An Upper Bound for the First Zero of Bessel Functions

By L. G. Chambers

Abstract. It is shown, using the Rayleigh-Ritz method of the calculus of variations, that an upper bound for the first zero $j_v$ of $x^{-
u} J_v(x)$, $\nu > -1$, is given by

$$(\nu + 1)^{1/2} \{ (\nu + 2)^{1/2} - 1 \},$$

and that for large $\nu$, $j_v = \nu + O(\nu^{1/2})$.

1. The following upper bound is given by Watson [4] for the first zero $j_v$ of $J_v(x)$ ($\nu > 0$)

$$(1) \quad j_v < \left\{ \frac{4}{3} (\nu + 1)(\nu + 5) \right\}^{1/2}.$$  

It may be shown that a better bound may be obtained, valid for $\nu > -1$, namely

$$(2) \quad (\nu + 1)^{1/2} \{ (\nu + 2)^{1/2} - 1 \}.$$  

2. Consider the function

$$(3) \quad u(z) = \Gamma(\nu + 1)(2/\gamma z)^{1/2} J_v(\gamma z).$$

The differential equation satisfied by $u(z)$ is given by Watson [3] to be

$$(4) \quad z^2 u'' + (2\nu + 1)zu' + \gamma^2 z^2 u = 0$$

with the boundary condition $u(0) = 1$, and if $\gamma$ is a zero of $J_v$, $u(1) = 0$.

Equation (4) can be written in Sturm-Liouville form

$$(5) \quad \frac{d}{dz} \left( z^{2\nu+1} \frac{du}{dz} \right) + \gamma^2 z^{2\nu+1} u = 0.$$  

Multiplying Eq. (5) by $u$ and integrating over $0 < z < 1$, it follows that

$$(6) \quad \gamma^2 = \frac{\int_0^1 z^{2\nu+1} u^2 dz}{\int_0^1 z^{2\nu+1} u^2 dx}.$$  

On integration by parts, $uu'z^{2\nu+1}$ will vanish at $z = 0$, if $\nu > -\frac{1}{2}$, and $u(1)$ vanishes. Thus the relation (6) provides a variational formulation, as indicated by Irving and Mullineux [1], for $\gamma^2$ which is an eigenvalue for the differential equation (5). The first eigenvalue will be $j_v^2$. The functional

$$(7) \quad \Lambda(\omega) = \frac{\int_0^1 z^{2\nu+1} \omega^2 dz}{\int_0^1 z^{2\nu+1} \omega^2 dx},$$

Received February 19, 1981; revised July 13, 1981.

1980 Mathematics Subject Classification. Primary 33A40, 49G10.

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0025-5718/81/0000-1069/$01.50

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as indicated by Irving and Mullineux [2], obeys the following relations

\[(8a) \quad \Lambda(u) = j_v^2,\]

\[(8b) \quad \Lambda(\omega) > \Lambda(u), \quad u \neq \omega.\]

Thus \(\Lambda(\omega)\) provides an upper bound to \(j_v^2\) when \(\omega(1) = 0\), by the Rayleigh-Ritz procedure.

3. Consider the approximating function

\[(9) \quad \omega = 1 - z^p,\]

where \(p\) is as yet unspecified.

\[(10) \quad \Lambda(\omega) = \frac{\int_0^1 z^{2v+1} p^{2v+2} z^{-2} \, dz}{\int_0^1 (1 - 2z^p + z^2p)z^{2v+1} \, dz}\]

\[(11) \quad = p^2 \left\{ \frac{1}{(2v + 2)} - \frac{1}{(2v + p + 2)} + \frac{1}{(2v + 2p + 2)} \right\}\]

\[(12) \quad = \frac{(v + 1)(2v + p + 2)(v + p + 1)}{v + p}\]

on simplification.

Up till now \(p\) has not been specified. \(\Lambda(\omega)\) may be regarded as a function of \(p\), and the best upper bound will follow by minimizing \(\Lambda(\omega)\) with respect to \(p\). It can be verified by differentiation that this happens when

\[(13) \quad p + v = (v + 2)^{1/2}.\]

Although \(p\) is negative outside of \(-1 < v < 2\), the process is still valid because all of the denominators in the expression (11) remain positive, and it can easily be seen that

\[\lim_{z \to 0} z^{2v+1} \omega' = 0,\]

so that the endpoint condition at \(z = 0\) remains satisfied. It follows that

\[j_v^2 < \frac{(v + 1)\left( (v + 2) + (v + 2)^{1/2} \right)\left( (v + 2)^{1/2} + 1 \right)}{(v + 2)^{1/2}},\]

which gives

\[(14) \quad j_v < (v + 1)^{1/2}\left( (v + 2)^{1/2} + 1 \right).\]

A straightforward reduction shows that, if \(v + 1 > 0\),

\[\left( v + 1 \right)^{1/2}\left( (v + 2)^{1/2} + 1 \right) < \left\{ \frac{4}{3} (v + 1)(v + 5) \right\}^{1/2}.\]

(For \(v = 7\) there is in fact equality.)

It thus follows that [4]

\[(15) \quad \left\{ v(v + 2) \right\}^{1/2} < j_v < (v + 1)^{1/2}\left( (v + 2)^{1/2} + 1 \right), \quad v > 1.\]

It follows from (15) that

\[(16) \quad j_v = v + O(v^{1/2}) \quad \text{for large } v.\]

As an example, the bound for \(v = 0\) is given by \(\sqrt{2} + 1 \approx 2.4142\) in comparison with the true value 2.4048.
Acknowledgement. I am indebted to a referee for some improvements in the presentation of this paper.

School of Mathematics and Computer Science
University College of North Wales
Bangor, Gwynedd LL57 2UW, Wales