MATHEMATICS OF COMPUTATION
VOLUME 38, NUMBER 157
JANUARY 1982

On Stieltjes Integral Transforms Involving Γ-Functions

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Abstract. After some methodological remarks on the theory of Stieltjes transforms, a systematic classification of transforms involving Γ-functions is presented. As a consequence, many new transforms are established and much simpler proofs for a few known transforms are obtained.

In circuit and system theory a real function $f(z)$, analytic in $\operatorname{Re} z \geqslant 0$ (hence finite at infinity), is deduced from its real part on the imaginary axis

$$u(y) = \operatorname{Re} f(iy) \tag{1}$$

by [5]

$$f(z) = \frac{2z}{\pi} \int_0^\infty \frac{u(y)\, dy}{z^2 + y^2}; \qquad \operatorname{Re} z > 0. \tag{2}$$

Relation (2) is even valid when $f(z)$ has sufficiently mild singularities on the imaginary axis and at infinity [8]. In the case of logarithmic singularities or branch points, however, cuts on the i-axis may be necessary to define $f(z)$ as single-valued in $\operatorname{Re} z > 0$, and the real part (1) must then be replaced by the real part on the right lip of the cut, i.e., by [1]

$$u(y) = \lim_{\varepsilon \to +0} \operatorname{Re} f(\varepsilon + iy). \tag{3}$$

The right-hand side of (2) is odd in z whereas $f(z)$ is not odd, else $u(y)$ would be zero. Consequently (2) does not hold for $\operatorname{Re} z \leqslant 0$; in any case, the integrand of (2) is singular for $z = \pm iy$. For $f(z) = u(y) = 1$ (2) reduces to the elementary integral

$$1 = \frac{2z}{\pi} \int_0^\infty \frac{dy}{z^2 + y^2}; \qquad \operatorname{Re} z > 0. \tag{4}$$

Subtracting $u(s)$ times (4) from (2), one obtains

$$f(z) - u(s) = \frac{2z}{\pi} \int_0^\infty \frac{u(y) - u(s)}{z^2 + y^2}\, dy, \tag{5}$$

where the integrand is no longer singular for $z = is$. With $f(is) = u(s) + iv(s)$, (5) divided by i yields

$$v(s) = \frac{2s}{\pi} \int_0^\infty \frac{u(y) - u(s)}{y^2 - s^2}\, dy.$$

Received February 25, 1981; revised June 8, 1981.
1980 *Mathematics Subject Classification.* Primary 44A15, 33A15.

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0025-5718/82/0000-0483/$02.00

This is equivalent [7] to the Hilbert transform

$$v(s) = \frac{2s}{\pi} \fint_0^\infty \frac{u(y)}{y^2 - s^2}\, dy,$$

where the integral is a Cauchy principal value.

In (2), change z into $\sqrt{z}$ and y into $\sqrt{y}$; next change $f(\sqrt{z})/\sqrt{z}$ into $f(z)$ and $u(\sqrt{y})/\sqrt{y}$ into $u(y)$. This yields the Stieltjes transform

$$f(z) = \frac{1}{\pi} \int_0^\infty \frac{u(y)\, dy}{y + z}, \tag{6}$$

holding in the whole z-plane cut on the negative real axis, and $u(y)$ is now related to the discontinuity of $\operatorname{Im} f(z)$ on the cut through

$$u(y) = -\tfrac{1}{2} \operatorname{Im}\left[f(ye^{i\pi}) - f(ye^{-i\pi}) \right], \tag{7}$$

a relation given in [4, p. 215, Eq. (5)] with a sign error. Relation (7) is less conspicuous than (1) or (3); moreover, the change of variables from (2) to (6) (corresponding in circuit theory to the transformation of an LC-impedance into an RC-impedance) is responsible for the many square-roots appearing in the table [4] of Stieltjes transforms. Finally additional transforms are deduced from (2), for $z = iy$, by

$$\operatorname{Re} z \frac{df(z)}{dz} = y \frac{du(y)}{dy}, \tag{8}$$

which is simpler than [4, p. 215, Eq. (9)] and by

$$\operatorname{Re} \frac{f(z) - f(0) - zf'(0)}{z^2} = \frac{u(0) - u(y)}{y^2}. \tag{9}$$

The idea that (2) is essentially simpler than (6) has been (somewhat unsystematically) exploited in two previous papers, thus generating a number of new transforms for Bessel functions [3] and for complete elliptic integrals [2]. In addition to the remarks just made, the purpose of this note is to derive some new transforms, and to present much simpler proofs for some known transforms, involving Γ-functions.

Owing to the complement relation for Γ-functions, the real or imaginary parts of some linear combinations of logarithms of Γ-functions have elementary expressions. For $z = iy$, and with the definition (3) of the real part whenever (1) is ambiguous (and similarly for the imaginary part), we have

$$\operatorname{Re}\left[\log \Gamma(z + a) + \log \Gamma(z + 1 - a)\right] = \frac{1}{2} \log \frac{2\pi^2}{\cosh(2\pi y) - \cos(2\pi a)}, \tag{10}$$

$$\operatorname{Im}\left[\log \Gamma(z + a) - \log \Gamma(z + 1 - a)\right] = -\arctan\left[\cot(\pi a)\tanh(\pi y)\right]. \tag{11}$$

A number of Stieltjes transforms corresponding to the definition (2) and resulting from (10) or (11) are given in Tables A to C. For $0 \leqslant a \leqslant 1$, the functions $\log \Gamma(z + a)$, $z \log \Gamma(z + a)$ and $z^{-1}[\log \Gamma(z + a) - \log \Gamma(a)]$ are analytic in $\operatorname{Re} z > 0$. They can be made finite at infinity by subtracting from $\log \Gamma(z + a)$ the necessary number of terms of its asymptotic expansion

$$\log \Gamma(z + a) \sim \left(z + a - \tfrac{1}{2}\right)\log z - z + \tfrac{1}{2} \log 2\pi + O(z^{-1}), \tag{12}$$

and the resulting differences have at most a logarithmic singularity on the i-axis (at $z = 0$ for $a = 0$). Transforms I to III are established by combining the resulting functions with parameters a and $1 - a$. Transform IV results from I by (8). Transforms V and VI result from II and III, respectively, by (8) after adding a multiple of the original transforms. Also VI is the derivative of I with respect to a; similarly IV is the derivative of II.

TABLE A ($0 \leqslant a \leqslant 1$)

	$f(z)$	$u(y)$
I	$\log \Gamma(z + a) + \log \Gamma(z + 1 - a) - 2z \log z + 2z$	$\frac{1}{2} \log \dfrac{2\pi^2 e^{2\pi y}}{\cosh(2\pi y) - \cos(2\pi a)}$
II	$z[\log \Gamma(z + a) - \log \Gamma(z + 1 - a) - (2a - 1)\log z]$	$y \arctan[\cot(\pi a)\tanh(\pi y)] + (a - \frac{1}{2})\pi y$
III	$\frac{1}{z}[\log \Gamma(z + a) - \log \Gamma(z + 1 - a) - \log \Gamma(a) + \log \Gamma(1 - a)]$	$-\dfrac{1}{y} \arctan[\cot(\pi a)\tanh(\pi y)]$
IV	$z[\psi(z + a) + \psi(z + 1 - a) - 2 \log z]$	$-\pi y \dfrac{\cos(2\pi a) - e^{-2\pi y}}{\cosh(2\pi y) - \cos(2\pi a)}$
V	$z^2\left[\psi(z + a) - \psi(z + 1 - a) - \dfrac{2a - 1}{z}\right]$	$\pi y^2 \dfrac{\sin(2\pi a)}{\cosh(2\pi y) - \cos(2\pi a)}$
VI	$\psi(z + a) - \psi(z + 1 - a)$	$-\dfrac{\pi \sin(2\pi a)}{\cosh(2\pi y) - \cos(2\pi a)}$

TABLE B ($0 < a < 1$)

VII	$\dfrac{1}{z^2}\left\{\log \Gamma(z + a) + \log \Gamma(z + 1 - a) - \log \dfrac{\pi}{\sin(\pi a)} - z[\psi(a) + \psi(1 - a)]\right\}$	$\dfrac{1}{2y^2} \log \dfrac{\cosh(2\pi y) - \cos(2\pi a)}{2 \sin^2(\pi a)}$
VIII	$\dfrac{1}{z}[\psi(z + a) + \psi(z + 1 - a) - \psi(a) - \psi(1 - a)]$	$\dfrac{\pi}{y} \dfrac{\sinh(2\pi y)}{\cosh(2\pi y) - \cos(2\pi a)}$

TABLE C ($0 \leqslant a < 1$)

IX	$\dfrac{1}{z^2}\left\{\log \Gamma(z + 1 + a) + \log \Gamma(z + 1 - a) - \log \dfrac{\pi a}{\sin(\pi a)} - z[\psi(1 + a) + \psi(1 - a)]\right\}$	$\dfrac{1}{2y^2} \log\left[\dfrac{a^2}{2 \sin^2(\pi a)} \cdot \dfrac{\cosh(2\pi y) - \cos(2\pi a)}{y^2 + a^2}\right]$
X	$\dfrac{1}{z}[\psi(z + 1 + a) + \psi(z + 1 - a) - \psi(1 + a) - \psi(1 - a)]$	$\dfrac{\pi}{y} \dfrac{\sinh(2\pi y)}{\cosh(2\pi y) - \cos(2\pi a)} - \dfrac{1}{y^2 + a^2}$

Transform VII results from I by (9). Since $f(z)$ is singular for $a = 0$ and $a = 1$, the transform is only valid in the range $0 < a < 1$. Transform VIII results from VII by (8) or from III by differentiation with respect to a. By adding to $f(z)$ of VII the function $\log(1 + z/a) - z/a$ whose real part is $(1/2)\log(1 + y^2/a^2)$, one suppresses the singularity for $a = 0$; the result is transform IX which now holds for $0 \leqslant a < 1$. Transform X is deduced from IX by (8) combined with a multiple of the original transform. Transform I is equivalent to a result presented in an annex

to Boersma's thesis (1964). Particular cases of transforms I and IV for $a = 0$ or 1 and for $a = \frac{1}{2}$ are equivalent (sometimes after integration by parts) to [6, pp. 181–182, Eqs. (10) to (14)]. In all these known cases, our proofs are much simpler than the original ones. All the other transforms are believed to be new. Further new transforms follow from those in Tables A, B, and C by specialization of a, by differentiating with respect to a, etc. Particularly simple results are obtained for $a = 0$ in IX and X, and for $a = \frac{1}{2}$ in IV, VII and VIII; even more elegant results are deduced by devising linear combinations in which various simplifications occur. Simple results are also derived for $a = \frac{1}{4}$ or $\frac{3}{4}$ in II, III, V, and VI. Finally, a new integral involving Catalan's constant is obtained by differentiating VII with respect to a and setting $a = \frac{3}{4}$.

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