Inequalities for Certain Hypergeometric Functions

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Abstract. Theorems on two-sided inequalities for Gauss and Kummer's hypergeometric functions as given by Buschman have been improved. Complex analogues of the said inequalities have been developed and it is pointed out that a similar analysis gives extensions of Luke's, Flett's, and Carlson's theorems.

1. Let $F(a)$ denote the hypergeometric function $\, _2F_1(\alpha, \beta; \gamma; x)$. A contiguous relation for $F(a)$, see, for instance, [6, 2.8(28)] wherein $\alpha$ is replaced by $\alpha + n$ can be rewritten as

$$F(\alpha + n + 1) = A_n F(\alpha + n) + B_n F(\alpha + n - 1),$$

where

(1.1) $A_n = \left( \frac{B_\gamma - B_\gamma + 2 - x}{\alpha + n} \right) / (1 - x)$, \quad $B_n = \left( \frac{\gamma}{\alpha + n} - 1 \right) / (1 - x)$.

The recursion formula (1.1) enables us to represent $F(\alpha + n + 1)$ by an $(n + 2) \times (n + 2)$ determinant

(1.2) $F(\alpha + n + 1) = \det \begin{bmatrix} F(\alpha) & -F(\alpha - 1) \\ B_0 & A_0 \end{bmatrix}$

$$\begin{bmatrix} & \ddots & \end{bmatrix}$$

$$\begin{bmatrix} B_{n-1} & A_{n-1} & -1 \\ B_n & A_n \end{bmatrix}$$

The determinant (1.2) will have a strictly dominant diagonal provided that

(1.3) $\begin{array}{l}
(i) \quad |F(\alpha)| > |F(\alpha - 1)|,
(ii) \quad |A_k| > |B_k| + 1, \quad \text{for } 0 < k < n,
(iii) \quad |A_n| > |B_n|.
\end{array}$

Assuming that $\alpha, \beta, \gamma$ and $x$ all are positive real numbers, an examination of power series representations of $F(\alpha)$ and $F(\alpha - 1)$ with respect to $x$ shows that

$$|F(\alpha)| > |F(\alpha - 1)| \quad \text{with } 0 < x < 1,$$
provided that
\[
\left| \frac{\alpha - 1}{\alpha + n - 1} \right| > 1 \quad \text{for } n > 1 \quad \text{and}
\]
\[
\left| \frac{\alpha - 1}{\alpha + n - 1} \right| < 1 \quad \text{for at least one value of } n.
\]

It therefore follows that 1.3(i) holds for \( \alpha > \frac{1}{2} \). Also, obviously 1.3(iii) will be valid under the same sets of conditions for which 1.3(ii) is valid. Now in order that 1.3(ii) may hold, in the first place, for \( \gamma < \alpha \), it is sufficient that \( \beta x > 0 \), which is obvious since \( \beta \) and \( x \) are both positive real numbers.

In the next place, consider the situation \( \gamma > \alpha \). Let \( \gamma > \alpha + k \) for some positive integer \( k \). The inequality
\[
|A_k| > |B_k| + 1
\]
will be satisfied for
\[
\beta > \beta x > \max\{\gamma, 2(\gamma - \alpha)\} > 0.
\]
Indeed this is so since in this case
\[
|A_k| = \left[ \frac{\beta x - \gamma}{\alpha + k} + 2 - x \right] / (1 - x), \quad |B_k| = \left( \frac{\gamma}{\alpha + k} - 1 \right) / (1 - x).
\]

If \( \gamma > \alpha + n \), nothing remains to say, but if \( \alpha < \gamma < \alpha + n \), there exists a nonnegative integer \( k_0 \) such that \( \alpha + k_0 < \gamma < \alpha + k_0 + 1 \). Thus when \( k > k_0 \), (1.5) holds for \( \beta x > 0 \), and when \( k < k_0 \), (1.5) holds under the conditions (1.6).

Thus, the sufficient conditions under which 1.3(i)-(1.3)(iii) hold may be summarized as
\[
\alpha > \frac{1}{2}, \quad \gamma < \alpha \quad \text{or} \quad \beta > \beta x > \max\{\gamma, 2(\gamma - \alpha)\} > 0.
\]

Buschman [3] has claimed that (1.3) holds if all \( \alpha, \beta, \gamma \), and \( x \) are real and positive and satisfy the set of conditions \( \alpha > 1, \beta > \beta x > 2\gamma > 0 \). A closer examination clearly reveals that our conditions are much weaker than those given by Buschman and hence one can expect to get estimates in a wider range.

Thus under the conditions (1.7), by the theorem of G. B. Price [9] we have
\[
A_n\left[ F(\alpha) - |F(\alpha - 1)| \right] \prod_{k=0}^{n-1} (A_k - 1) < _2F_1(\alpha + n + 1, \beta; \gamma; x)
\]
\[
< A_n\left[ F(\alpha) + |F(\alpha - 1)| \right] \prod_{k=0}^{n-1} (A_k + 1),
\]
where the absolute value symbols on \( F(\alpha) \) and \( A_k \)'s, \( k = 0, \ldots, n \), have been dropped because of our assumptions. Further, the absolute value symbol on \( F(\alpha - 1) \) can also be dropped by recourse to Erber's formula [5, (11)], which for real parameters and variables can be rewritten as
\[
|_2F_1(a, b; c; z)| < _2F_1(|a|, |b|; |c|; |z|); \quad |z| < 1.
\]

Consequently
\[
|F(\alpha - 1)| = |_2F_1(\alpha - 1, \beta; \gamma; x)| < _2F_1(|\alpha - 1|, \beta; \gamma; x).
\]
Hence converting products into gamma-functions, the result (1.8) along with (1.10) enables us to write the modified version of Buschman's Theorem 1 in the following form.

**Theorem 1.** If \( \alpha > \frac{1}{2} \), \( \gamma < \alpha \) or \( \beta > \beta x > \max\{\gamma, 2(\gamma - \alpha)\} > 0 \), then

\[
g(x)L < _2 F_1(\alpha + n + 1, \beta; \gamma; x) < g(x)U,
\]

where

\[
g(x) = (1 - x)^{n-1}(\beta x - \gamma + (2 - x)(\alpha + n))\Gamma(\alpha)/\Gamma(\alpha + n + 1),
\]

\[
(1.11)\quad U = [F(\alpha) + F(\alpha - 1)](3 - 2x)^{\frac{\beta x - \gamma}{3 - 2x} + \alpha + n}/\Gamma(\frac{\beta x - \gamma}{3 - 2x} + \alpha),
\]

\[
L = [F(\alpha) - F(\alpha - 1)]\Gamma(\beta x - \gamma + \alpha + n)/\Gamma(\beta x - \gamma + \alpha).
\]

It is observed here that for \( \alpha > 1 \), \( F(\alpha - 1) = F(\alpha - 1) \), and therefore \( U \) and \( L \) of (1.11) correspond to those of Theorem 1 of Buschman [3]. Further, by using the bounds for the determinant (1.2) as given by Brenner [1], we have

**Theorem 2.** If \( \alpha > \frac{1}{2} \), \( \gamma < \alpha \) or \( \beta > \beta x > \max\{\gamma, \gamma - \alpha\} > 0 \), then

\[
L' < _2 F_1(\alpha + n + 1, \beta; \gamma; x) < U',
\]

where

\[
(1.12)\quad L' = \left(\frac{F^2(\alpha) - F^2(\alpha - 1)}{F(\alpha)}\right)^{n-1}\prod_{k=0}^{n-1}\left(A_k - \frac{|B_k| + 1}{A_k}\right),
\]

\[
U' = A_n\left(\frac{F^2(\alpha) + F^2(\alpha - 1)}{F(\alpha)}\right)^{n-1}\prod_{k=0}^{n-1}\left(A_k + \frac{|B_k| + 1}{A_k}\right).
\]

It should be noted that the absolute value symbol on \( B_k \)'s can be dropped when \( \gamma > \alpha + n \). Note also that whereas Theorem 2 gives improved lower and upper bounds for \( _2 F_1(\alpha + n + 1, \beta; \gamma; x) \) over Theorem 1, Theorem 1 is more suitable in applications because of its simplicity.

Improvements over Theorems 1 and 2, though in a restrictive domain, may further be obtained in the light of the suggestions made by Srivastava and Brenner [10] by writing the determinant (1.2) in the alternate form

\[
(1.13)\quad F(\alpha + n + 1) = \det\begin{bmatrix}
F(\alpha) & -F(\alpha - 1)\sqrt{B_0} & 0 & & \\
\sqrt{B} & A_0 & -\sqrt{B_1} & & \\
0 & \vdots & \ddots & \ddots & \\
0 & 0 & \ddots & \ddots & \sqrt{B_n-1} A_{n-1} & -\sqrt{B_n} \\
0 & 0 & 0 & \vdots & \sqrt{B_n} & A_n
\end{bmatrix}.
\]

Thus, for example, the inequality corresponding to Theorem 1 may be stated as follows:

**Theorem 3.** If \( \alpha > \frac{1}{2} \), and either \( \alpha \beta > \beta \gamma > \alpha \beta x > \gamma > 0 \), or \( \alpha(1 - x) > \gamma - \alpha > 0 \), \( \beta x > \max\{\gamma, 2(\gamma - \alpha)\} > 0 \), then

\[
L'' < _2 F_1(\alpha + n + 1; \beta; \gamma; x) < U''.
\]
where

\[ L'' = A_n \left[ F(\alpha) - F(\alpha - 1) \right] \sqrt{B_0} \prod_{k=0}^{n-1} \left( A_k - \sqrt{B_{k+1}} \right), \]

(1.14)

\[ U'' = A_n \left[ F(\alpha) + F(\alpha - 1) \right] \sqrt{B_0} \prod_{k=0}^{n-1} \left( A_k + \sqrt{B_{k+1}} \right). \]

If \( 1 < \alpha < \gamma \), the \( _2F_1 \)'s in the bounds of the above listed theorems can further be approximated by application of Luke's [8, 4.21, 4.23], Carlson's [4] or Flett's [7] theorems to obtain inequalities in terms of parameters and variables.

Proceeding as before, an improved version of Theorem 2 of Buschman [3] can be stated as

**Theorem 4.** If \( \alpha > \gamma > 0 \) or \( x > \max \{ \gamma, 2(\gamma - \alpha) \} > 0 \), then

\[ h(x)B < \, _1F_1(\alpha + n + 1; \gamma; x) < h(x)A, \]

where

\[ h(x) = (x - \gamma + 2(\alpha + n))\Gamma(\alpha)/\Gamma(\alpha + n + 1), \]

\[ A = \left[ \, _1F_1(\alpha; \gamma; x) + \, _1F_1(\alpha - 1; \gamma; x) \right] \Gamma \left( \frac{x - \gamma}{3} + \alpha + n \right)/\Gamma \left( \frac{x - \gamma}{3} + \alpha + n + 1 \right), \]

\[ B = \left[ \, _1F_1(\alpha; \gamma; x) - \, _1F_1(\alpha - 1; \gamma; x) \right] \Gamma(x - \gamma + \alpha + n)/\Gamma(x - \gamma + \alpha + n + 1). \]

Also, by the same analysis, it is found that Theorem 3 of Buschman, which gives bounds for the confluent hypergeometric function \( \Psi \), is valid in a larger domain \( 2c - 1 > \alpha > 0, \gamma > 0 \).

2. The Case of Complex Parameters and Variables. Erber [5] observed that for \( n > 0 \),

\[ |(\alpha)_n| < |(\alpha)|_n, \quad |(\alpha)_n| > (\cos(\theta/2))^{n-1}|(\alpha)|_n, \quad \theta = \arg \alpha, |\theta| < \pi, \]

and used these to obtain

\[ |_2F_1(\alpha, \beta; \gamma; z)| < \cos(\theta/2)_2F_1(|\alpha|, |\beta|; |\gamma|; |z|\sec \theta/2), \]

(2.2) where \( \theta = \arg \gamma, |\theta| < \pi, \) and \( |z| < \cos(\theta/2) \). From (2.1) we can also have

\[ |_pF_q(\alpha_p; \beta_q; z)| < \Pi \cos(\theta_q/2)_pF_q(|\alpha_p|; |\beta_q|; |z|\sec(\theta_q/2)), \]

(2.3) where \( \theta_q = \arg(\beta_q), |\theta_q| < \pi, |z| < \Pi \cos(\theta_q/2), \) and as usual \( \Pi \) stands for the product symbol. If \( p < q \), the condition \( |z| < \Pi \cos(\theta_q/2) \) in (2.3) can be dropped.

With the help of (2.2) and the triangle inequality \( |\alpha + n| < |\alpha| + n, n \) being any nonnegative integer, extensions of Theorems 1, 3, and 4 for complex parameters and arguments can be obtained. For reasons of brevity we shall however state only the extension of Theorem 1.

**Theorem 5.** If \( a, b, c, \) and \( z \) are complex numbers and \( \theta = \arg c, |\theta| < \pi, |z| < \cos(\theta/2) \), then

\[ |_2F_1(a + n + 1, b; c; z)| < \cos(\theta/2)U \cdot g(z), \]
where
\[ g(z) = \frac{[1 - |z|\sec(\theta/2)]^{-n-1}}{(|a|)_{n+1}} \left[ |bz|\sec(\theta/2) - |c| + (2 - |z|\sec(\theta/2))(|a| + n) \right], \]
\[ U = \left\{ _2F_1([|a|, |b|; |c|; |z|\sec(\theta/2)) + _2F_1([|a| - 1, |b|; |c|; |z|\sec(\theta/2)) \right. \]
\[ \cdot (3 - 2|z|\sec(\theta/2))^{\frac{1}{4}}((|bz|\sec(\theta/2) - |c|)/(3 - 2|z|\sec(\theta/2)) + |a|)_n, \]
provided
\[ |a| > \frac{1}{2}, \quad |c| < |a| \quad \text{or} \quad |b| > |bz|\sec(\theta/2) > \max\{|c|, 2(|c| - |a|)\} > 0. \]

In the sequel, complex analogues of inequalities of Luke [8, 4.21, 4.23, 5.6, 5.8] and those of Flett [7] and Carlson [4] could also be given similarly.

Acknowledgements. Thanks are due to the referee for his very valuable suggestions.

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