 1 \), \( k = 2 \), \( n_2 = 1 \) that the partial derivative is greater than \( p^{n_j-1}\log p_j(p_1 - p_2)^2 \).

In all other cases we examine the effect on the partial of replacing \( p_j^{2n_j} - p_i^{n_i-1} \) by \( p_i^{2n_i} \) for all \( i > 2 \) and (when \( j > 2 \)) replacing \( 2p_j^{2n_j} - p_j^{n_j-1} \) by \( 2p_j^{2n_j} \). It is then immediately clear that in all cases the partial derivative is positive.

**Proposition 6.** In the case \( k = 2 \) and \( n_1 = 1 \), \( f \) is an increasing function of \( p_1 \) but is an increasing function of \( p_2 \). In all other cases \( f \) is an increasing function of \( p_j \) for all \( j \).

**Proof.** When \( k = 2 \) and \( n_1 = 1 \), we find that the partial derivative \( f_{p_1} = 2p_2^{2n_2 - 1}(p_1 - p_2) < 0 \). To show that all other partials are positive we examine (for the cases \( k > 3 \) or \( k = 2 \) and \( j > 2 \)) the effect of replacing in \( f_{p_j} \) the term \( 2n_jp_j^{2n_j - 1} - (n_j - 1)p_j^{n_j - 2} \) by \( 2n_jp_j^{2n_j - 1} \) and replacing \( p_i^{2n_i} - p_i^{n_i-1} \) by \( p_i^{2n_i} \) for all \( i > 2 \) when \( j > 2 \), and for all \( i > 3 \) when \( j = 1 \) and \( k > 3 \). Finally in the case \( k = 2 \), \( n_1 > 2 \) we show directly that

\[
f_{p_1} > p_1^{n_1-2}p_2^{n_2-1}[4p_1^3 + p_2^2 - 4p_1p_2] > p_1^{n_1-2}p_2^{n_2-1}(2p_1 - p_2)^2.
\]

**Proposition 7.** \( f \) increases with \( k \) in the sense that if \( p \) is a prime not dividing \( a \) then \( f(ap^h) > f(a) \) for all \( h > 1 \).

**Proof.** Let \( b = \sigma(a)\phi(a) \). Then

\[
f(ap^h) = (ap^h - 1)^2 - b(p^{2h} - p^{h-1}) > p^{2h}f(a).
\]
The Computations. In calculating nonstandard solutions \( n = \prod p_i^{n_i} \) of (*) it follows from Propositions 3 and 4 that \( k > 2 \) and if \( k = 2 \) we do not need to consider the case \( n_1 = 1 \). Therefore from Propositions 5–7, we can regard \( f \) as always an increasing function in all variables. Thus, for any upper limit \( L \), there is clearly a systematic way of calculating for all \( \sqrt{f} < L \), namely for each increasing \( k \) (starting with \( k = 2 \)) and each increasing choice of the \( n_i \) (starting with \( n_1 = 2 \) and \( n_2 = 1 \)) we calculate for all \( p_1 < p_2 < \cdots < p_k \) in each case up until \( \sqrt{f} > L \), recording all those \( n \) in which \( m = \sqrt{f} \) is an integer.

Note that in the following table we have separated the solutions for odd and even \( m \) since the odd \( m \) appear to have somewhat different properties. In fact, to say \( m \) is an odd \(*\)-number is simply to say that \( m + 2 \) is prime and (*) has the sole solution \( n = 2(m + 2) \) or that (*) has no solutions at all.

The following is the set of all nonstandard solutions of (*) for \( m < 1000 \). Note that the solutions marked \# are those guaranteed by Theorem 3.

<table>
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<th>m</th>
<th>n</th>
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Note. The only values of \( m < 5000 \) for which (*) has a solution with \( k = 4 \) are:

\[
\begin{align*}
m & \quad n \\
1744 & \quad 3.5.7.41 \\
3216 & \quad 5.11.13.19 \\
4516 & \quad 3.5.19.41
\end{align*}
\]
### PRIMES DIFFERING BY A FIXED INTEGER

#### EVEN

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