Spline Interpolation at Knot Averages
on a Two-Sided Geometric Mesh*

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Abstract. For splines of degree \( k > 1 \) with knots \( -t_i = t_{2m+1-i} = 1 + q + q^2 + \cdots + q^{m-i}, \ i = 1, \ldots, m, \) where \( 0 < q < 1, \) it is shown that spline interpolation to continuous functions at nodes \( \tau_i = \sum_{l=1}^{k} w_{l+i}, \ i = 1, \ldots, n = 2m - k - 1, \) has operator norm \( \| P \| \) which is bounded independently of \( q \) and \( m \) as \( q \) tends to zero if and only if

\[
(1 - w_1) < \frac{1}{2}, \quad (1 - w_k) < \frac{1}{2}, \quad \text{and} \quad w_j > 0, \ j = 1, \ldots, k.
\]

The choice of nodes \( \tau_i = \sum_{l=1}^{k} w_{l+i} \) and the growth rate of \( \| P \| \) with \( k \) are also discussed.

1. Two-Sided \( q \)-Splines. To integers \( n > 0, k > 0, \) and a nondecreasing sequence \( t = (t_i)_{i=1}^{n+k+1} \) with \( t_i < t_{i+k+1}, \ i = 1, \ldots, n, \) is associated \( S_{k+1,t} \), the space of polynomial splines of order \( k + 1 \) with knot sequence \( t, \) defined by \( S_{k+1,t} = \text{span}\{N_1, \ldots, N_n\}, \) where each \( N_i = N_{i,k+1} \) is an appropriate normalized \( B \)-spline. See [1] for specific details.

With \( q > 0, m \) a positive integer, \( n = 2m - k - 1, \) and

\[
t_i = - (1 + q + \cdots + q^{m-i}), \quad i = 1, \ldots, m,
\]

\[
t_i = 1 + q + \cdots + q^{i-1}, \quad i = m + 1, \ldots, 2m,
\]

\( S_{k+1,t} \) is the space of two-sided \( q \)-splines.

Each two-sided \( q \)-spline can be represented as

\[
s(t) = \sum_{i=1}^{m-1} A_j[q^{-m}(t_{j+1} - t)_+]^k + \sum_{l=0}^{k} A_{m+l} t^l
\]

\[
+ \sum_{l=1}^{m-1} A_{m+k+l} [q^{-l}(t - t_{m+l})_+]^k,
\]

where \( u_+ = \max\{u, 0\}, \) with the endpoint conditions

\[
s^{(i)}(t_1) = s^{(i)}(t_{2m}) = 0, \quad i = 0, \ldots, k - 1.
\]

Conversely, each function of the form (1.2) which satisfies (1.3) is a two-sided \( q \)-spline.

With the notation

\[
[i] = 1 + q + \cdots + q^{i-1}, \quad i = 0, 1, \ldots,
\]

relations such as

\[
t_{j+1} - t_i = q^{m-j}[j + 1 - i], \quad 0 < i < j < m,
\]

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and
\[ t_{i+1} - t_j = q^{-m} \left[ i + 1 - j \right], \quad m < j < i < 2m, \]
can be stated in a compact form. The notation
\[ [i]! = [i][i - 1] \cdots [2][1] \quad \text{and} \quad \left[ \frac{j}{i} \right] = \frac{[j]!}{[i]![j - i]!} \]
will also be useful.

The clause “as \( q \) tends to zero” appears throughout this paper. It will always mean “for all \( q \) satisfying \( 0 < q < q_0 \)”. The specific choice of \( q_0 \) will vary from instance to instance. However, \( q_0 \) will never depend on \( m \).

**Lemma 1.1.** With \( k \) and \( m \) fixed, let \( \{s\} \) be a set of two-sided \( q \)-splines with \( \{(A_1, \ldots, A_{2m+k-1})\} \) the corresponding set of coefficient vectors in (1.2). Then \( \{s\} \) is uniformly bounded as \( q \) tends to zero if and only if \( \{(A_j)\} \) is uniformly bounded as \( q \) tends to zero. Moreover, if the bound on \( \{s\} \) is independent of \( m \), then so is the bound on \( \{(A_j)\} \).

**Proof.** Let \( 1 > q_0 > 0 \) and \( C \) be such that
\[ |A_j| \leq C, \quad \text{all } j \text{ and } 0 < q < q_0. \]
Then, for each real \( t \) and \( 0 < q < q_0 \),
\[
|s(t)| \leq C \left( \sum_{j=0}^{m-1} q^{-m} (t_{j+1} - t_j)^k + \sum_{j=0}^{k} t_{2m}^j + \sum_{j=0}^{m-1} [q^{-j}(t_{2m} - t_{m+j})]^k \right)
\]
\[
= C \left( \sum_{j=0}^{m-1} [j]^k + \sum_{j=0}^{k} [m]^j + \sum_{j=0}^{m-1} [m - j]^k \right) \leq (2m + k - 1)C[m]^k.
\]
Conversely, let \( 1 > q_0 > 0 \) and \( B \) be such that
\[ |s(t)| < B, \quad \text{all real } t \text{ and } 0 < q < q_0. \]

Since
\[ \sum_{j=0}^{k} A_{m+j}(i/k)^j = s(i/k), \quad i = 0, \ldots, k, \]
is a matrix equation with nonsingular coefficient matrix \( V = ((i/k)^j) \) depending only on \( k \),
\[ |A_{m+j}| \leq (k + 1)B_k B, \quad j = 0, \ldots, k, \]
where \( B_k \) is a bound on the entries of \( V^{-1} \). Set \( C_0 = (k + 1)B_k B \) and assume inductively that \( q_i \) is such that \( |A_{m-j}| \leq C_j \) for \( j = 0, 1, \ldots, i - 1 \) for \( q < q_i \).

From (1.2)
\[
s(t_{m-i}) - s(t_{m-i+1}) = A_{m-i} + \sum_{j=0}^{i-1} A_{m-j}([i - j + 1]^k - [i - j]^k) + \sum_{j=0}^{k} A_{m+j}(-1)^j([i + 1]^j - [i]^j),
\]
so that

\[ |A_{m-i}| \leq 2B + \sum_{i=1}^{i-1} C_j \left( [i-j+1]^k - [i-j]^k \right) + C_0 \sum_{i=1}^k \left( [i+1]' - [i]' \right) \]

\[ \leq 2B + \sum_{i=1}^{i-1} C_j q^{i-j/k} (1 - q_0)^{-k} + C_0 \sum_{i=1}^k q^j (1 - q_0)^{-j} \]

\[ \leq 2B + \sum_{j=0}^{i-1} C_j q^{i-j/k} R_k \quad \text{with } R_k = k^2 (1 - q_0)^{-k}. \]

Setting \( C_i = 2B + \sum_{j=0}^{i-1} C_j q^{i-j/k} R_k \) allows the induction to proceed. Then \( C_1 = 2B + C_0 q_1 R_k \), and \( C_{i+1} = q_1 (1 + R_k) C_i + 2B (1 - q_1), \) \( i = 1, \ldots, m - 2 \). This recurrence solves as

\[ C_i = \frac{2B (1 - q_1)}{1 - q_1 - q_1 R_k} \left[ 1 - (q_1 + q_1 R_k)^{i-1} \right] + C_1 (q_1 + q_1 R_k)^{i-1}, \]

\( i = 1, \ldots, m - 1, \)

if \( q_1 + q_1 R_k \neq 1 \). Imposing the added restriction \( q_1 + q_1 R_k < \frac{1}{2} \) and noting that a symmetric argument will yield \( |A_{m+k+j}| \leq C_j, j = 1, \ldots, m - 1 \), establishes that

\[ \max_j |A_j| \leq \max_i C_i \leq 4B + C_1 + C_0. \]

This bound is independent of \( m \) if \( B \) is independent of \( m \). \( \square \)

**Lemma 1.2.** Let \( k \) and \( m \) be fixed. As \( q \) tends to zero, the coefficients \( (A_{j}) \) satisfy

\[ A_i + \sum_{i+1}^{m-1} A_j q^{j-i/k} \left[ \frac{j}{i} \right] + \sum_{k-i}^k A_{m+j} O(q^{(m-i)(k-i)}) = 0, \quad i = 1, \ldots, k - 1, \]

and

\[ A_k + \sum_{k+1}^{m-1} A_j \left[ \frac{j}{k} \right] + \sum_{k+1}^k A_{m+j} \left( \frac{j!}{k!} + O(q^{m-k+1}) \right) = 0. \]

**Proof.** This follows from (1.3). Let functionals \( \Lambda_{i,v}, 1 < i < v < k, \) be defined by

\[ \Lambda_{i,v}s = q^{(m-i)(k-v)/k} \frac{y!}{k!} (-1)^{k-v} s^{(k-v)}(t_i) \]

and, recursively,

\[ \Lambda_{i,v}s = q^{v-k} (\Lambda_{i-1,v}s - [i - 1] \Lambda_{i-1,v-1}s) / [i]. \]

From (1.2)

\[ s^{(k-v)}(t_i) = \sum_{1}^{m-1} A_j q^{(j-m)(k-v)/v!} (-1)^{k-v} [j]^v \]

\[ + \sum_{k-v}^k A_{m+j} \frac{j!}{(j-k+v)!} \frac{j!}{i!} s^{(k+1)}(t_i), \]

whence

\[ \Lambda_{i,v}s = \sum_{1}^{m-1} A_j q^{(j-1)(k-v)} [j]^v + \sum_{k-v}^k A_{m+j} q^{(m-1)(k-v)} C_{1,v}, \]
where
\[ C_{ij\nu} = \frac{\nu^{j!}}{j! (j - k + \nu)!} (-1)^{k-\nu} t_i^{j-k+\nu}. \]

The recursion formula gives
\[ \Lambda_{ij\nu} s = \sum_{i} A_j q^{(j-k)(k-\nu)} [j-1] + \sum_{i} A_{m+j} q^{(m-j)(k-\nu)} C_{ij\nu}, \]

where
\[ C_{ij\nu} = (C_{i-1,j\nu} - [j-1] q^{m-j+1} C_{i-1,j\nu-1})/[i]. \]

From (1.3) each \( \Lambda_{ij\nu} s = 0 \) and, in particular, \( \Lambda_{ii\nu} s = 0 \). This fact, along with the observation that \( C_{ijk} = C_{ij\nu}/[k]! + O(q^{m-k+1}) \) completes the proof. □

Combining Lemmas 1.1, 1.2, and a symmetric counterpart of Lemma 1.2 yields

**Lemma 1.3.** Let \( k \) and \( m \) be fixed and let \( \{s\} \) be a set of two-sided \( q \)-splines which is bounded as \( q \) tends to zero. Then the corresponding set of coefficient vectors \( \{(A_i)\} \) satisfies
\[ A_i = O(q^{k-i}), \quad i = 1, \ldots, k-1, \]
\[ A_i = O(1), \quad i = k, \ldots, 2m, \]
\[ A_{2m+i} = O(q^i), \quad i = 1, \ldots, k-1, \]

as \( q \) tends to zero. If the bound on \( \{s\} \) is independent of \( m \), then so are the bounds on the \( A_i \).

The independence of \( m \) in the \( O(q^{k-i}) \) and \( O(q^i) \) bounds follows from the exponential decay of the coefficients in the first \( k-1 \) equations of Lemma 1.2.

**2. Spline Interpolation.** Let \( \tau = (\tau_j)^n_i \) be a strictly increasing sequence. It is known [1] that: For each function \( f \) defined on \( \tau \) there is exactly one \( s \in \mathbb{S}_{k+1}\) such that \( s(\tau_j) = f(\tau_j), i = 1, \ldots, n \), if and only if \( N_i(\tau_j) > 0, i = 1, \ldots, n \), or, equivalently, if and only if
\[ \tau_i < \tau_j < \tau_{i+k+1}, \quad i = 1, \ldots, n. \]

When \( \tau \) satisfies (2.1) a linear map \( P \) into \( \mathbb{S}_{k+1}\) which reproduces \( \mathbb{S}_{k+1}\) may be defined by: For each function \( f \) defined on \( \tau \), \( Pf \in \mathbb{S}_{k+1}\) and \( (Pf)(\tau_j) = f(\tau_j), i = 1, \ldots, n \). In fact, \( Pf = \sum f(\tau_j) L_j \) where \( (L_j)^\tau_i \) is defined by \( L_j(\tau_i) = \delta_{ij}, i, j = 1, \ldots, n \). The operator norm of \( P \) is
\[ \|P\| = \sup_{f} \frac{\|Pf\|}{\|f\|}, \]
where the sup is taken over all \( f \in C[\tau_1, \tau_{n+k+1}] \) and
\[ \|f\| = \sup \{|f(t)|: t_1 < t < t_{n+k+1}\}. \]

It is well known that
\[ \|P\| = \max \sum_{1}^{n} |L_j(t)| = \max \left( \max_{0 < \mu < n} \tau_{\mu} \leq t \leq \tau_{\mu+1} \right) s_{\mu}(t), \]
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where \( \tau_0 = t_1, \tau_{n+1} = t_{n+k+1} \) and \((s_\mu)_0^n\) is defined by

\[
\begin{align*}
  s_\mu(t_i) &= (-1)^{i+\mu}, \quad i = 1, \ldots, \mu, \\
  &= (-1)^{i+\mu}, \quad i = \mu + 1, \ldots, n.
\end{align*}
\]

For each \( \mu \), the so-called Lebesgue function \( \Sigma |L_\mu(t)| \) coincides with \( s_\mu(t) \) on the interval \( [\tau_{\mu}, \tau_{\mu+1}] \).

One way of specifying \( \tau \) is to require that the nodes be knot averages, i.e.,

\[
\tau_i = \sum_{j=0}^{k+1} w_j t_{i+j}, \quad i = 1, \ldots, n,
\]

where the \( w_j \) are fixed nonnegative numbers which sum to one.

**Theorem 1.** Let \( k \geq 2, m, \text{ and } (w_j)_{j=0}^{k+1} \) be fixed. Let \( t \) be given by (1.1) and \( \tau \) be given by (2.3). If \( \|P\| \) is bounded as \( q \) tends to zero, then

\[
w_i > 0, \quad i = 1, \ldots, k.
\]

If the bound on \( \|P\| \) is also independent of \( m \), then either

\[
w_0 = 0 \quad \text{and} \quad (1 - w_1)^k < \frac{1}{2}
\]

or

\[
w_0 > 0 \quad \text{and} \quad \frac{1}{2} < (1 - w_0)^k
\]

and, either

\[
w_{k+1} = 0 \quad \text{and} \quad (1 - w_k)^k < \frac{1}{2}
\]

or

\[
w_{k+1} > 0 \quad \text{and} \quad \frac{1}{2} < (1 - w_{k+1})^k.
\]

Conversely, if (2.4), (2.5), (2.6) hold, then \( \|P\| \) is bounded independently of \( m \) as \( q \) tends to zero.

**Proof.** Let \( w_a \) be the first positive weight and \( w_b \) be the last positive weight, so that \( \tau_i = \sum_{j=0}^b w_j t_{i+j} \), and set

\[
\begin{align*}
  \theta_1 &= (1 - w_a) + (1 - w_a - w_{a+1})q + \cdots + w_{b}q^{b-a-1}, \\
  \theta_2 &= (1 - w_b) + (1 - w_b - w_{b-1})q + \cdots + w_{a}q^{b-a-1}.
\end{align*}
\]

If \( a = b \), then \( \theta_1 = \theta_2 = 0 \). If \( a < b \), then \( 0 < \theta_1 < 1 \) and \( 0 < \theta_2 < 1 \) as \( q \) tends to zero. Therefore,

\[
\begin{align*}
  t_{i+b-1} - \theta_2 q^{m+1-b-i} < t_{i+b}, \quad i = 1, \ldots, m - b, \\
  t_{i+a} - \theta_1 q^{l+1-a-i} < t_{i+a+1}, \quad i = m - a + 1, \ldots, n,
\end{align*}
\]

for all sufficiently small \( q > 0 \). Since

\[
\tau_i = 1 - 2 \sum_{j=a}^{m-i} w_j + O(q), \quad i = m - b + 1, \ldots, m - a,
\]

as \( q \) tends to zero, it follows that also

\[
-1 < \tau_{m-b+1} < \tau_{m-b+2} < \cdots < \tau_{m-a} < +1
\]

for all sufficiently small \( q > 0 \).
Henceforth, we require that $q$ be such that the inequalities in (2.7) and (2.9) hold. This requirement is independent of $m$.

Now let $\|P\|$ be bounded independently of $m$ as $q$ tends to zero. We shall prove that (2.4) and (2.6) must hold. A symmetric argument, which we omit, will give (2.5).

Let $s = s_\mu$ be defined by (2.2) with $\mu < m - b + 1$ or $\mu > m - a - 1$. There is a constant $C$ which bounds $\|P\|$ so that $\|s\| \leq C$ as $q$ tends to zero. Since the restriction of $s$ to $[-1, +1]$ is a polynomial of degree $k$, it follows from a theorem of A. A. Markov (see [7]) that

$$\max\{|s'(t)|: -1 < t < 1\} \leq Ck^2.$$  

Thus, (2.8), (2.9), and the mean-value theorem imply that

$$2 = |s(\tau_i) - s(\tau_{i+1})| \leq Ck^2(\tau_{i+1} - \tau_i) = 2Ck^2w_m - i + O(q)$$

for $i = m - b + 1, \ldots, m - a - 1$ as $q$ tends to zero. Thus, $w_i > 1/Ck^2 > 0$, $i = a + 1, \ldots, b - 1$.

Suppose that $b < k$. Then, on the one hand, (1.2) gives

$$\pm 1 = s(\tau_i) = \sum_{b}^{m-1} A_j(\lfloor j - b \rfloor + \theta_2q^{j-b})^k + \sum_{0}^{k} A_{m+j}(\lfloor -m - b \rfloor - \theta_2q^{m-b})^j$$

whereas, on the other hand, with $\Lambda_{\mu}s$ as in the proof of Lemma 1.2,

$$0 = \theta_2^k \Lambda_{BB} s + \sum_{b+1}^{k-1} [i - b]^k \Lambda_{\mu}s + [k]! \Lambda_{kk} s$$

$$= A_b \theta_2^k + \sum_{b+1}^{m-1} A_j(\lfloor j - b \rfloor)^k + O(q^{i-b})$$

$$+ \sum_{0}^{k} A_{m+j}(\lfloor -m - b \rfloor)^j + O(q^{m-b}).$$

Subtraction yields

$$\pm 1 = \sum_{b+1}^{m-1} A_j O(q^{i-b}) + \sum_{0}^{k} A_{m+j} O(q^{m-b}),$$

so that $(A_j)$ cannot be bounded as $q$ tends to zero. This contradiction to Lemma 1.3 shows that $b > k$.

A similar argument with $s(\tau_n)$ shows that $a < 1$, so that (2.4) is proved.

To prove (2.6), we first suppose that $w_{k+1} = 0$. We must show that $(1 - w_k)^k < \frac{1}{2}$ or, equivalently, that

$$r_2 = \theta_2^k / (1 - \theta_2^k) < 1 \quad \text{as } q \text{ tends to zero.}$$

Again, let $s = s_\mu$ be defined by (2.2). Then Lemma 1.2 and (1.2) give

$$-s(\tau_1) = [k]! \Lambda_{kk} s - s(\tau_1) = \sum_{0}^{m-1} M_0 A_{k-j} + \sum_{0}^{k} R_0 A_{m+j}$$
and

\begin{equation}
(2.12) \quad s(\tau_i) - s(\tau_{i+1}) = \sum_{i=1}^{m-k-1} M_{ij} A_{k+j} + \sum_{j=0}^{k} R_{ij} A_{m+j}, \quad i = 1, \ldots, m - k - 1,
\end{equation}

where

\begin{align*}
M_{0j} &= \frac{[k + j]!/[j]! - ([j] + \theta_2 q^j)^k}{j = 0, \ldots, m - k - 1}, \\
M_{ij-1} &= \theta_2^k, \quad i = 1, \ldots, m - k - 1, \\
M_{ij} &= ([j - i + 1] + \theta_2 q^{j-i})^k - ([j - i] + \theta_2 q^{j-i})^k, \\
R_{ij} &= [k]! C_{ji} - \tau_i, \quad j = 0, \ldots, m - k - 1; j = i, \ldots, m - k - 1,
\end{align*}

with $C_{ij} = t_i^k/[k]! + O(q^{m-k+1})$ as in the proof of Lemma 1.2.

Since the $A_j$ are bounded and

\begin{align*}
M_{ii} &= 1 - \theta_2^k + O(q), \quad i = 0, \ldots, m - k - 1, \\
M_{ij} < \frac{[k + j]!/[j]!}{q^j [k] [k + j]^{k-1}} < q^j k(1 - q)^{-k}, \\
M_{ij} < \frac{[j - i + 2]^k - [j - i]^k}{q^{j-i} k(1 - q)^{-k}},
\end{align*}

the system (2.11) and (2.12) has the form

\begin{align*}
(1 - \theta_2^k) A_k &= -s(\tau_i) + O(q), \\
\theta_2^k A_{k+i-1} + (1 - \theta_2^k) A_{k+i} &= s(\tau_i) - s(\tau_{i+1}) + O(q), \quad i = 1, \ldots, m - k - 1,
\end{align*}

which solves as

\begin{align}
A_{k+i} &= \frac{2(-1)^{i+\mu}}{1 - 2\theta_2^k} \left[ 1 - \frac{r_{2i+1}}{2\theta_2^k} \right] + O(q), \quad i = 0, \ldots, \min(\mu - 1, m - k - 1), \\
A_{k+\mu+i} &= \frac{2(-1)^{i+\mu+1}}{1 - 2\theta_2^k} \left[ 1 - \frac{r_{2i+1}^\mu}{\theta_2^k} + \frac{r_{2i+1}^{i+\mu+1}}{2\theta_2^k} \right] + O(q), \quad i = 0, \ldots, m - k - 1 - \mu,
\end{align}

if $r_2 \neq 1$ and as

\begin{align*}
A_{k+i} &= (-1)^{i+1}(2 + 4i) + O(q), \quad i = 0, \ldots, \min(\mu - 1, m - k - 1), \\
A_{k+\mu+i} &= (-1)^{i+1}(2 + 4i - 4\mu) + O(q), \quad i = 0, \ldots, m - k - 1 - \mu,
\end{align*}

if $r_2 = 1$ provided that the buildup of $O(q)$ terms is bounded independently of $m$. This will be the case if $qr_2 < 1$ as $q$ tends to zero, a condition that can be met independently of $m$. By Lemma 1.3 these $A_j$ are bounded independently of $m$. Therefore, (2.10) must hold and $(1 - \theta_k)^k < \frac{1}{2}$. 

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To complete the proof of (2.6) we now suppose that \( w_{k+1} > 0 \). Since \( \theta_2 = 1 - w_{k+1} + O(q) \), we must now show that \( \theta_2^k > \frac{1}{2} \) as \( q \) tends to zero, that is

\[
(2.14) \quad r_2 = \frac{\theta_2^k}{(1 - \theta_2^k)} > 1 \quad \text{as} \quad q \text{ tends to zero.}
\]

Since the \( \tau_i \) have "moved over one interval", Eqs. (2.11) and (2.12) are replaced by

\[
-s(\tau_i) = [k]! A_k + \sum_{\ell=1}^{m-k-1} \left( [k+j]/[j]! - ([j-1] + \theta_2q^{r-1})^k \right) A_{k+j}
\]

\[
+ \sum_{0}^{k} R_{0j} A_{m+j}
\]

and

\[
s(\tau_i) - s(\tau_{i+1}) = \sum_{i}^{m-k-1} M_{i,j-1} A_{k+j} + \sum_{0}^{k} R_{0j} A_{m+j},
\]

(2.16)

and the bounds on \( M_{0j} \) and \( R_{0j} \) are replaced by

\[
| [k+j]/[j]! - ([j-1] + \theta_2q^{r-1})^k | < q^{r-1}k(1 - q)^k,
\]

\[
|R_{0j}| < q^{m-k-1}(1+1)(1-q)^j,
\]

\[
|R_{0j}| < q^{m-k-1-i}(1-q)^{i}.
\]

This incomplete system now has the form

\[
A_k + (1 - \theta_2^k) A_{k+1} + s(\tau_1) + O(q),
\]

\[
\theta_2^k A_{k+i} + (1 - \theta_2^k) A_{k+i+1} = s(\tau_i) - s(\tau_{i+1}) + O(q), \quad i = 1, \ldots, m - k - 2.
\]

Adding the equation

\[
(2.17) \quad s(\tau_{m-k-1}) = A_{m-k} \theta_2^k + \sum_{0}^{k} A_{m+j} \tau_{m-k-1} = A_{m-k} \theta_2^k + s(-1) + O(q)
\]

and imposing the restriction \( qr_2^{-1} < 1 \) permits us to solve this system backwards in terms of \( s(-1) \) as

\[
A_{m-i} = \frac{(-1)^{m-i-1-k-\mu}}{2\theta_2^{i}} \left[ 2 - (1 + r_2) r_2^{-i} \right] + (1 + r_2) (-r_2)^{-i} s(-1)
\]

\[
+ O(q), \quad i = 1, \ldots, m - k - 1 - \mu,
\]

(2.18)

\[
A_{k+\mu+1-i} = \frac{(-1)^{\mu-i}}{2\theta_2^{\mu+1-i}} \left[ 2 - (1 + r_2) r_2^{-i} (2 - r_2^{-m+k+\mu+1}) \right]
\]

\[
+ (1 + r_2) (-r_2)^{-m+k+\mu+1-i} s(-1) + O(q), \quad i = 1, \ldots, \mu,
\]

if \( 0 < \mu < m - k - 2 \) and as

\[
A_{m-i} = \frac{(-1)^{\mu-m+k-i}}{2\theta_2^{i}} \left[ 2 - (1 + r_2) r_2^{-i} \right]
\]

\[
+ (1 + r_2) (-r_2)^{-i} s(-1) + O(q), \quad i = 1, \ldots, m - k - 1,
\]

if \( \mu \geq m - k - 1 \). Since the \( A_j \) are bounded independently of \( m \), (2.14) must hold and \( (1 - w_{k+1})^k > \frac{1}{2} \).
The proof that (2.4), (2.5), (2.6) are necessary conditions for \( \| P \| \) to be bounded independently of \( m \) as \( q \) tends to zero is complete.

To prove that (2.4), (2.5), (2.6) are sufficient that \( \| P \| \) be bounded independently of \( m \) as \( q \) tends to zero, we will use the approach outlined in the proof of Lemma 1.1. That is, we will first show that, for each \( s = s_\mu \), the block \( A_m, \ldots, A_{m+k} \) is bounded and then argue recursively from bounds on \( s(t_i) \) (replacing \( s(t_i) \) in the proof of Lemma 1.1) that \( A_{m+i} \) (and \( A_{m+k+i} \), \( i = 1, \ldots, m-1 \), are bounded independently of \( m \). Finally, we will use (1.2) and (2.13) or (2.18) or (2.19) to bound \( s_\mu(t) \) for all \( t \) and all \( \mu \).

If \( a = 0 \) and \( b = k + 1 \), the first step, bounding the block \( A_m, \ldots, A_{m+k} \) is easy since (2.9) implies that

\[
(2.20) \quad \sum_{i=k+1-b}^{k} A_{m+j} r_{m-k+i} = \pm 1, \quad i = k + 1 - b, \ldots, k - a,
\]

and (2.4), (2.8) give a bounded inverse for the Vandermonde matrix \( (r_{m-k+i}) \).

However, if \( b = k \) then the \( i = 0 \) equation of (2.20) is replaced by

\[
(2.21) \quad \theta_k^r A_{m-1} + \sum_{i=0}^{k} A_{m+j} r_{m-k} = \pm 1.
\]

If \( a = 1 \), there is a similar replacement of

\[
(2.22) \quad \sum_{i=0}^{k} A_{m+j} r_{m} + \theta_1^k A_{m+k+1} = \pm 1
\]

for the \( i = k \) equation of (2.20).

Therefore, if \( b = k \) (and/or \( a = 1 \)), a preliminary step to eliminate \( \theta_k^r A_{m-1} \) from (2.21), at the expense of adding a bounded quantity to the right member, is necessary. While eliminating \( \theta_k^r A_{m-1} \) through a sequence of upper triangulation steps on (2.11), (2.12), (2.21) is straightforward, there must be an argument that \( \theta_k^r A_{m-1} \) is bounded independently of \( m \) as \( q \) tends to zero independently of \( m \). The following lines supply this argument.

Let \( b = k \) and let \( s \) be any of the \( s_\mu \) given by (2.2). Using the bounds on \( M = (M_{ij}) \), we see that this matrix is diagonally dominant if \( q \) is such that

\[ 1 - \theta_2^k > \theta_2^k + kq(1 - q)^{-k-1}. \]

But (2.5a) is equivalent to \( 1 - \theta_2^k > \theta_2^k \) for sufficiently small \( q \), so that this condition can be met by imposing a further restriction on \( q \).

Let \( q_0 > 0 \) and \( \delta > 0 \) be such that \( \delta = 1 - 2\theta_2^k - kq_0(1 - q_0)^{-k-1} \). Then the solutions of a system

\[ Mx = b \]

satisfy \( \max|x_i| < \delta^{-1}\max|b_i| \) by the usual diagonal dominance argument. Applying this fact with

\[
\begin{align*}
b_0 &= \lfloor k \rfloor! \Lambda_{kk} s - s(t_1) = -s(t_1), \\
b_i &= s(t_i) - s(t_{i+1}), & i = 1, \ldots, m - k - 1,
\end{align*}
\]

as well as with

\[
b_i = -R_{ij}, & i = 0, \ldots, m - k - 1,
\]
for each $j = 0, \ldots, k$, yields

$$A_{m-1} = C + \sum_{0}^{k} C_j A_{m+j}$$

with

$$|C| < \delta^{-1} \max\{ |s(\tau_1)|, |s(\tau_2) - s(\tau_{i+1})| : i = 1, \ldots, m - k - 1 \} = 2/\delta,$$

$$|C_0| < \delta^{-1} \max_{i} |R_0| = |R_{00}| / \delta = O(q^{m-k}),$$

$$|C_j| < \delta^{-1} \max_{i} |R_j| = |\tau_{m-k-1}^j - \tau_{m-k}^j| / \delta$$

$$= \left| (1 + q + \theta_2 q^2)^j - (1 + \theta_2 q)^j \right| / \delta < q^j [2] / \delta = O(q), \quad j = 1, \ldots, k.$$

Combining these deductions with (2.21) gives the equation

$$(2.23) \quad \sum_{0}^{k} A_{m+j}(\tau_{m-k}^j + C_j \theta_2^j) = s(\tau_{m-k}) - C \theta_2^k,$$

which can be adjoined to (2.20). Since $C_j = O(q)$ and $\tau_{m-k+1} - \tau_{m-k} = 2w_k + O(q)$, the resulting system has a bounded solution as $q$ tends to zero. We have assumed that $a = 0$. If $a = 1$, a similar argument at $\tau_m$ is needed.

We have completed the first step in the proof of sufficiency, i.e., we have shown that the set $A_m, \ldots, A_{m+k}$ is bounded. But now (2.12) or (2.16) and their symmetric counterparts imply immediately that the set $A_k, \ldots, A_{2m}$ is bounded. An argument similar to the proof of Lemma 1.2 gives $O(q')$ bounds on $A_{k-i}$ and $A_{2m+i}, i = 1, \ldots, k - 1$. The second step in the proof is completed.

Now we must bound $s_\mu(t)$ for all $t$ and all $\mu$. For $-1 < t < +1$, the boundedness of $A_m, \ldots, A_{m+k}$ and (2.4) give a uniform bound on $s_\mu(t)$. If $t_1 < t < t_m$, there is a $\theta_i$ in $[0, 1]$ and an $i > 0$ such that $t_{m-i} < t = t_{m-i+1} - \theta_i q^j = \theta_i q^j < t_{m-i+1}$. Then

$$s(t) = \sum_{m-i}^{m-1} A_j [i + j - m] + \theta_i q^j - m] + \sum_{0}^{k} A_{m+j}.$$ 

If $i < m - b$, then $\tau_{m+1-b-i} = [i] - \theta_2 q^j$ and

$$|s(t)| < |s(\tau_{m+1-b-i})| + |A_{m-i}| + O(q) = 1 + |A_{m-i}| + O(q)$$

can be easily shown. If $i > m - b$, a modified argument gives

$$|s(t)| < |s(\tau_i)| + \sum_{1}^{k} |A_i| + O(q) = 1 + |A_k| + O(q).$$

Thus, the $s_\mu(t)$ are uniformly bounded for all $\mu$ and all $t$ so that $\|P\|$ is bounded independently of $m$ as $q$ tends to zero. \(\square\)

3. Two Special Cases. Theorem 1 provides counterexamples when (2.4), (2.5), (2.6) are not satisfied, e.g., interpolation at the knots with $k > 2$ or interpolation at weighted two-knot averages with $k > 3$. The condition that $q$ tend to zero compares (contrasts?) with the often-used condition that the local mesh ratios $(t_{i+1} - t_i)/(t_{i+1} - t_j), |i - j| = 1$ be bounded.
For $k > 3$ and $q = 1$, it is easy to select weights $w_j$ satisfying (2.4), (2.5), (2.6) which still produce unbounded spline interpolation. Thus, even for two-sided $q$-splines, these conditions are not sufficient to guarantee bounded interpolation. Indeed, the method of their derivation suggests that they are linked quite closely to the tendency of $q$ to zero.

For the two special cases which follow it is not clear that we need $q$ to tend to zero. Computational evidence with small $k$ suggests, in fact, that $q$ tending to zero gives "worst-case" results. Thus, Theorems 2 and 3 are imperfect in that the condition that $q$ tend to zero may be superfluous.

**Theorem 2.** Let $t$ be given by (1.1) and, for each $k > 1$ and $m > k$, let $\tau$ be given by

$$
\tau_i = (t_{i+1} + t_{i+2} + \cdots + t_{i+k})/k, \quad i = 1, \ldots, n.
$$

Then, $\|P\|$ is bounded as $q$ tends to zero. Moreover, there exist absolute constants $1 < C_1 < C_2$ such that, for each $k > 2$,

$$
C_1^k < \|P\| < C_2^k \quad \text{as } q \text{ tends to zero.}
$$

**Theorem 3.** Let $t$ be given by (1.1) and, for each $k > 1$ and $m > k$, let $\tau$ be given by (2.3) with

$$
w_0 = w_{k+1} = \sin^2(\alpha_k/2),
$$

$$
w_j = \sin(\alpha_k)\sin(2\alpha_k), \quad j = 1, \ldots, k,
$$

where $\alpha_k = \pi/(2k + 2)$. Then, $\|P\|$ is bounded as $q$ tends to zero. Moreover, there exist absolute constants $0 < C_3 < C_4$ such that, for each $k > 2$,

$$
C_3 \log k < \|P\| < C_4 \log k \quad \text{as } q \text{ tends to zero.}
$$

Proof of Theorems 2 and 3. The assertions that $\|P\|$ is bounded as $q$ tends to zero are proved by showing that (2.4), (2.5), (2.6) hold. These follow readily, since, in Theorem 2,

$$
(1 - w_k)^k = (1 - w_1)^k = (k - 1)^k/k^k < 1/e < 3/8
$$

while, in Theorem 3,

$$
(1 - w_{k+1})^k = (1 - w_0)^k = \cos^{2k}(\alpha_k/2) > (1 - \alpha_k^2/8)^{2k}
$$

$$
> 1 - \pi \alpha_k/8 > (8k + 3)/(8k + 8) > 3/4.
$$

In Theorem 2, the lower bound on $\|P\|$ follows from the fact that, as $q$ tends to zero, the nodes

$$
\tau_{m-k+1}, \tau_{m-k+2}, \ldots, \tau_{m-k+j}, \ldots, \tau_{m-1}
$$

tend to

$$
(2 - k)/k, (4 - k)/k, \ldots, (2j - k)/k, \ldots, (k - 2)/k,
$$

and that, for $s = s_\mu$ with $m - k \leq \mu \leq m - 1$,

$$
|s(\pm 1)| = (1 - \theta_z^{m-k})/ (1 - 2\theta_z^k) + O(q) \geq 1/ (1 - \theta_z^k) + O(q) > 1,
$$

so that $\|P\|$ is bounded below by any lower bound for polynomial interpolation on $[-1, + 1]$ at the equally-spaced nodes

$$-1, (2 - k)/k, (4 - k)/k, \ldots, (2j - k)/k, \ldots, (k - 2)/k, +1.$$
See Rivlin \[7, \text{pp. 96–99}\] for a proof that such polynomial interpolation grows exponentially.

Similarly, in Theorem 3, the lower bound on $\|P\|$ follows from the fact that $\tau_{m-k}, \ldots, \tau_m$ approach the Chebyshev nodes $-\cos(2j\alpha_k - \alpha_k), j = 1, \ldots, k + 1$, as $q$ tends to zero and the fact that polynomial interpolation on these nodes has logarithmic growth. See \[7, \text{pp. 93–96}\].

To complete the proof that $\|P\|$ grows exponentially or logarithmically in Theorem 2 or Theorem 3, respectively, it is necessary only to show that, for each $\mu$, $s_\mu(t)$ is "controlled" outside $(-1, +1)$. This fact follows from the closing lines of the proof of Theorem 1, where it was noted that, for $t_1 < t < t_m$, there is a $j < m$ such that $|s(t)| < 1 + |A_j| + O(q)$. For Theorem 2, (3.1) and (2.13) imply that

$$
\max_{j < m} |A_j| < \frac{2}{1 - 2\theta_2^k} + O(q) < \frac{2e}{e - 2} + O(q) < 8,
$$

so that $|s(t)| < 10$ for $t < -1$ as $q$ tends to zero. For Theorem 3, (3.2) and (2.18), (2.19) imply that

$$
\max_{j < m} |A_j| < \frac{2}{2\theta_2^k - 1} + 2|s(-1)| + O(q)
= \frac{2}{2e^{2k} - 1} + 2|s(-1)| + O(q)
< 4 + 2|s(-1)| + O(q),
$$

so that $|s(t)| < 6 + 2|s(-1)|$ for $t < -1$ as $q$ tends to zero. Symmetry considerations give like bounds for $|s(t)|$ on $+1 = t_{m+1} < t < t_{2m}$.

The proof of Theorem 2 and Theorem 3 is complete. □

If $q = 1$ (not covered by these theorems), two-sided $q$-spline interpolation is essentially the same as cardinal spline interpolation, for which logarithmic growth of $\|P\|$ with $k$ has been demonstrated; see \[6\]. This fact supports the conjecture that $q$ tending to zero gives "worst-case" results for the nodes (3.1).

For cubic spline interpolation with arbitrary knot spacing and the nodes (3.1), de Boor \[2\] has shown that $\|P\| < 27$. He conjectures that $\|P\| < 3$ or $4$ may be true. The following supplies a lower bound on $\limsup \|P\|$, where the $\limsup$ is taken over all ordered knot spacings.

**Theorem 4.** Let $k = 3$ and let $t$ and $\tau$ be given by (1.1) and (3.1), respectively. Then

$$
\lim \|P\| = \frac{(222\sqrt{111} + 999)/1331 = 2.507825\ldots},
$$

where $\lim \|P\|$ denotes the limiting value of $\|P\|$ as $q$ tends to zero and $m$ tends to infinity.

**Proof.** Let $s = s_\mu$ with $\mu = m - 1$. From (2.21) and (2.13)

$$
(3.3) \quad s(-1) = \frac{(-1)^{m-k} + k}{1 - 2\theta_2^k} (1 - r_2^{m-k}) + O(q)
$$
for each \( k > 2 \) and \( \mu > m - k \). Similarly,

\[
s(+1) = \frac{(-1)^m - \mu}{1 - 2q} (1 - r_i^m - k) + O(q)
\]

for \( k > 2 \) and \( \mu < m - 1 \). Thus, for the case presently under consideration, \( s(t) \) tends, on \([-1, +1]\), to the cubic \( p(t) \) satisfying \( p(\pm 1) = 27/11 \) and \( p(\pm 1/3) = \pm 1 \). This cubic is

\[
p(t) = (-297t^3 + 243t^2 + 297t - 27)/88.
\]

It has a maximum on \([-1, +1]\) of \((222\sqrt{111} + 999)/1331\) at \( t = (9 + 2\sqrt{111})/33 \). Showing that \( \lim ||P|| \) exists and is equal to this maximum requires a discussion (which we omit) similar to the last paragraph in the proof of Theorem 1 above. \( \square \)

For arbitrary \( k \) it is easy to find \( p(t) \), the polynomial which \( s_{m-1}(t) \) approaches as \( q \) tends to zero and \( m \) tends to infinity. From (3.1) and (3.4)

\[
\lim s(+1) = z_k = \frac{1}{1 - 2((k - 1)/k)^k}.
\]

From (3.3), \( \lim s(-1) = (-1)^{k-1}z_k \). Then standard combinatorial formulas give (see Gould [4, p. 59])

\[
p(t) = -(-1)^{l} \sum_{j=0}^{l} \frac{(-4)^{j}T}{T + j} \left( T + j \right) + \frac{2T + k z_k}{T + l} \left( T + l \right),
\]

if \( k \) is even with \( l = k/2 \) and \( T = lt \), and

\[
p(t) = -(-1)^{l} \sum_{j=0}^{l} \frac{(-4)^{j}2T}{2j + 1} \left( T + j - 1/2 \right) + \frac{2T + k z_k}{k} \left( T + l - 1/2 \right),
\]

if \( k \) is odd with \( l = (k - 1)/2 \) and \( T = kt/2 \). The maximum of \( p(t) \) on \((k - 2)/k < t < +1\) is a good lower bound on \( ||P|| \) as \( q \) tends to zero and \( m \) tends to infinity.

The following table was computed via double-precision arithmetic in FORTRAN on an Amdahl 470/V7 computer. All entries are rounded down.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \max p(t) )</th>
<th>( k )</th>
<th>( \max p(t) )</th>
<th>( k )</th>
<th>( \max p(t) )</th>
<th>( k )</th>
<th>( \max p(t) )</th>
</tr>
</thead>
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<td>2.0000</td>
<td>7</td>
<td>7.7939</td>
<td>12</td>
<td>9.02 \times 10^9</td>
<td>27</td>
<td>9.45 \times 10^5</td>
</tr>
<tr>
<td>3</td>
<td>2.5078</td>
<td>8</td>
<td>11.8194</td>
<td>15</td>
<td>5.13 \times 10^2</td>
<td>30</td>
<td>6.60 \times 10^6</td>
</tr>
<tr>
<td>4</td>
<td>3.0814</td>
<td>9</td>
<td>18.7344</td>
<td>18</td>
<td>3.17 \times 10^3</td>
<td>33</td>
<td>4.67 \times 10^7</td>
</tr>
<tr>
<td>5</td>
<td>3.9686</td>
<td>10</td>
<td>30.7986</td>
<td>21</td>
<td>2.05 \times 10^4</td>
<td>36</td>
<td>3.34 \times 10^8</td>
</tr>
<tr>
<td>6</td>
<td>5.4087</td>
<td>11</td>
<td>52.1254</td>
<td>24</td>
<td>1.37 \times 10^5</td>
<td>39</td>
<td>2.42 \times 10^9</td>
</tr>
</tbody>
</table>

This table, in which the exponential growth is clear, is associated with Theorem 2 above. A corresponding table of lower bounds on \( \lim \sup ||P|| \) for the node assignment of Theorem 3 can be computed from the fact that the Lebesgue function for polynomial interpolation on the Chebyshev nodes attains its maximum.
at $t = 1$; see [7, Eq. (4.2.19)]. The first few entries of such a table are:

$$
(1,1.414) \quad (2,1.666) \quad (3,1.847) \quad (4,1.988) \quad (5,2.104) \quad (6,2.202).
$$

A later entry is (35,3.243). The logarithmic growth is clear. For $k = 1$ with arbitrary knots it can be shown that $\|P\| \leq \sqrt{2}$ when nodes are specified by (3.2) above.

Whether the other bounds are "good" bounds for the arbitrary knot case is problematical.

4. Remarks. For one-sided $q$-splines with spline knots $t_i = (1 - q^i)/(1 - q)$, $i = \ldots , -1, 0, 1, 2, \ldots$, and interpolation nodes $\tau_i = t_i + \theta q^i$, where $\theta$ is fixed, $0 < \theta < 1$, S. L. Lee [5] has considered eigensplines, i.e., nontrivial splines $s(t)$ satisfying $s(t) = \lambda s(1 + qt)$ for some fixed eigenvalue $\lambda$. Setting $\lambda = -1$ yields, for each $k \geq 2$, a certain equation $F_k(q, \theta) = 0$. If $q$ and either $\theta_1$ or $\theta_2$ defined above satisfy this equation, then two-sided $q$-spline interpolation is unbounded. Lee [5] has shown that $F_k(0 + , \theta) = C[2^{1-k} - 1][2(1 - \theta)^k - 1]$.

For quadratic splines with arbitrary knots $t_i$, Demko [3] has shown that interpolation is bounded independently of $t_i$ and $\tau_i$ if the nodes $\tau_i$ satisfy $\tau_i = t_i + 2 - \lambda y(t_{i+2} - t_{i+1})$ with $\lambda^2 \leq \gamma < \frac{1}{2}$ and $(1 - \lambda)^2 \leq \gamma < \frac{1}{4}$. Consequently, for $k = 2$, the results of Theorem 1 above with (2.5)a and (2.6)a are valid for all $q$ and not just as $q$ tends to zero.

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5. S. L. LEE, private communication.