Spline Interpolation at Knot Averages on a Two-Sided Geometric Mesh*

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Abstract. For splines of degree \( k > 1 \) with knots \( -t_i = t_{2m+1-i} = 1 + q + q^2 + \cdots + q^{m-i}, \) \( i = 1, \ldots, m, \) where \( 0 < q < 1, \) it is shown that spline interpolation to continuous functions at nodes \( \tau_i = \sum_{j=1}^i w_j t_{i+j}^k, \) \( i = 1, \ldots, n = 2m - k - 1, \) has operator norm \( \|P\| \) which is bounded independently of \( q \) and \( m \) as \( q \) tends to zero if and only if \( 1 - w_k < \frac{1}{2}, \) \( 1 - w_k^k < \frac{1}{2}, \) and \( w_j > 0, j = 1, \ldots, k. \) The choice of nodes \( \tau_i = \sum_{j=1}^k w_{j+i} t_{j+i} \) and the growth rate of \( \|P\| \) with \( k \) are also discussed.

1. Two-Sided \( q \)-Splines. To integers \( n > 0, k > 0, \) and a nondecreasing sequence \( t = (t_i)_{i=1}^{n+k+1} \) with \( t_i < t_{i+k+1}, i = 1, \ldots, n, \) is associated \( S_{k+1,t}, \) the space of polynomial splines of order \( k + 1 \) with knot sequence \( t, \) defined by \( S_{k+1,t} = \text{span}\{N_1, \ldots, N_n\}, \) where each \( N_i = N_{i,k+1} \) is an appropriate normalized \( B \)-spline. See [1] for specific details.

With \( q > 0, m \) a positive integer, \( n = 2m - k - 1, \) and

\[
\begin{align*}
t_i &= - (1 + q + \cdots + q^{m-i}), \quad i = 1, \ldots, m, \\
&= 1 + q + \cdots + q^{i-k-1}, \quad i = m + 1, \ldots, 2m,
\end{align*}
\]

(1.1)

\( S_{k+1,t} \) is the space of two-sided \( q \)-splines.

Each two-sided \( q \)-spline can be represented as

\[
s(t) = \sum_{j=0}^{m-1} A_j [q^{j-m}(t_{j+1} - t)_+]^k + \sum_{j=0}^{k} A_{m+j} t^j
\]

(1.2)

\[
+ \sum_{j=1}^{m-1} A_{m+k+j} [q^{-j}(t - t_{m+j})_+]^k,
\]

where \( u_+ = \max\{u, 0\}, \) with the endpoint conditions

\[
s^{(i)}(t_i) = s^{(i)}(t_{2m}) = 0, \quad i = 0, \ldots, k - 1.
\]

(1.3)

Conversely, each function of the form (1.2) which satisfies (1.3) is a two-sided \( q \)-spline.

With the notation

\[
[i] = 1 + q + \cdots + q^{i-1}, \quad i = 0, 1, \ldots,
\]

relations such as

\[
t_{j+1} - t_i = q^{m-j}[j + 1 - i], \quad 0 < i < j < m,
\]

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and\[t_{i+1} - t_j = q^{j-m}[i + 1 - j], \quad m < j < i < 2m,\]
can be stated in a compact form. The notation
\[
[i]! = [i][i-1] \cdots [2][1] \quad \text{and} \quad \begin{bmatrix} j \\ i \end{bmatrix} = \begin{bmatrix} j! \\ i! \end{bmatrix}
\]
will also be useful.

The clause “as \( q \) tends to zero” appears throughout this paper. It will always mean “for all \( q \) satisfying \( 0 < q < q_0 \)”. The specific choice of \( q_0 \) will vary from instance to instance. However, \( q_0 \) will never depend on \( m \).

**Lemma 1.1.** With \( k \) and \( m \) fixed, let \( \{s\} \) be a set of two-sided \( q \)-splines with \( \{(A_1, \ldots, A_{2m+k-1})\} \) the corresponding set of coefficient vectors in (1.2). Then \( \{s\} \) is uniformly bounded as \( q \) tends to zero if and only if \( \{(A_j)\} \) is uniformly bounded as \( q \) tends to zero. Moreover, if the bound on \( \{s\} \) is independent of \( m \), then so is the bound on \( \{(A_j)\} \).

**Proof.** Let \( 1 > q_0 > 0 \) and \( C \) be such that
\[|A_j| < C, \quad \text{all } j \text{ and } 0 < q < q_0.\]
Then, for each real \( t \) and \( 0 < q < q_0 \),
\[
|s(t)| < C\left( \sum_{l=1}^{m-1} |q^{l-m}(t_{j+1} - t_i)|^k + \sum_{l=0}^k t_{2m} + \sum_{l=1}^{m-1} |q^{-j}(t_{2m} - t_{m+j})|^k \right)
= C\left( \sum_{l=1}^{m-1} [j]^k + \sum_{l=0}^k [m]^j + \sum_{l=1}^{m-1} [m - j]^k \right) < (2m + k - 1)C[m]^k
\]
Conversely, let \( 1 > q_0 > 0 \) and \( B \) be such that
\[|s(t)| < B, \quad \text{all real } t \text{ and } 0 < q < q_0.\]
Since
\[
\sum_{l=0}^k A_{m+j}(i/k)^l = s(i/k), \quad i = 0, \ldots, k,
\]
is a matrix equation with nonsingular coefficient matrix \( V = (i/k)^l \) depending only on \( k \),
\[|A_{m+j}| < (k + 1)B_kB, \quad j = 0, \ldots, k,
\]
where \( B_k \) is a bound on the entries of \( V^{-1} \). Set \( C_0 = (k + 1)B_kB \) and assume inductively that \( q_1 \) is such that \( |A_{m+j}| < C_j \) for \( j = 0, 1, \ldots, i - 1 \) for \( q < q_1 \). From (1.2)
\[
s(t_m) - s(t_{m-1}) = A_{m-i} + \sum_{l=1}^{i-1} A_{m-j}([i - j + 1]^k - [i - j]^k)
+ \sum_{l=1}^k A_{m+j}(-1)^l([i + 1]^l - [i]^l),
\]
so that

$$|A_{m-i}| < 2B + \sum_{i=1}^{i-1} C_j (i - j + 1)^k - [i - j]^k + C_0 \sum_{i=1}^{i-1} ([i + 1]! - [i]!)$$

$$< 2B + \sum_{i=1}^{i-1} C_j q^{i-j}k(1 - q_0)^{-k} + C_0 \sum_{i=1}^{i-1} q^j(1 - q_0)^{-j}$$

$$< 2B + \sum_{i=0}^{i-1} C_j q^{i-j}R_k \quad \text{with} \quad R_k = k^2(1 - q_0)^{-k}.$$ 

Setting $C_i = 2B + \sum_{i=0}^{i-1} C_j q^{i-j}R_k$ allows the induction to proceed. Then $C_1 = 2B + C_0 q_1 R_k$, and $C_{i+1} = q_1 (1 + R_k) C_i + 2B(1 - q_1)$, $i = 1, \ldots, m - 2$. This recurrence solves as

$$C_i = \frac{2B(1 - q_1)}{1 - q_1 - q_1 R_k} \left[ 1 - (q_1 + q_1 R_k)^{i-1} \right] + C_1 (q_1 + q_1 R_k)^{i-1},$$

$$i = 1, \ldots, m - 1,$$

if $q_1 + q_1 R_k \neq 1$. Imposing the added restriction $q_1 + q_1 R_k < \frac{1}{2}$ and noting that a symmetric argument will yield $|A_{m+k+j}| < C_j$, $j = 1, \ldots, m - 1$, establishes that

$$\max_j |A_j| < \max_i C_i < 4B + C_1 + C_0.$$

This bound is independent of $m$ if $B$ is independent of $m$. □

**Lemma 1.2.** Let $k$ and $m$ be fixed. As $q$ tends to zero, the coefficients $(A_j)$ satisfy

$$A_i + \sum_{i+1}^{m-1} A_j q^{j-i}(k-i) \left[ \begin{array}{c} j \\ i \end{array} \right] + \sum_{i-1}^{k-i} A_{m+j} O(q^{m-j}(k-j)) = 0, \quad i = 1, \ldots, k - 1,$$

and

$$A_k + \sum_{k+1}^{m-1} A_j \left[ \begin{array}{c} j \\ k \end{array} \right] + \sum_{0}^{k} A_{m+j} \left( \frac{j!}{[k]!} + O(q^{m-k+1}) \right) = 0.$$

**Proof.** This follows from (1.3). Let functionals $\Lambda_{it}$, $1 \leq i < \nu < k$, be defined by

$$\Lambda_{it} = q^{(m-1)(k-\nu)} \frac{j!}{k!} (-1)^{k-\nu} s^{(k-\nu)}(t_1)$$

and, recursively,

$$\Lambda_{iv} = q^{i-k} (\Lambda_{i-1,v} - [i - 1] \Lambda_{i-1,v-1}) / [i].$$

From (1.2)

$$s^{(k-\nu)}(t_1) = \sum_{i=1}^{m-1} A_j q^{(j-m)(k-\nu)} \frac{j!}{\nu!} (-1)^{j-\nu} \left[ \begin{array}{c} j \\ \nu \end{array} \right]$$

$$+ \sum_{k-\nu}^{k} A_{m+j} \frac{j!}{(j-k+\nu)!} \left[ \begin{array}{c} j \\ k \end{array} \right],$$

whence

$$\Lambda_{1v} = \sum_{i=1}^{m-1} A_j q^{(i+1)(k-v)} \left[ \begin{array}{c} i \\ \nu \end{array} \right] + \sum_{k-\nu}^{k} A_{m+j} q^{(m-1)(k-v)} C_{1v},$$
where

\[ C_{ij\nu} = \frac{\nu j!}{k! (j - k + \nu)!} (-1)^{k-n} t_i^{k-n} \]

The recursion formula gives

\[ \lambda_{ij\nu} = \sum_{i=1}^{m-1} A_j q^{(i)(k-\nu)} \left[ \begin{array}{c} j \\ i \end{array} \right] + \sum_{k=1}^{k-\nu} A_{m+j} q^{(m-i)(k-\nu)} C_{ij\nu}, \]

where

\[ C_{ij\nu} = \left( C_{i-1,j,\nu} - [i - 1] q^{m-i+1} C_{i-1,j,\nu-1} \right)[i]. \]

From (1.3) each \( \lambda_{ij\nu} = 0 \) and, in particular, \( \lambda_{ii\nu} = 0 \). This fact, along with the observation that \( C_{ijk} = C_{ij\nu}/[k]! + O(q^{m-k+1}) \) completes the proof. □

Combining Lemmas 1.1, 1.2, and a symmetric counterpart of Lemma 1.2 yields

**Lemma 1.3.** Let \( k \) and \( m \) be fixed and let \( \{s\} \) be a set of two-sided q-splines which is bounded as \( q \) tends to zero. Then the corresponding set of coefficient vectors \( \{(A_i)\} \) satisfies

\[ A_i = O(q^{k-i}), \quad i = 1, \ldots, k-1, \]
\[ A_i = O(1), \quad i = k, \ldots, 2m, \]
\[ A_{2m+i} = O(q^i), \quad i = 1, \ldots, k-1, \]

as \( q \) tends to zero. If the bound on \( \{s\} \) is independent of \( m \), then so are the bounds on the \( A_i \).

The independence of \( m \) in the \( O(q^{k-i}) \) and \( O(q^i) \) bounds follows from the exponential decay of the coefficients in the first \( k-1 \) equations of Lemma 1.2.

2. **Spline Interpolation.** Let \( \tau = (\tau_i)^n \) be a strictly increasing sequence. It is known [1] that: For each function \( f \) defined on \( \tau \) there is exactly one \( s \in \mathbb{S}_{k+1,n} \) such that \( s(\tau_i) = f(\tau_i), i = 1, \ldots, n, \) if and only if \( N_i(\tau_i) > 0, i = 1, \ldots, n, \) or, equivalently, if and only if

\[ t_i < \tau_i < t_{i+k+1}, \quad i = 1, \ldots, n. \]

When \( \tau \) satisfies (2.1) a linear map \( P \) into \( \mathbb{S}_{k+1,n} \) which reproduces \( \mathbb{S}_{k+1,n} \) may be defined by: For each function \( f \) defined on \( \tau \), \( P f \in \mathbb{S}_{k+1,n} \) and \( (P f)(\tau_i) = f(\tau_i), \)

\[ i = 1, \ldots, n. \]

In fact, \( P f = \sum f(\tau_j)L_j \) where \( (L_j)^n \) is defined by \( L_j(\tau_i) = \delta_{ij}, i, j = 1, \ldots, n. \) The operator norm of \( P \) is

\[ \| P \| = \sup_{f} \frac{\| P f \|}{\| f \|}, \]

where the sup is taken over all \( f \in C[t_1, t_{n+k+1}] \) and

\[ \| f \| = \sup \{|f(t)|: t_1 < t < t_{n+k+1}\}. \]

It is well known that

\[ \| P \| = \max_{i} \sum_{t} |L_j(t)| = \max_{0 < \mu < n} \left( \max_{\tau_\mu < t < \tau_{\mu+1}} s_\mu(t) \right). \]
where \( \tau_0 = t_1, \tau_{n+1} = t_{n+k+1} \) and \( s_{\mu}^0 \) is defined by

\[
\begin{align*}
  s_{\mu}(t_i) &= (-1)^{i+\mu}, & i &= 1, \ldots, \mu, \\
  &= -(-1)^{i+\mu}, & i &= \mu + 1, \ldots, n.
\end{align*}
\]

(2.2)

For each \( \mu \), the so-called Lebesgue function \( \sum |L_\mu(t)| \) coincides with \( s_{\mu}(t) \) on the interval \([\tau_\mu, \tau_{\mu+1}]\).

One way of specifying \( \tau \) is to require that the nodes be knot averages, i.e.,

\[
\tau_i = \sum_{j=0}^{k+1} w_j t_{i+j}, \quad i = 1, \ldots, n,
\]

(2.3)

where the \( w_j \) are fixed nonnegative numbers which sum to one.

**Theorem 1.** Let \( k \geq 2, m, \) and \((w_0)^{k+1} \) be fixed. Let \( t \) be given by (1.1) and \( \tau \) be given by (2.3). If \( \|P\| \) is bounded as \( q \) tends to zero, then

\[
w_i > 0, \quad i = 1, \ldots, k.
\]

(2.4)

If the bound on \( \|P\| \) is also independent of \( m \), then either

\[
w_0 = 0 \quad \text{and} \quad (1 - w_1)k < \frac{1}{2}
\]

(2.5)a

or

\[
w_0 > 0 \quad \text{and} \quad \frac{1}{2} < (1 - w_0)k
\]

(2.5)b

and, either

\[
w_{k+1} = 0 \quad \text{and} \quad (1 - w_k)k < \frac{1}{2}
\]

(2.6)a

or

\[
w_{k+1} > 0 \quad \text{and} \quad \frac{1}{2} < (1 - w_{k+1})k.
\]

(2.6)b

Conversely, if (2.4), (2.5), (2.6) hold, then \( \|P\| \) is bounded independently of \( m \) as \( q \) tends to zero.

**Proof.** Let \( w_a \) be the first positive weight and \( w_b \) be the last positive weight, so that \( \tau_\mu = \sum_{j=a}^{b} w_j t_{i+j} \), and set

\[
\theta_1 = (1 - w_a) + (1 - w_a - w_{a+1})q + \cdots + w_b q^{b-a-1},
\]

\[
\theta_2 = (1 - w_b) + (1 - w_b - w_{b-1})q + \cdots + w_a q^{b-a-1}.
\]

If \( a = b \), then \( \theta_1 = \theta_2 = 0 \). If \( a < b \), then \( 0 < \theta_1 < 1 \) and \( 0 < \theta_2 < 1 \) as \( q \) tends to zero. Therefore,

\[
t_{i+b-1} < \tau_i = t_{i+b} - \theta_2 q^{m+1-b-i} < t_{i+b}, \quad i = 1, \ldots, m - b,
\]

(2.7)

\[
t_{i+a} < \tau_i = t_{i+a} + \theta_1 q^{i+a-m} < t_{i+a+1}, \quad i = m - a + 1, \ldots, n,
\]

for all sufficiently small \( q > 0 \). Since

\[
\tau_i = 1 - 2 \sum_{j=0}^{m-i} w_j + O(q), \quad i = m - b + 1, \ldots, m - a,
\]

(2.8)

as \( q \) tends to zero, it follows that also

\[
-1 < \tau_{m-b+1} < \tau_{m-b+2} < \cdots < \tau_{m-a} < +1
\]

(2.9)

for all sufficiently small \( q > 0 \).
Henceforth, we require that \( q \) be such that the inequalities in (2.7) and (2.9) hold. This requirement is independent of \( m \).

Now let \( \|P\| \) be bounded independently of \( m \) as \( q \) tends to zero. We shall prove that (2.4) and (2.6) must hold. A symmetric argument, which we omit, will give (2.5).

Let \( s = s_\mu \) be defined by (2.2) with \( \mu < m - b + 1 \) or \( \mu > m - a - 1 \). There is a constant \( C \) which bounds \( \|P\| \) so that \( \|s\| \leq C \) as \( q \) tends to zero. Since the restriction of \( s \) to \([-1, +1]\) is a polynomial of degree \( k \), it follows from a theorem of A. A. Markov (see [7]) that

\[
\max\{|s'(t)|: -1 < t < 1\} \leq Ck^2.
\]

Thus, (2.8), (2.9), and the mean-value theorem imply that

\[
2 = |s(\tau_i) - s(\tau_{i+1})| \leq Ck^2(\tau_{i+1} - \tau_i) \leq 2Ck^2w_{m-i} + O(q)
\]

for \( i = m - b + 1, \ldots, m - a - 1 \) as \( q \) tends to zero. Thus, \( w_i > 1/Ck^2 > 0 \), \( i = a + 1, \ldots, b - 1 \).

Suppose that \( b < k \). Then, on the one hand, (1.2) gives

\[
\pm 1 = s(\tau_i) = \sum_{b}^{m-1} A_j([j - b] + \theta_2q^j - b)^k + \sum_{0}^{k} A_{m+j}([-m - b] - \theta_2q^{m-b})^j
\]

\[
= A_b\theta_2^k + \sum_{b+1}^{m-1} A_j([j - b]^k + O(q^j-b))
\]

\[
+ \sum_{0}^{k} A_{m+j}([-m - b])^j + O(q^{m-b}),
\]

whereas, on the other hand, with \( \Lambda_\mu s \) as in the proof of Lemma 1.2,

\[
0 = \theta_2^k \Lambda_{bb}s + \sum_{b+1}^{k-1} [i - b]^k \Lambda_\mu s + [k]! \Lambda_{kk}s
\]

\[
= A_b\theta_2^k + \sum_{b+1}^{m-1} A_j([j - b]^k + O(q^j-b))
\]

\[
+ \sum_{0}^{k} A_{m+j}([-m - b])^j + O(q^{m-b}).
\]

Subtraction yields

\[
\pm 1 = \sum_{b+1}^{m-1} A_jO(q^j-b) + \sum_{0}^{k} A_{m+j}O(q^{m-b}),
\]

so that \( (A_j) \) cannot be bounded as \( q \) tends to zero. This contradiction to Lemma 1.3 shows that \( b > k \).

A similar argument with \( s(\tau_\mu) \) shows that \( a < 1 \), so that (2.4) is proved.

To prove (2.6), we first suppose that \( w_{k+1} = 0 \). We must show that \( (1 - w_k)^k < \frac{1}{2} \)

or, equivalently, that

\[
r_2 = \theta_2^k / (1 - \theta_2^k) < 1 \quad \text{as } q \text{ tends to zero.}
\]

Again, let \( s = s_\mu \) be defined by (2.2). Then Lemma 1.2 and (1.2) give

\[
-s(\tau_i) = [k]! \Lambda_{kk}s - s(\tau_\mu) = \sum_{0}^{m-k-1} M_{0j}A_{k+j} + \sum_{0}^{k} R_{0j}A_{m+j}
\]
and
\[ s(\tau_i) - s(\tau_{i+1}) = \sum_{i=1}^{m-k-1} M_{ij} A_{k+j} + \sum_{j=0}^{k} R_{ij} A_{m+j}, \quad i = 1, \ldots, m - k - 1, \]

where
\[
M_{0j} = \frac{[k+j]!/[j]! - ([j] + \theta_2 q^j)^k}{j = 0, \ldots, m - k - 1,} \\
M_{ij} = \theta_2^k, \quad i = 1, \ldots, m - k - 1, \\
M_{ij} = ([j - i + 1] + \theta_2 q^j)^k - ([j - i] + \theta_2 q^{j-i})^k, \quad i = 1, \ldots, m - k - 1; j = i, \ldots, m - k - 1, \\
R_{0j} = \frac{[k]! C_{kk} - \tau_i}{j = 0, \ldots, k,} \\
R_{ij} = \tau_i - \tau_{i+1}, \quad i = 1, \ldots, m - k - 1; j = 0, \ldots, k, \\
with \quad C_{kk} = t_i^k/[k]! + O(q^{m-k+1}) as in the proof of Lemma 1.2.

Since the \( A_j \) are bounded and
\[
M_{ij} < [k+j]^k - [j]^k < q^k [k+j]^{k-1} < q^k (1-q)^{-k}, \\
M_{ij} < [j-i+2]^k - [j-i]^k < q^i (1-q)^{-k}, \\
\quad j = i + 1, \ldots, m - k - 1, \\
|R_{ij}| < q^{m-k}(j+1)(1-q)^{-j}, \\
|R_{ij}| < q^{m-k-i}(1-q)^{-j},
\]
the system (2.11) and (2.12) has the form
\[
(1 - \theta_2^k) A_k = -s(\tau_i) + O(q), \\
\theta_2^k A_{k+i} + (1 - \theta_2^k) A_{k+i} = s(\tau_i) - s(\tau_{i+1}) + O(q), \quad i = 1, \ldots, m - k - 1,
\]
which solves as
\[
A_{k+i} = \frac{2(-1)^{i+1}}{1 - 2\theta_2^k} \left[ 1 - \frac{r_{i+1}^k}{2\theta_2^k} \right] + O(q), \quad i = 0, \ldots, \min(\mu - 1, m - k - 1), \\
(2.13)
A_{k+i} = \frac{2(-1)^{i+1}}{1 - 2\theta_2^k} \left[ 1 - \frac{r_{i+1}^k}{2\theta_2^k} \right] + O(q), \quad i = 0, \ldots, m - k - 1 - \mu,
\]
if \( r_2 \neq 1 \) and as
\[
A_{k+1} = (-1)^{i+1}(2 + 4i) + O(q), \quad i = 0, \ldots, \min(\mu - 1, m - k - 1), \\
A_{k+i} = (-1)^{i+1}(2 + 4i - 4\mu) + O(q), \quad i = 0, \ldots, m - k - 1 - \mu,
\]
if \( r_2 = 1 \) provided that the buildup of \( O(q) \) terms is bounded independently of \( m \). This will be the case if \( qr_2 < 1 \) as \( q \) tends to zero, a condition that can be met independently of \( m \). By Lemma 1.3 these \( A_j \) are bounded independently of \( m \). Therefore, (2.10) must hold and \((1 - \omega_k)^k < \frac{1}{2} \).
To complete the proof of (2.6) we now suppose that \( w_{k+1} > 0 \). Since \( \theta_2 = 1 - w_{k+1} + O(q) \), we must now show that \( \theta_2^k > \frac{1}{2} \) as \( q \) tends to zero, that is

\[
(2.14) \quad r_2 = \frac{\theta_2^k}{1 - \theta_2^k} > 1 \quad \text{as } q \text{ tends to zero.}
\]

Since the \( \tau_i \) have "moved over one interval", Eqs. (2.11) and (2.12) are replaced by

\[
-s(\tau_i) = [k]!A_k + \sum_{j=1}^{m-k-1} \left( [k + j]/[j]! - \left( [j - 1] + \theta_2 q^{(-1)^j}\right)\right)A_{k+j}
\]

(2.15)

\[
+ \sum_{j=0}^{k} R_{ij} A_{m+j}
\]

and

\[
s(\tau_i) - s(\tau_i+1) = \sum_{j=1}^{m-k-1} M_{ij} A_k + \sum_{j=0}^{k} R_{ij} A_{m+j},
\]

(2.16)

\[i = 1, \ldots, m - k - 2,\]

and the bounds on \( M_{ij} \) and \( R_{ij} \) are replaced by

\[
| [k + j]/[j]! - \left( [j - 1] + \theta_2 q^{(-1)^j}\right)\left| < q^{(-1)^j}(1 - q)^k, \right.
\]

\[
|R_{ij}| < q^{m-k-1}(j + 1)(1 - q)^j,
\]

\[
|R_{ij}| < q^{m-k-1}(1 - q)^j.
\]

This incomplete system now has the form

\[
A_k + (1 - \theta_2^k)A_{k+1} = -s(\tau_1) + O(q),
\]

\[
\theta_2^k A_{k+i} + (1 - \theta_2^k)A_{k+i+1} = s(\tau_i) - s(\tau_{i+1}) + O(q), \quad i = 1, \ldots, m - k - 2.
\]

Adding the equation

\[
(2.17) \quad s(\tau_{m-k-1}) = A_{m-1} \theta_2 A_{k+i} + \sum_{j=0}^{k} A_{m+j} \tau_{m-k-1} = A_{m-1} \theta_2^k + s(-1) + O(q)
\]

and imposing the restriction \( qr_2^{-1} < 1 \) permits us to solve this system backwards in terms of \( s(-1) \) as

\[
A_{m-i} = \frac{(-1)^{m-1-k-\mu-i}}{2\theta_2^i - 1} \left[ 2 - (1 + r_2)r_2^{-i} \right] + (1 + r_2)(-r_2)^{-i}s(-1) + O(q), \quad i = 1, \ldots, m - k - 1 - \mu,
\]

(2.18)

\[
A_{k+i+1} = \frac{(-1)^{i-1}}{2\theta_2^k - 1} \left[ 2 - (1 + r_2)r_2^{-i} \left( 2 - r_2^{-m+k+\mu+1}\right) \right] + (1 + r_2)(-r_2)^{-m+k+\mu+1-i}s(-1) + O(q), \quad i = 1, \ldots, \mu,
\]

if \( 0 < \mu < m - k - 2 \) and as

\[
(2.19) \quad A_{m-i} = \frac{(-1)^{m-1-k-\mu-i}}{2\theta_2^i - 1} \left[ 2 - (1 + r_2)r_2^{-i} \right] + (1 + r_2)(-r_2)^{-i}s(-1) + O(q), \quad i = 1, \ldots, m - k - 1,
\]

if \( \mu > m - k - 1 \). Since the \( A_j \) are bounded independently of \( m \), (2.14) must hold and \( (1 - w_{k+1})^k > \frac{1}{2} \).
The proof that (2.4), (2.5), (2.6) are necessary conditions for \( \| P \| \) to be bounded independently of \( m \) as \( q \) tends to zero is complete.

To prove that (2.4), (2.5), (2.6) are sufficient that \( \| P \| \) be bounded independently of \( m \) as \( q \) tends to zero, we will use the approach outlined in the proof of Lemma 1.1. That is, we will first show that, for each \( s = s_\mu \), the block \( A_m, \ldots, A_{m+k} \) is bounded and then argue recursively from bounds on \( s(t) \) (replacing \( s(t) \) in the proof of Lemma 1.1) that \( A_{m-i} \) (and \( A_{m+k+i} \), \( i = 1, \ldots, m - 1 \), are bounded independently of \( m \). Finally, we will use (1.2) and (2.13) or (2.18) or (2.19) to bound \( s_\mu(t) \) for all \( t \) and all \( \mu \).

If \( a = 0 \) and \( b = k + 1 \), the first step, bounding the block \( A_m, \ldots, A_{m+k} \) is easy since (2.9) implies that

\[
\sum_{i=k+1-b}^{k} A_{m+i-k+i} = \pm 1, \quad i = k + 1 - b, \ldots, k - a,
\]

and (2.4), (2.8) give a bounded inverse for the Vandermonde matrix \((r_{m-k+i})\). However, if \( b = k \) then the \( i = 0 \) equation of (2.20) is replaced by

\[
\theta_z^k A_{m-1} + \sum_{i=0}^{k} A_{m+i} r_{m-k} = \pm 1.
\]

If \( a = 1 \), there is a similar replacement of

\[
\sum_{i=0}^{k} A_{m+i} r_{m} + \theta_z^k A_{m+k+1} = \pm 1
\]

for the \( i = k \) equation of (2.20).

Therefore, if \( b = k \) (and/or \( a = 1 \)), a preliminary step to eliminate \( \theta_z^k A_{m-1} \) from (2.21), at the expense of adding a bounded quantity to the right member, is necessary. While eliminating \( \theta_z^k A_{m-1} \) through a sequence of upper triangulation steps on (2.11), (2.12), (2.21) is straightforward, there must be an argument that \( \theta_z^k A_{m-1} \) is bounded independently of \( m \) as \( q \) tends to zero independently of \( m \). The following lines supply this argument.

Let \( b = k \) and let \( s \) be any of the \( s_\mu \) given by (2.2). Using the bounds on \( M = (M_{ij}) \), we see that this matrix is diagonally dominant if \( q \) is such that

\[1 - \theta_2^k > \theta_2^k + kq(1 - q)^{-k-1} \]

But (2.5)\( a \) is equivalent to \( 1 - \theta_2^k > \theta_2^k \) for sufficiently small \( q \), so that this condition can be met by imposing a further restriction on \( q \).

Let \( q_0 > 0 \) and \( \delta > 0 \) be such that \( \delta = 1 - 2\theta_2^k - kq_0(1 - q_0)^{-k-1} \). Then the solutions of a system

\[Mx = b\]

satisfy \( \max_{x_j} |x_j| \leq \delta^{-1} \max_{|b_j|} \) by the usual diagonal dominance argument. Applying this fact with

\[
b_0 = \lfloor k \rfloor! A_{kk}s - s(t), \quad s(t) = -s(t_1),
\]

\[
b_i = s(t_i) - s(t_{i+1}), \quad i = 1, \ldots, m - k - 1,
\]
as well as with

\[
b_i = -R_y, \quad i = 0, \ldots, m - k - 1,
\]
for each \( j = 0, \ldots, k \), yields

\[ A_{m-1} = C + \sum_{0}^{k} C_j A_{m+j} \]

with

\[
|C| < \delta^{-1} \max\{|s(\tau_i)|, |s(\tau_i) - s(\tau_{i+1})|: i = 1, \ldots, m - k - 1\} = 2/\delta,
\]

\[
|C_0| < \delta^{-1} \max_i |R_{00}| = |R_{00}|/\delta = O(q^{m-k}),
\]

\[
|C_j| < \delta^{-1} \max_i |R_{ij}| = |\tau_{m-k-j}| / \delta
\]

\[
= |(1 + q + \theta_2 q^2)^j - (1 + \theta_2 q^j)|/\delta < q[2]j[3]^{j-1}/\delta
\]

\[
< q[3]j/\delta = O(q), \quad j = 1, \ldots, k.
\]

Combining these deductions with (2.21) gives the equation

\[
(2.23) \quad \sum_{0}^{k} A_{m+j}(\tau_{m+j} - C_j \theta_2^j) = s(\tau_{m-k}) - C \theta_2^k,
\]

which can be adjoined to (2.20). Since \( C_j = O(q) \) and \( \tau_{m-k+1} - \tau_{m-k} = 2w_k + O(q) \), the resulting system has a bounded solution as \( q \) tends to zero. We have assumed that \( a = 0 \). If \( a = 1 \), a similar argument at \( \tau_m \) is needed.

We have completed the first step in the proof of sufficiency, i.e., we have shown that the set \( A_m, \ldots, A_{m+k} \) is bounded. But now (2.12) or (2.16) and their symmetric counterparts imply immediately that the set \( A_k, \ldots, A_{2m} \) is bounded. An argument similar to the proof of Lemma 1.2 gives \( O(q^i) \) bounds on \( A_{k-i} \) and \( A_{2m+k-i}, i = 1, \ldots, k - 1 \). The second step in the proof is completed.

Now we must bound \( s_\mu(t) \) for all \( t \) and all \( \mu \). For \( -1 < t < +1 \), the boundedness of \( A_m, \ldots, A_{m+k} \) and (2.4) give a uniform bound on \( s_\mu(t) \). If \( t_1 < t < t_m \), there is a \( \theta_i \) in \([0, 1]\) and an \( i > 0 \) such that \( t_{m-i} < t = t_{m-i+1} - \theta_i q^i = -[i] - \theta_i q^i < t_{m-i+1} \). Then

\[
s(t) = \sum_{m-i}^{m-1} A_i([i + j - m] + \theta_i q^{i+j-m})^k + \sum_{0}^{k} A_{m+j} t^i.
\]

If \( i < m - b \), then \( \tau_{m+1-b-i} = -[i] - \theta_2 q^i \) and

\[
|s(t)| < |s(\tau_{m+1-b-i})| + |A_{m-i}| + O(q) = 1 + |A_{m-i}| + O(q)
\]

can be easily shown. If \( i > m - b \), a modified argument gives

\[
|s(t)| < |s(\tau_i)| + \sum_{1}^{k} |A_j| + O(q) = 1 + |A_k| + O(q).
\]

Thus, the \( s_\mu(t) \) are uniformly bounded for all \( \mu \) and all \( t \) so that \( \|P\| \) is bounded independently of \( m \) as \( q \) tends to zero. \( \square \)

3. Two Special Cases. Theorem 1 provides counterexamples when (2.4), (2.5), (2.6) are not satisfied, e.g., interpolation at the knots with \( k > 2 \) or interpolation at weighted two-knot averages with \( k > 3 \). The condition that \( q \) tend to zero compares (contrasts?) with the often-used condition that the local mesh ratios \( (t_{j+1} - t_j)/(t_{i+1} - t_i), |i - j| = 1 \) be bounded.
For \( k > 3 \) and \( q = 1 \), it is easy to select weights \( w_j \) satisfying (2.4), (2.5), (2.6) which still produce unbounded spline interpolation. Thus, even for two-sided \( q \)-splines, these conditions are not sufficient to guarantee bounded interpolation. Indeed, the method of their derivation suggests that they are linked quite closely to the tendency of \( q \) to zero.

For the two special cases which follow it is not clear that we need \( q \) to tend to zero. Computational evidence with small \( k \) suggests, in fact, that \( q \) tending to zero gives "worst-case" results. Thus, Theorems 2 and 3 are imperfect in that the condition that \( q \) tend to zero may be superfluous.

**Theorem 2.** Let \( t \) be given by (1.1) and, for each \( k > 1 \) and \( m > k \), let \( \tau \) be given by

\[
\tau_i = (t_{i+1} + t_{i+2} + \cdots + t_{i+k})/k, \quad i = 1, \ldots, n.
\]

Then, \( \|P\| \) is bounded as \( q \) tends to zero. Moreover, there exist absolute constants \( 1 < C_1 < C_2 \) such that, for each \( k > 2 \),

\[
C_1^k < \|P\| < C_2^k \quad \text{as } q \text{ tends to zero.}
\]

**Theorem 3.** Let \( t \) be given by (1.1) and, for each \( k > 1 \) and \( m > k \), let \( t \) be given by (2.3) with

\[
w_0 = w_{k+1} = \sin^2(\alpha_k/2),
\]

\[
w_j = \sin(\alpha_k)\sin(2\alpha_k), \quad j = 1, \ldots, k,
\]

where \( \alpha_k = \pi/(2k + 2) \). Then, \( \|P\| \) is bounded as \( q \) tends to zero. Moreover, there exist absolute constants \( 0 < C_3 < C_4 \) such that, for each \( k > 2 \),

\[
C_3 \log k < \|P\| < C_4 \log k \quad \text{as } q \text{ tends to zero.}
\]

**Proof of Theorems 2 and 3.** The assertions that \( \|P\| \) is bounded as \( q \) tends to zero are proved by showing that (2.4), (2.5), (2.6) hold. These follow readily, since, in Theorem 2,

\[
(1 - w_k)^k = (1 - w_1)^k = (k - 1)^k/k^k < 1/e < 3/8
\]

while, in Theorem 3,

\[
(1 - w_{k+1})^k = (1 - w_0)^k = \cos^{2k}(\alpha_k/2) > (1 - \alpha_k^2/8)^{2k} > 1 - \pi\alpha_k/8 > (8k + 3)/(8k + 8) > 3/4.
\]

In Theorem 2, the lower bound on \( \|P\| \) follows from the fact that, as \( q \) tends to zero, the nodes

\[
\tau_m - k + 1, \tau_m - k + 2, \ldots, \tau_m - k + j, \ldots, \tau_m - 1
\]

tend to

\[
(2 - k)/k, (4 - k)/k, \ldots, (2j - k)/k, \ldots, (k - 2)/k,
\]

and that, for \( s = s_\mu \) with \( m - k \leq \mu < m - 1 \),

\[
|s(\pm 1)| = (1 - r_{2m-k})/ (1 - 2\theta_{2}^k) + O(q) > 1/(1 - \theta_{2}^k) + O(q) > 1,
\]

so that \( \|P\| \) is bounded below by any lower bound for polynomial interpolation on \([-1, +1]\) at the equally-spaced nodes

\[-1, (2 - k)/k, (4 - k)/k, \ldots, (2j - k)/k, \ldots, (k - 2)/k, +1.
\]
See Rivlin [7, pp. 96–99] for a proof that such polynomial interpolation grows exponentially.

Similarly, in Theorem 3, the lower bound on \( \|P\| \) follows from the fact that \( \tau_{m-k}, \ldots, \tau_m \) approach the Chebyshev nodes \(-\cos(2j\alpha_k - \alpha_k), j = 1, \ldots, k + 1, \) as \( q \) tends to zero and the fact that polynomial interpolation on these nodes has logarithmic growth. See [7, pp. 93–96].

To complete the proof that \( \|P\| \) grows exponentially or logarithmically in Theorem 2 or Theorem 3, respectively, it is necessary only to show that, for each \( \mu \), \( s_\mu(t) \) is "controlled" outside \((-1, +1)\). This fact follows from the closing lines of the proof of Theorem 1, where it was noted that, for \( t_1 < t < t_m \), there is a \( j < m \) such that \( |s(t)| < 1 + |A_j| + O(q) \). For Theorem 2, (3.1) and (2.13) imply that

\[
\max_{j < m} |A_j| < \frac{2}{1 - 2\theta_s^k} + O(q) < \frac{2e}{e - 2} + O(q) < 8,
\]

so that \( |s(t)| < 10 \) for \( t < -1 \) as \( q \) tends to zero. For Theorem 3, (3.2) and (2.18), (2.19) imply that

\[
\max_{j < m} |A_j| < \frac{2}{2\theta_s^k - 1} + 2|s(-1)| + O(q) < \frac{2 \cos^{2k}(\alpha_k/2) - 1}{2 \cos^{2k}(\alpha_k) - 1} + 2|s(-1)| + O(q) < 4 + 2|s(-1)| + O(q),
\]

so that \( |s(t)| < 6 + 2|s(-1)| \) for \( t < -1 \) as \( q \) tends to zero. Symmetry considerations give like bounds for \( |s(t)| \) on \(+1 = t_{m+1} < t < t_{2m}\).

The proof of Theorem 2 and Theorem 3 is complete. □

If \( q = 1 \) (not covered by these theorems), two-sided \( q \)-spline interpolation is essentially the same as cardinal spline interpolation, for which logarithmic growth of \( \|P\| \) with \( k \) has been demonstrated; see [6]. This fact supports the conjecture that \( q \) tending to zero gives "worst-case" results for the nodes (3.1).

For cubic spline interpolation with arbitrary knot spacing and the nodes (3.1), de Boor [2] has shown that \( \|P\| < 27 \). He conjectures that \( \|P\| < 3 \) or \( 4 \) may be true. The following supplies a lower bound on \( \limsup \|P\| \), where the \( \limsup \) is taken over all ordered knot spacings.

**Theorem 4.** Let \( k = 3 \) and let \( t \) and \( \tau \) be given by (1.1) and (3.1), respectively. Then

\[
\lim \|P\| = (222\sqrt{111} + 999)/1331 = 2.507825 \ldots,
\]

where \( \lim \|P\| \) denotes the limiting value of \( \|P\| \) as \( q \) tends to zero and \( m \) tends to infinity.

**Proof.** Let \( s = s_\mu \) with \( \mu = m - 1 \). From (2.21) and (2.13)

\[
s(-1) = (-1)^{\mu-m+k} \frac{1 - r_2^{m-k}}{1 - 2\theta_s^k} + O(q)
\]
for each \( k > 2 \) and \( \mu > m - k \). Similarly,

\[
(3.4) \quad s(\pm 1) = \frac{(-1)^{m-1-\mu}}{1 - 2\theta_k^m} (1 - r_k^m) + O(q)
\]

for \( k > 2 \) and \( \mu < m - 1 \). Thus, for the case presently under consideration, \( s(t) \) tends, on \([-1, +1]\), to the cubic \( p(t) \) satisfying \( p(\pm 1) = 27/11 \) and \( p(\pm 1/3) = \pm 1 \). This cubic is

\[
p(t) = (-297t^3 + 243t^2 + 297t - 27)/88.
\]

It has a maximum on \([-1, +1]\) of \((222\sqrt{111} + 999)/1331\) at \( t = (9 + 2\sqrt{111})/33 \). Showing that \( \lim \|P\| \) exists and is equal to this maximum requires a discussion (which we omit) similar to the last paragraph in the proof of Theorem 1 above.

For arbitrary \( k \) it is easy to find \( p(t) \), the polynomial which \( s_{m-1}(t) \) approaches as \( q \) tends to zero and \( m \) tends to infinity. From (3.1) and (3.4)

\[
\lim s(+1) = z_k = \frac{1}{1 - 2((k - 1)/k)^k}.
\]

From (3.3), \( \lim s(-1) = (-1)^{k-1}z_k \). Then standard combinatorial formulas give (see Gould \[4, p. 59\])

\[
p(t) = -(-1)^j \sum_{0}^{l} \frac{(-4)^j}{T + j} \left( \frac{T + j}{2j} \right) + \frac{2T + kz_k}{T + l} \left( \frac{T + l}{2l} \right),
\]

if \( k \) is even with \( l = k/2 \) and \( T = lt \), and

\[
p(t) = -(-1)^j \sum_{0}^{l} \frac{(-4)^j T}{2j + 1} \left( \frac{T + j - 1/2}{2j} \right) + \frac{2T + kz_k}{k} \left( \frac{T + l - 1/2}{2l} \right),
\]

if \( k \) is odd with \( l = (k - 1)/2 \) and \( T = kt/2 \). The maximum of \( p(t) \) on \((k - 2)/k < t < +1\) is a good lower bound on \( \|P\| \) as \( q \) tends to zero and \( m \) tends to infinity.

The following table was computed via double-precision arithmetic in FORTRAN on an Amdahl 470/V7 computer. All entries are rounded down.

<table>
<thead>
<tr>
<th>(k)</th>
<th>(\max p(t))</th>
<th>(k)</th>
<th>(\max p(t))</th>
<th>(k)</th>
<th>(\max p(t))</th>
<th>(k)</th>
<th>(\max p(t))</th>
</tr>
</thead>
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<td>7</td>
<td>7.7939</td>
<td>12</td>
<td>9.02 \times 10^2</td>
<td>27</td>
<td>9.45 \times 10^5</td>
</tr>
<tr>
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<td>2.5078</td>
<td>8</td>
<td>11.8194</td>
<td>15</td>
<td>5.13 \times 10^3</td>
<td>30</td>
<td>6.60 \times 10^6</td>
</tr>
<tr>
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<td>3.0814</td>
<td>9</td>
<td>18.7344</td>
<td>18</td>
<td>3.17 \times 10^4</td>
<td>33</td>
<td>4.67 \times 10^7</td>
</tr>
<tr>
<td>5</td>
<td>3.9686</td>
<td>10</td>
<td>30.7986</td>
<td>21</td>
<td>2.05 \times 10^5</td>
<td>36</td>
<td>3.34 \times 10^8</td>
</tr>
<tr>
<td>6</td>
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<td>11</td>
<td>52.1254</td>
<td>24</td>
<td>1.37 \times 10^6</td>
<td>39</td>
<td>2.42 \times 10^9</td>
</tr>
</tbody>
</table>

This table, in which the exponential growth is clear, is associated with Theorem 2 above. A corresponding table of lower bounds on \( \lim \sup \|P\| \) for the node assignment of Theorem 3 can be computed from the fact that the Lebesgue function for polynomial interpolation on the Chebyshev nodes attains its maximum
at $t = 1$; see [7, Eq. (4.2.19)]. The first few entries of such a table are:

$$(1,1.414) \quad (2,1.666) \quad (3,1.847) \quad (4,1.988) \quad (5,2.104) \quad (6,2.202).$$

A later entry is (35,3.243). The logarithmic growth is clear. For $k = 1$ with arbitrary knots it can be shown that $||P|| < \sqrt{2}$ when nodes are specified by (3.2) above. Whether the other bounds are “good” bounds for the arbitrary knot case is problematical.

4. Remarks. For one-sided $q$-splines with spline knots $t_i = (1 - q^i)/(1 - q)$, $i = \ldots, -1, 0, 1, 2, \ldots$, and interpolation nodes $\tau_i = t_i + \theta q^i$, where $\theta$ is fixed, $0 < \theta < 1$, S. L. Lee [5] has considered eigensplines, i.e., nontrivial splines $s(t)$ satisfying $s(t) = \lambda s(1 + qt)$ for some fixed eigenvalue $\lambda$. Setting $\lambda = -1$ yields, for each $k > 2$, a certain equation $F_k(q, \theta) = 0$. If $q$ and either $\theta_1$ or $\theta_2$ defined above satisfy this equation, then two-sided $q$-spline interpolation is unbounded. Lee [5] has shown that $F_k(0 + , \theta) = C[2\theta^k - 1][2(1 - \theta)^k - 1]$.

For quadratic splines with arbitrary knots $t_i$, Demko [3] has shown that interpolation is bounded independently of $t_i$ and $\tau_i$ if the nodes $\tau_i$ satisfy $\tau_i = t_{i+2} - \lambda_2(t_{i+2} - t_{i+1})$ with $\lambda_2^2 < \gamma < \frac{1}{2}$ and $(1 - \lambda_2)^2 < \gamma < \frac{1}{2}$. Consequently, for $k = 2$, the results of Theorem 1 above with (2.5)$a$ and (2.6)$a$ are valid for all $q$ and not just as $q$ tends to zero.

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5. S. L. Lee, private communication.