MATHEMATICS OF COMPUTATION
VOLUME 39, NUMBER 159
JULY 1982, PAGES 29-52

# Vortex Methods. II: Higher Order Accuracy in Two and Three Dimensions

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**Abstract.** In an earlier paper the authors introduced a new version of the vortex method for three-dimensional, incompressible flows and proved that it converges to arbitrarily high order accuracy, provided we assume the consistency of a discrete approximation to the Biot-Savart Law. We prove this consistency statement here, and also derive substantially sharper results for two-dimensional flows. A complete, simplified proof of convergence in two dimensions is included.

**1. Introduction.** This paper continues an analysis of the accuracy and convergence of vortex methods begun earlier by the authors in [1]. The principle of such methods is to simulate inviscid, incompressible fluid flow in two or three space dimensions by computing the paths of representative particles in the fluid. In [1] a version of the vortex method was introduced for three-dimensional flows which can be designed to represent the flow with arbitrary accuracy. It was shown that this three-dimensional method converges, provided a certain discrete integral approximation to the velocity field is consistent to the specified order of accuracy. In the present paper we verify this last condition, as stated in the Consistency Lemma below, thereby completing the convergence argument. In addition, we discuss two-dimensional methods in detail, since our techniques lead to sharper results in this case and allow more flexibility than the previous work [7], [8]. Given the consistency lemma, the proof of convergence is drastically simpler in the two-dimensional case. We include this shorter convergence argument here, so that this paper provides a complete self-contained treatment of two-dimensional vortex methods which can be read independently of [1]. Also the proof of stability which we give in Section 5 both generalizes and simplifies the earlier stability proof in the two-dimensional case due to Hald [7].

Before stating the results more precisely, we review the formulation of the vortex method in two dimensions. Let $z = (z_1, z_2)$ denote a point in the plane, $u(z, t) = (u_1, u_2)$ the fluid velocity, and $\omega(z, t)$ the scalar vorticity,

$$\omega = u_{2,1} - u_{1,2}.$$

The basis of the vortex method is the fact that, for incompressible flows, the vorticity determines the velocity: since div $u = 0$, we can express the velocity $u$ in terms of a stream function $\Psi$,

$$u = (\partial_2\Psi, -\partial_1\Psi)$$

Received October 7, 1981.
1980 *Mathematics Subject Classification.* Primary 65M15, 76C05.
*Key words and phrases.* Vortex method, incompressible flow.
* Partially supported by National Science Foundation Grant MCS-81-01639.
** Partially supported by National Science Foundation Grant MCS-81-02360.

which satisfies

$$\Delta\Psi = -\omega.$$

If we write $\Psi$ as the convolution of the Green's function with $\omega$, we have

$$u(z,t) = \int_{\mathbf{R}^2} K(z-z')\omega(z')\,dz', \tag{1.1}$$

where $K(z) = -(2\pi)^{-1}(\partial_2, -\partial_1)\log|z|$, or

$$K(z) = -\left(2\pi|z|^2\right)^{-1}(z_2, -z_1). \tag{1.2}$$

We will use $\alpha = (\alpha_1, \alpha_2)$ for the Lagrangian coordinates of a fluid particle. Thus, a particle starting at the position $\alpha$ at time 0 follows a trajectory $z(t;\alpha)$ determined by the equation

$$\frac{dz}{dt} = u(z,t), \qquad z(0;\alpha) = \alpha. \tag{1.3}$$

We denote the solution of the ordinary differential equation (1.3) by $z(t;\alpha) = \Phi^t(\alpha)$; thus $\Phi^t$ is the coordinate transformation from time 0 to time $t$ determined by the flow. It is well known that in two dimensions the momentum equation leads to the vorticity equation

$$\omega_t + (u\cdot\nabla)\omega = 0.$$

Hence $\omega$ is conserved on particle paths, i.e.,

$$\omega(\Phi^t(\alpha), t) = \omega(\alpha, 0). \tag{1.4}$$

Now suppose the initial vorticity $\omega_0(\alpha) = \omega(\alpha, 0)$ has support inside a bounded set, say

$$B(R_0) = \{z: |z| < R_0\}.$$

We introduce a square grid in the $\alpha$-plane with squares of side $h$ centered about the lattice points $hj \in \Lambda^h$, where $\Lambda^h = h\Lambda$ and $\Lambda = \{(j_1, j_2): j_1, j_2 \text{ integers}\}$. We will write $z_j(t) = \Phi^t(jh)$, $u_j(t) = u(z_j(t), t)$ for the position and velocity at time $t$ of a particle in the ideal flow beginning at a grid point $jh \in \Lambda^h$. According to (1.4), $\omega(z_j(t), t) = \omega_0(jh) \equiv \omega_j$. Thus, the vorticity at $z_j(t)$ is nonzero only for $jh \in \Lambda^0$, where $\Lambda^0 = \Lambda^h \cap B(R_0)$.

In writing equations to approximate $z_j(t)$, we will smooth out the singularity of the kernel $K$ in (1.1), as was done in [1] as well as in earlier treatments of the two-dimensional vortex method. Let $\psi(z)$ be a function with $\int \psi(z)\,dz = 1$, and let $\psi_\delta(z) = \delta^{-2}\psi(z/\delta)$, where $\delta$ is a parameter to be determined in relation to $h$. We will replace $K$ with

$$K_\delta(z) = \int K(z-z')\psi_\delta(z')\,dz'. \tag{1.5}$$

Further conditions on $\psi$ are specified below. Finally, with the initial vorticity $\omega_0(\alpha)$ prescribed and $h$ fixed, we compute approximate particle paths $\tilde{z}_j(t)$, $jh \in \Lambda_0^h$, as solutions of the system of ordinary differential equations

$$\frac{d\tilde{z}_j}{dt} = \tilde{u}_j^h(t), \qquad \tilde{z}_j(0) = jh. \tag{1.6}$$

Here $\tilde{u}_j^h$ is a discrete approximation to the velocity expression (1.1) computed from $\{z_j(t)\}$,

$$\tilde{u}_j^h(t) = \sum_{kh \in \Lambda_0^h} K_\delta\big(\tilde{z}_j(t) - \tilde{z}_k(t)\big)\omega_k h^2. \tag{1.7}$$

Having determined $\{\tilde{z}_j\}$ from (1.6), (1.7), we can compute a continuous approximation to the velocity field defined, as in (1.7), by

$$\tilde{u}^h(z, t) = \sum_{jh \in \Lambda_0^h} K_\delta\big(z - \tilde{z}_j(t)\big)\omega_j h^2. \tag{1.8}$$

Theorem 1 below asserts that, subject to certain conditions, $\{\tilde{z}_j(t)\}$ and $\tilde{u}^h(z, t)$ accurately approximate $\{z_j(t)\}$ and $u(z, t)$.

The accuracy of the vortex method is controlled by the choice of the cutoff function $\psi$ and the relative size of $\delta$ and $h$. As in [1], we will say that $\psi$ belongs to the class $FeS^{-L,p}$ provided three conditions are satisfied:

(1) $\psi(z)$ belongs to $C^2(\mathbf{R}^2)$;

(2) $\int \psi(z)\,dz = 1$ and $\int z^\gamma \psi(z)\,dz = 0$, where $\gamma$ is any multi-index with $1 \leqslant |\gamma| \leqslant p - 1$;

(3) $L$ is a positive number, and, for any multi-index $\beta$, the Fourier transform $\hat{\psi}(\zeta)$ satisfies

$$\sup_{\zeta \in \mathbf{R}^2} |D_\zeta^\beta \hat{\psi}(\zeta)| \leqslant C_\beta (1 + |\zeta|)^{-L-|\beta|}.$$

Condition (3) means that $\hat{\psi}$ belongs to the symbol class $S_{1,0}^{-L}$. It implies that $\psi$ is smooth and rapidly decreasing away from $z = 0$; see Lemma 2.1. We say $\psi \in FeS^{-\infty,p}$ if $\psi \in FeS^{-L,p}$ for every $L > 0$. (The notation $FeS$ means the Fourier transform of a symbol class.) If $\psi(z)$ is a function only of $|z|$, the moment condition (2) is automatically satisfied for $|\gamma|$ odd, so that in this case we might as well take $p$ to be an even integer, $p \geqslant 2$. For example, we can define $\psi \in FeS^{-\infty,p}$ by setting

$$\hat{\psi}(z) = c_p \exp(-|\zeta|^p), \tag{1.9}$$

normalized so that $\psi$ has integral 1. If $p = 2$, this is the familiar Gaussian distribution.

We will measure the error in $\{z_j(t)\}$, $\{u(z_j, t)\}$ by the discrete integral norm

$$|f_j|_{L_h^\mu} = \left\{ \sum_{j \in \Lambda_0^h} |f_j|^\mu h^2 \right\}^{1/\mu},$$

where $1 < \mu < \infty$ and the error in $u(z, t)$ on any ball $B(R_0)$ by the corresponding norm in $L^\mu(B(R_0))$. In studying vortex methods with a cutoff such as the Gaussian, i.e., $\psi \in FeS^{-L,2}$, it is crucial that we use the $L_h^\mu$ norms with $\mu > 2$ and sufficiently large in the arguments in Section 5. It is an underlying assumption of the analysis that the flow being approximated is sufficiently smooth. It has long been known that, with mild regularity assumptions on the initial vorticity, the Euler equations of ideal flow in two dimensions have a classical solution for all time; for a recent treatment see McGrath [13]. Moreover, if the initial vorticity is smooth, the solution

is also, and bounds for derivatives of the velocity on a time interval $0 \leqslant t \leqslant T$, $T$ arbitrary, can be obtained from bounds on the initial data (e.g., see Lemmas 3.1 and 3.2 of [3]). Our main result for two-dimensional flows is the following.

THEOREM 1 (CONVERGENCE IN TWO DIMENSIONS). *Assume that the velocity field $u(z, t)$ is sufficiently smooth for $z \in \mathbf{R}^2$, $0 \leqslant t \leqslant T$, and that the initial vorticity has bounded support. Also assume*

(i) *The cutoff $\psi$ belongs to $FeS^{-L,p}$ for some $L$, $p$ with $2 < L \leqslant \infty$ and $p \geqslant 2$.*

(ii) *We choose $\delta = h^q$, where $0 < q < 1$ if $L = \infty$ and $0 < q < (L-1)/(p+L)$ if $L < \infty$.*

*Then, with $1 < \mu < \infty$, we have the following estimates for the quantities computed by the vortex algorithm* (1.6)–(1.8):

(1) *Convergence of the particle paths*

$$\max_{0 \leqslant t \leqslant T} |\tilde{z}_j(t) - z_j(t)|_{L_h^\mu} \leqslant Ch^{pq}, \tag{1.10}$$

(2) *Convergence of the discrete velocity*

$$\max_{0 \leqslant t \leqslant T} |\tilde{u}_j^h(t) - u(z_j, t)|_{L_h^\mu} \leqslant Ch^{pq}, \tag{1.11}$$

(3) *Convergence of the continuous velocity*

$$\max_{0 \leqslant t \leqslant T} |\tilde{u}^h(\cdot, t) - u(\cdot, t)|_{L^\mu(B(R_0))} \leqslant Ch^{pq}. \tag{1.12}$$

*Here $R_0 > 0$ is an arbitrary finite radius. The constant $C$ depends on $T, L, p, q, \mu, R_0$, the diameter of* $\operatorname{supp} \omega_0$, *and bounds for a finite number of derivatives of the velocity field.*

Our analysis builds on the earlier convergence proof of Hald [7] in two dimensions, although the techniques used in the consistency argument given here are quite different from those in [7]. Hald's main result was essentially that, with $p = 4$ and $q = \frac{1}{2}$ in the notation above, the errors are of order $h^2$ in $L^2$ norms. His assumptions on $\psi$ are rather different, however, and do not require as much smoothness. Besides conditions (1) and (2) above, he assumes that $\psi$ has support in $\{|z| \leqslant 1\}$ and is $C^3$ for $|z| \leqslant 1$. As indicated by his work and our statement above, when $\psi$ is not very smooth, it seems to be necessary to take $\delta$ considerably larger than $h$, thereby reducing the accuracy. If, on the other hand, we use $\psi \in FeS^{-\infty,p}$, we can take $\delta$ essentially of the order of $h$ (i.e., $q$ near 1) and conclude that the errors are $O(h^{p-\varepsilon})$. Thus, with this class of cutoff functions, which are no more difficult to implement than others which have been used, we obtain substantial improvement in accuracy. With $p = 4$, as in Hald's case, we find essentially fourth order accuracy; with no moment conditions at all ($p = 2$ in the radially symmetric case) we have essentially second order accuracy. This last case includes the Gaussian distribution, showing that it is possible to obtain second order accuracy with a positive $\psi$. The fact that $\mu$ can be taken arbitrarily large in Theorem 1 means that the convergence is nearly uniform.

The smoothing of the velocity kernel has the effect of replacing point vortices by finite cores of vorticity. This technique has been used for some time [5], [11] to overcome instabilities that arise with point vortices. The numerical experiments of Hald and Del Prete [8] supported the conclusion that the error is of order $\delta^p = h^{pq}$

in the case $p = 2$; it would be interesting to try test problems such as theirs with higher order accuracy. Leonard [12] suggested the use of the generalized Gaussians (1.9) to obtain increased accuracy. An excellent survey of vortex methods, including extensive references, may be found in [12]. Additional discussion of the use of two-dimensional vortex methods was given in [2].

In discussing the consistency and convergence we use discrete velocity approximations as in (1.7), (1.8), determined by the exact positions of the particles in the actual flow,

$$u^h(z,t) = \sum_{jh \in \Lambda_0^h} K_\delta\big(z - z_j(t)\big)\omega_j h^2, \tag{1.13}$$

$$u_j^h(t) = \sum_{kh \in \Lambda_0^h} K_\delta\big(z_j(t) - z_k(t)\big)\omega_k h^2. \tag{1.14}$$

The consistency argument is independent of the space dimension, $N$, and we combine the two cases in the statement below. In three dimensions, the Biot-Savart Law expresses the velocity $u(z, t)$ in the form (1.1), where $\omega = \operatorname{curl} u$ is now a vector and $K$ is a $3 \times 3$-matrix-valued function; see (1.2), (1.3) of [1]. With $N = 3$, the definition of $u^h(z, t)$ corresponding to (1.13) above is (1.11) of [1]; the difference is that $\omega_j$ is replaced by $\omega(\Phi^t(z_j), t)$ and $h^2$ by $h^3$. The consistency statement is as follows.

CONSISTENCY LEMMA. *Assume the hypothesis of Theorem* 1 *above if* $N = 2$ *or the hypothesis of Theorem* 1 *of* [1] *if* $N = 3$. *Then for any* $R_0 > 0$ *we have*

$$\max_{\substack{|z| \leq R_0 \\ 0 \leq t \leq T}} |u^h(z,t) - u(z,t)| \leq Ch^{pq}. \tag{1.15}$$

For $N = 3$, this lemma completes the convergence proof of [1]; for $N = 2$, it is used in Section 5 to prove Theorem 1 above. In the rest of this section we discuss the estimate (1.15), which is derived in Sections 2–4. The relation between $h$ and $\delta$ expressed in the theorems comes about from balancing the errors in smoothing and in discretizing, which are somewhat opposite in character. The error due to smoothing is of the order of $\delta^p$, and is therefore worse for larger $\delta$. But the error in (1.15) from the discretization is improved by increasing $\delta$. (The stability estimate also requires that $\delta$ is at least of the order of $h$.)

To describe these two errors further, we express $u^h$ as $K_\delta * \omega^h$, where $*$ is convolution and

$$\omega^h(z,t) = \sum_{jh \in \Lambda_0^h} \delta\big(z - z_j(t)\big)\omega\big(z_j(t), t\big)h^N.$$

Here $\delta$ is the usual Dirac measure. (The double use of the letter $\delta$ should not cause confusion.) Then,

$$u^h - u = K_\delta * \omega^h - K * \omega = (K_\delta * \omega - K * \omega) + K_\delta * (\omega^h - \omega).$$

Since $K_\delta = K * \psi_\delta$, the first term is $K * (\psi_\delta * \omega - \omega)$. Applying the Fourier transform, we have

$$\hat{\psi}_\delta(\zeta) = \hat{\psi}(\delta\zeta) \tag{1.16}$$

and $|\hat{K}(\zeta)| \leq |\zeta|^{-1}$, so that

$$|(K_\delta * \omega - K * \omega)\hat{}(\zeta)| \leq |\zeta|^{-1} |\hat{\psi}(\delta\zeta) - 1| |\hat{\omega}(\zeta)| .$$

Now for $\psi \in S^{-L,p}$, $\hat{\psi}(\zeta) = O(|\zeta|^p)$ at $\zeta = 0$, and in fact $|\hat{\psi}(\delta\zeta) - 1| \leq C(\delta|\zeta|)^p$; see the proof of Lemma 4.1. If $\omega$ is sufficiently smooth, then $|\hat{\omega}(\zeta)| \leq C_R(1 + |\zeta|)^{-R}$, so that by taking $R$ large enough we have

$$|(K_\delta * \omega - K * \omega)\hat{}(\zeta)| \leq C\delta^p(1 + |\zeta|)^{-N-1}.$$

It follows that $K_\delta * \omega - K * \omega$ is uniformly of order $\delta^p$.

The second term, $K_\delta * (\omega^h - \omega)$ or $K * [\psi_\delta * (\omega^h - \omega)]$, is more subtle and easiest to discuss at time zero. Except for a negligible part, (see (4.2)–(4.6)), this term can be estimated (Lemma 2.2) by a Sobolev norm of $\psi_\delta * (\omega^h - \omega)$, i.e., the square root of

$$\int (1 + |\zeta|)^{2M} |\hat{\psi}_\delta(\zeta)|^2 |\hat{\omega}^h(\zeta) - \hat{\omega}(\zeta)|^2 \, d\zeta$$

for suitable $M$, or, using (1.16),

$$\int (1 + |\zeta|)^{2M} (1 + \delta|\zeta|)^{-2L} |\hat{f}_h(\zeta)|^2 \, d\zeta, \tag{1.17}$$

where $f_h = \omega^h - \omega$. To estimate this integral we reexpress $f^h$ using the Poisson summation formula (e.g., see [14, pp. 251–252]). Applying the formula to $\hat{\omega}$, we know that

$$\sum_{j \in \Lambda} \hat{\omega}(\zeta - 2\pi j/h) \tag{1.18}$$

has a Fourier series expansion whose coefficients are $\omega(jh)$. But then the inverse transform of (1.18) is just $\omega^h$, and $\hat{f}_h = (\omega^h)\hat{} - \hat{\omega}$ is the sum of (1.18) with the $j = 0$ term taken out. Using this representation we estimate the integral (1.17) in Proposition 3.2 and find that it is bounded by $C_\varepsilon \delta^{-2L} h^{2(L-M)-\varepsilon}$. Provided $\delta$ is of larger order than $h$, and $L$ is sufficiently large, this error improves as $h \to 0$.

For later time we verify that this second term has an error of the same character as for time zero. Again the important point is to estimate $\psi_\delta * (\omega^h - \omega)$ in a bounded region. Using $\Phi^t$ to change variables, we write this as

$$\int \psi_\delta(\Phi^t(\alpha) - \Phi^t(\tilde{\alpha})) f_h^t(\tilde{\alpha}) \, d\tilde{\alpha},$$

where now

$$f_h^t(\alpha) = \sum_j \delta(\alpha - jh)\omega(\Phi^t(jh), t)h^N.$$

Next, we express $\psi_\delta$ as the inverse transform of $\hat{\psi}$. The part of the integral away from the diagonal $\alpha = \tilde{\alpha}$ is negligible (Lemma 4.3), and for the rest we can change the $\zeta$ variable and rewrite the integral as

$$\iint e^{i(\alpha - \tilde{\alpha})\zeta} p_\delta(\alpha, \zeta, \tilde{\alpha}) f_h^t(\tilde{\alpha}) \, d\tilde{\alpha} \, d\zeta,$$

with some symbol $p_\delta$. We regard this integral as a certain pseudodifferential operator applied to $f_h^t$. By expressing the operator in a more standard form, we reduce the estimation of this term to the previous case (1.17). Only the most elementary facts

about pseudodifferential operators are needed, but we have to be careful to take into account the dependence on the parameter $\delta$ and the interaction with $f_h^t$ as $\delta$, $h \to 0$ (see Lemmas 2.4 and 4.4).

The necessary properties of pseudodifferential operators are developed in Section 2; the treatment is largely self-contained. The estimation of (1.17) is carried out in Section 3. As noted above, this essentially proves the consistency at time zero. In Section 4, we perform the reduction to time zero. The proofs of certain lemmas stated in Section 4 are deferred to an Appendix, in order to avoid interrupting the argument. Finally, in Section 5 we present the convergence proof in two dimensions.

**2. Some Facts About Integral and Pseudodifferential Operators.** First we introduce the Sobolev spaces $H^s(\mathbf{R}^N)$, $s$ any real number, with corresponding norm $\|\cdot\|_s$, defined by

$$\|g\|_s^2 = \int_{\mathbf{R}^N} \left(1 + |\zeta|^2\right)^s |\hat{g}(\zeta)|^2 \, d\zeta, \tag{2.1}$$

where $\hat{g}(\zeta)$ is the Fourier transform of $g$. When $s$ is a positive integer, we also use the fact that the norm defined in (2.1) is equivalent to the norm defined by the expression

$$\sum_{0 \leq |\beta| \leq s} \int_{\mathbf{R}^N} |D^\beta g|^2 \, dx. \tag{2.2}$$

For $s$ a positive integer, we denote the $H^s$ norm over an open set $\Omega$ by $\|\cdot\|_{s,\Omega}$, where the integration corresponding to (2.2) is over the set $\Omega$.

The first simple fact which we need below is the following one:

LEMMA 2.1. *Assume $\psi$ is a tempered distribution with $\hat{\psi}(\zeta) \in S_{1,0}^M$; then $\psi(z)$ is $C^\infty$ in $\mathbf{R}^N - \{0\}$, and, for any fixed $\varepsilon_0 > 0$ and indices $\gamma$, $\beta$ with $\beta > 0$,*

$$\sup_{|z| \geq \varepsilon_0} |z|^\beta |D_z^\gamma \psi(z)| \leq C_{\beta,\gamma}$$

*with a finite constant $C_{\beta,\gamma}$.*

The proof of Lemma 2.1 follows immediately from the identity

$$\left(i|z|^2\right)^{-1} \sum_{j=1}^N z_j \frac{\partial}{\partial \zeta_j} e^{iz\cdot\zeta} = e^{iz\cdot\zeta}$$

and repeated integration by parts in the Fourier integral formula

$$\psi(z) = \int e^{iz\cdot\zeta} \hat{\psi}(\zeta) \, d\zeta.$$

We also need to study convolution operators, which are given by

$$\mathcal{K}f = \int K(x-y) f(y) \, dy,$$

where, as we see from (1.2) and the Biot-Savart formula in Part I,

$$K \text{ is } C^\infty \text{ in } \mathbf{R}^N \sim \{0\} \text{ and homogeneous of degree } 1 - N. \tag{2.3}$$

We need the following simple variant of Lemma 3.8 in Part I (we omit the proof).

LEMMA 2.2. *Assume that $f(x)$ satisfies* $\operatorname{supp} f(x) \subseteq \{x \mid |x| \leq R'\}$, *$s$ is a nonnegative integer, and $\rho(x)$ belongs to $C_0^\infty(\mathbf{R}^N)$. Then there is a constant $C_\rho$ so that*

$$\|\rho \mathcal{K} f\|_{s+1} \leq C_\rho \|f\|_{s,R'}.$$

In Section 4, we need some elementary facts regarding pseudodifferential operators and related kernels. We need to consider multiple symbols $p_\delta(\alpha, \zeta, \tilde{\alpha})$ with $(\alpha, \zeta, \tilde{\alpha}) \in \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N$ and depending on a parameter $\delta$, $0 < \delta \leq 1$.

*Definition.* The family of multiple symbols $p_\delta(\alpha, \zeta, \tilde{\alpha})$ is *uniformly bounded* in the symbol class $S_{1,0}^M$, provided for any multi-index $(\beta, \gamma, \tilde{\beta})$ there is a constant $C_{\beta,\gamma,\tilde{\beta}}$ independent of $\delta$ so that

$$|D_\alpha^\beta D_{\tilde{\alpha}}^{\tilde{\beta}} D_\zeta^\gamma p_\delta| \leq C_{\beta,\gamma,\tilde{\beta}} (1 + |\zeta|)^{M-|\gamma|}. \tag{2.4}$$

Such a family of symbols $p_\delta(\alpha, \zeta, \tilde{\alpha})$ is properly supported provided that $p_\delta(\alpha, \zeta, \tilde{\alpha})$ vanishes for $|\alpha - \tilde{\alpha}| > \varepsilon_0$ for some fixed $\varepsilon_0 > 0$. Associated with a properly supported multiple symbol is a related operator $P_\delta$ defined by

$$P_\delta f = \iint_{\mathbf{R}^N \times \mathbf{R}^N} e^{i(\alpha - \tilde{\alpha}) \cdot \zeta} p_\delta(\alpha, \zeta, \tilde{\alpha}) f(\tilde{\alpha})\, d\tilde{\alpha}\, d\zeta. \tag{2.5}$$

(See Chapter 2 of [15] or pages 102–109 of [10] for a complete discussion.) Given a symbol $q(\alpha, \zeta) \in S_{1,0}^{-L}$, with no $\tilde{\alpha}$ dependence, a classical pseudodifferential operator is defined by

$$q(\alpha, D_x) f = \int e^{i\alpha \cdot \zeta} q(\alpha, \zeta) \hat{f}(\zeta)\, d\zeta. \tag{2.6}$$

The main fact regarding the operators in (2.5) defined by multiple symbols which we use in Section 4 is that such operators can be expanded as a finite sum of classical pseudodifferential operators, as defined in (2.6), within a remainder operator of negative order. More precisely, the following proposition is proved in pages 102–107 of [10] or Section 2 of [15]. (One only needs to remark that these proofs given for a single operator remain valid for families of operators defined by uniformly bounded symbols.)

LEMMA 2.3. *Consider a family of properly supported multiple symbols $p_\delta(\alpha, \zeta, \tilde{\alpha})$, uniformly bounded in $S_{1,0}^M$, $M$ any number, with corresponding operators $P_\delta$ defined in* (2.5). *Let $R$ be an arbitrary integer. Then there are symbols $p_\delta^\gamma(\alpha, \zeta)$ uniformly bounded in $S_{1,0}^{M-\gamma}$, a remainder operator $\mathcal{R}_\delta$, and an integer $K(R)$ so that*

$$P_\delta f = \sum_{-K \leq \gamma \leq M} p_\delta^\gamma(\alpha, D_x) f + \mathcal{R}_\delta f, \tag{2.7}$$

*where $\mathcal{R}_\delta$ has order $-R$, i.e.,*

$$\|\mathcal{R}_\delta f\|_0 \leq C_R \|f\|_{-R} \tag{2.8}$$

*for a fixed constant $C_R$ independent of $\delta$. In fact, $p_\delta^{M-\gamma}(\alpha, \zeta)$ can be computed by the formula*

$$p_\delta^{M-\gamma}(\alpha, \zeta) = \sum_{|\tilde{\gamma}| = \gamma} \frac{1}{\tilde{\gamma}!} D_{\tilde{\alpha}}^{\tilde{\gamma}} D_\zeta^{\tilde{\gamma}} p_\delta(\alpha, \zeta, \tilde{\alpha}) \Big|_{\alpha = \tilde{\alpha}}. \tag{2.9}$$

The last general fact regarding distributions related to pseudodifferential operators which we need in Section 4 is the following statement.

LEMMA 2.4. *Consider a function $Q_\delta(\alpha, \zeta)$ satisfying the condition*

$$\sum_{|\beta| \leq M} \int_{\mathbf{R}^N} | D_\alpha^\beta Q_\delta(\alpha, \zeta) | \, d\alpha \leq g_\delta(\zeta) \tag{2.10}$$

*for some $M > N$ with $g_\delta(\zeta) \in L^2(\mathbf{R}^N)$. Define a function $w(\alpha)$ by the formula*

$$w(\alpha) = \int_{\mathbf{R}^N} e^{i\alpha \cdot \zeta} Q_\delta(\alpha, \zeta) \, d\zeta. \tag{2.11}$$

*Then $w(\alpha) \in L^2(\mathbf{R}^N)$ and*

$$\| w \|_0 \leq C \| g_\delta(\zeta) \|_0, \tag{2.12}$$

*where $C$ is independent of $\delta$.*

We prove Lemma 2.4 in the following standard fashion. We define $\hat{Q}_\delta(\eta, \zeta)$ by

$$\hat{Q}_\delta(\eta, \zeta) = \int e^{-i\eta \cdot \alpha} Q_\delta(\alpha, \zeta) \, d\alpha.$$

The condition (2.10) implies that for any multi-index $\beta$ with $|\beta| \leq M$

$$| \eta^\beta | \, | \hat{Q}_\delta(\eta, \zeta) | = \left| \int e^{i\eta \cdot \alpha} D_\alpha^\beta Q_\delta(\alpha, \zeta) \, d\alpha \right| \leq g_\delta(\zeta)$$

and, therefore,

$$| \hat{Q}_\delta(\eta, \zeta) | \leq C(1 + |\eta|)^{-M} g_\delta(\zeta).$$

If $w$ is defined as in (2.11), $\hat{w}(\eta)$ satisfies

$$| \hat{w}(\eta) | \leq \int | \hat{Q}_\delta(\eta - \zeta, \zeta) | d\zeta \leq C \int (1 + |\eta - \zeta|)^{-M} | g_\delta(\zeta) | \, d\zeta.$$

Since $M > N$ and the convolution of an $L^1$ and $L^2$ function is in $L^2$, we have

$$\| w \|_0 = \| \hat{w} \|_0 \leq C \| (1 + |\zeta|)^{-M} \|_{L^1} \| g_\delta(\zeta) \|_0 \leq C \| g_\delta \|_0.$$

**3. The Error in Approximation at Time Zero.** In this section, our main proposition estimates the difference in the approximation of the vorticity by discrete and continuous convolutions with $\psi_\delta$ at time $t = 0$ where $\psi \in FeS_{1,0}^{-L}$. More precisely, we consider the function

$$F_{h,\delta}(z) \equiv \sum_{jh \in \Lambda_0^h} \psi_\delta(z - z_j) \omega(jh) h^N - \int \psi_\delta(z - \tilde{z}) \omega(\tilde{z}) \, d\tilde{z} \tag{3.1}$$

for a given smooth function $\omega(z)$ with compact support and measure $F_{h,\delta}(z)$ in positively indexed Sobolev spaces. In the proof in Section 4, we reduce the main error in the consistency at later times to a finite sum of error terms like the one on the right-hand side of (3.5) below which estimates $\| F_{h,\delta}(z) \|_M$.

First, we introduce the distribution

$$f_h(\alpha) = \sum_{jh \in \Lambda^h} \delta(\alpha - jh) \omega(jh) h^N - \omega(\alpha), \tag{3.2}$$

where $\delta(\alpha - jh)$ is the Dirac measure centered at $jh$. By Poisson's summation formula, as discussed in the Introduction, the Fourier transform of $f_h(\alpha)$ is given by

$$\hat{f}_h(\zeta) = \sum_{\substack{j \in \Lambda \\ |j|>0}} \hat{\omega}\left(\zeta - \frac{2\pi j}{h}\right). \tag{3.3}$$

Since $\hat{\psi}(\zeta)$ belongs to $S_{1,0}^{-L}$, we have

$$|\hat{\psi}(\delta\zeta)| \leqslant C_1(1 + |\delta\zeta|)^{-L}. \tag{3.4}$$

Thus, since $F_{h,\delta}(z)$ is the convolution of $\psi_\delta$ with $f_h$, we obtain, using the definition (2.1),

$$\|F_{h,\delta}(z)\|_M^2 \leqslant C \int (1 + |\zeta|^2)^M (1 + |\delta\zeta|)^{-2L} |\hat{f}_h(\zeta)|^2 \, d\zeta. \tag{3.5}$$

As remarked above, we prove in Section 4 that the main error in consistency at later time can be bounded by the right-hand side of (3.5) with $M = [N/2]$. The two facts which we prove in this section are the following ones:

PROPOSITION 3.1. *For a given $M \geqslant 0$, assume that $L$ satisfies $L > M + N/2$ and also that $\delta$ satisfies $\delta \geqslant C_0 h$, $C_0 > 0$. Then for any $\varepsilon > 0$, fixed but arbitrarily small,*

$$\int (1 + |\zeta|^2)^M (1 + |\delta\zeta|)^{-2L} |\hat{f}_h(\zeta)|^2 \, d\zeta \leqslant C_\varepsilon \delta^{-2L} h^{2(L-M)-\varepsilon}. \tag{3.6}$$

*Here the constant $C$ depends on $\varepsilon$ and a fixed finite number of derivatives of $\omega$. This number of derivatives also depends on $\varepsilon$.*

PROPOSITION 3.2. *Assume that $M$ satisfies $M < -N/2$; then*

$$\int (1 + |\zeta|)^{2M} |\hat{f}_h(\zeta)|^2 \, d\zeta \leqslant C_M h^{-2M-N},$$

*where $C_M$ involves finitely many derivatives of $\omega$.*

*Remark* 1. In Section 4, we use Proposition 3.1 with $M = [N/2]$ and Proposition 3.2 with $M \ll 0$.

*Remark* 2. Proposition 3.1 indicates the necessity of choosing $\delta = h^q$, $q < 1$, in the basic vortex method. After examination of the proof below, it will be evident that

$$Ch^{-2M} \leqslant \int (1 + |\zeta|^2)^M (1 + |\delta\zeta|)^{-2L} |\hat{f}_h(\zeta)|^2 \, d\zeta$$

for appropriate functions $\omega(\alpha)$ when $\delta \equiv C_0 h$, so there is a substantial error in approximation in nonnegative Sobolev norms unless $\delta = h^q$, $q < 1$.

The proof of Proposition 3.2 mimics the proof of Proposition 3.1 but is simpler, so we concentrate on the *proof of Proposition* 3.1:

To begin the argument, for simplicity, we assume that $\hat{\omega}(z)$ is rapidly decreasing. Thus, for any positive integer $q$, we have

$$|\hat{\omega}(\zeta)| \leqslant C_q(1 + |\zeta|)^{-q}. \tag{3.7}$$

The proofs below require only a finite number of constants $C_q$ and, therefore, only a fixed finite number of derivatives of $\omega$. We introduce the sets $R_k$, $k \in \Lambda$, where

$$R_k = \left\{ \zeta \,\middle|\, \left| \zeta_i - \frac{2\pi k_i}{h} \right| \leqslant \frac{\pi}{h},\ 1 \leqslant i \leqslant N \right\}.$$

By utilizing (3.3) and the triangle inequality, we estimate the expression on the right-hand side of (3.5) by twice the sum of two different groups of terms,

$$\begin{aligned} &\left\{ \sum_{k \in \Lambda} \int_{R_k} (1 + |\zeta|)^{2M} (1 + |\delta\zeta|)^{-2L} \left| \sum_{\substack{|j-k|>0 \\ j \in \Lambda}} \hat{\omega}\left(\zeta - \frac{2\pi j}{h}\right) \right|^2 d\zeta \right\} \\ &\quad + \left\{ \sum_{\substack{k \in \Lambda \\ |k|>0}} \int_{R_k} |\zeta|^{2M} (1 + |\delta\zeta|)^{-2L} \left| \hat{\omega}\left(\zeta - \frac{2\pi k}{h}\right) \right|^2 d\zeta \right\} \equiv \{1\} + \{2\}. \end{aligned} \tag{3.8}$$

Our first objective is to deduce that the terms in $\{1\}$ satisfy

$$\{1\} \leqslant C_R h^R \quad \text{for any } R > 0.$$

The following elementary lower bound is crucial for the argument:

(3.9) There is a fixed constant $C_0 > 0$ so that if $\zeta \in R_k$ and $|j - k| > 0$, $|\zeta - 2\pi j/h| \geqslant C_0 h^{-1} |j - k|$.

From (3.7) and (3.9), we deduce that, for $\zeta \in R_k$,

$$\begin{aligned} \left| \sum_{|j-k|>0} \hat{\omega}\left(\zeta - \frac{2\pi j}{h}\right) \right| &\leqslant C_q \sum_{|j-k|>0} \left| \zeta - \frac{2\pi j}{h} \right|^{-q} \\ &\leqslant C_q h^q \sum_{|j|>0} |j|^{-q} \leqslant C_q h^q \end{aligned} \tag{3.10}$$

provided that $q$ satisfies $q > N$; this requirement on $q$ guarantees that $\Sigma_{|j|>0} |j|^{-q} < \infty$. From (3.8) and (3.10), we see that the term in $\{1\}$ is bounded by

$$h^{2q} C_q \int (1 + |\zeta|)^{2M} (1 + |\delta\zeta|)^{-2L} \, d\zeta.$$

Since $M \geqslant 0$, the above expression is estimated by

$$\begin{aligned} &h^{2q} C_q \left( 1 + \int_{|\zeta| \geqslant 1} |\zeta|^{2M} (1 + |\delta\zeta|)^{-2L} \, d\zeta \right) \\ &\quad \leqslant h^{2q} C_q \left( 1 + \delta^{-2L-N} \int_{\mathbf{R}^N} (|\zeta| + 1)^{2M-2L} \, d\zeta \right), \end{aligned} \tag{3.11}$$

and the last integral is convergent provided $L$ satisfies $L > M + N/2$. Since we have $\delta \geqslant C_0 h$, the right-hand side of the last inequality in (3.11) is estimated by

$$C_q h^{2(q-L)-N} = C_q h^{2R} \quad \text{for any } R > 0,$$

by choosing $q$ large enough.

Next we estimate the principal error term {2} in (3.8). Since $1 + |\delta\zeta| \geqslant |\delta\zeta|$ and $|\zeta| \geqslant \pi/h$, this error term is dominated by

$$\delta^{-2L} \sum_{\substack{|k|\neq 0\\ k\in\Lambda}} \int_{R_k} |\zeta|^{2(M-L)} \left|\hat{\omega}\left(\zeta - \frac{2\pi k}{h}\right)\right|^2 d\zeta. \tag{3.12}$$

For each $k$, we split the integral over $R_k$ into integrals over two disjoint sets,

$$R_k^1 = \left\{\zeta \in R_k \,\middle|\, \left|\zeta - \frac{2\pi k}{h}\right| > h^{-\theta}\right\},$$

$$R_k^2 = \left\{\zeta \,\middle|\, \left|\zeta - \frac{2\pi k}{h}\right| \leqslant h^{-\theta}\right\},$$

where $\theta$, $0 < \theta < 1$, is arbitrarily small but fixed. Because $\hat{\omega}(\zeta)$ is rapidly decreasing, the estimate in (3.7) applies, and we have

$$\left|\hat{\omega}\left(\zeta - \frac{2\pi k}{h}\right)\right|^2 \leqslant C_q \left|\zeta - \frac{2\pi k}{h}\right|^{-2q} \leqslant C_q h^{2q\theta} \quad \text{for } \zeta \in R_k^1. \tag{3.13}$$

Recall that $L$ satisfies $L > M + N/2$; therefore

$$\begin{aligned} &\delta^{-2L} \sum_{\substack{|k|\neq 0\\ k\in\Lambda}} \int_{R_k^1} |\zeta|^{2(M-L)} \left|\hat{\omega}\left(\zeta - \frac{2\pi k}{h}\right)\right|^2 d\zeta \\ &\qquad \leqslant C_q \delta^{-2L} h^{2q\theta} \int_{|\zeta|\geqslant \pi/h} |\zeta|^{2(M-L)}\, d\zeta \leqslant C_q \delta^{-2L} h^{2q\theta + 2(L-M) - N}. \end{aligned} \tag{3.14}$$

Since $M$, $L$ are both fixed, $\delta \geqslant C_0 h$, and $q$ can be chosen arbitrarily large, the last term in (3.14) above is $O(h^R)$ for any $R > 0$. What remains in the proof of Proposition 3.1 is to estimate

$$\begin{aligned} &\sum_{|k|\neq 0} \delta^{-2L} \int_{|\zeta - 2\pi k/h| \leqslant h^{-\theta}} |\zeta|^{2(M-L)} \left|\hat{\omega}\left(\zeta - \frac{2\pi k}{h}\right)\right|^2 d\zeta \\ &\qquad \leqslant \sum_{|k|\neq 0} \delta^{-2L} C \int_{|\zeta - 2\pi k/h| \leqslant h^{-\theta}} |\zeta|^{2(M-L)}\, d\zeta, \end{aligned} \tag{3.15}$$

and this term yields the principal error on the right-hand side of (3.6). By rescaling (3.15), we have

$$\begin{aligned} &\sum_{|k|\neq 0} \delta^{-2L} \int_{|\zeta - 2\pi k/h| \leqslant h^{-\theta}} |\zeta|^{2(M-L)}\, d\zeta \\ &\qquad \leqslant \delta^{-2L} h^{2(L-M)-N} \sum_{|k|\neq 0} \int_{|\tilde{\zeta} - 2\pi k| \leqslant h^{1-\theta}} |\tilde{\zeta}|^{2(M-L)}\, d\tilde{\zeta} \\ &\qquad \leqslant C\delta^{-2L} h^{2(L-M)-N\theta} \sum_{|k|\neq 0} |k|^{2(M-L)} \leqslant C\delta^{-2L} h^{2(L-M)-N\theta}. \end{aligned} \tag{3.16}$$

In deducing the last inequality above, we have used the condition $L > M + N/2$ which, as in (3.10) above, implies the convergence of the infinite series. By setting $\varepsilon = N\theta$, we observe that the estimate in (3.16) completes the proof of Proposition 3.1.

The *proof of Proposition* 3.2 is similar but simpler. We use the same splitting of terms into $\{1\}$ and $\{2\}$ in this case, where $L = 0$ but $M$ satisfies $M < -N/2$. For term $\{1\}$, we repeat the argument in (3.9)–(3.11), arriving at an estimate of the form

$$\{1\} \leqslant C_q h^{2q} \int (1 + |\zeta|)^{2M} d\zeta,$$

where $q$ is arbitrary. However, since $M < -N/2$, the above integral is convergent, so that

$$\{1\} \leqslant C_R h^R \quad \text{for any } R > 0.$$

For term $\{2\}$, this time we are very crude in our estimates and get directly,

$$\{2\} \leqslant C \int_{|\zeta| \geqslant C_0 h^{-1}} (1 + |\zeta|)^{2M} d\zeta \leqslant C h^{-2M-N}$$

as required in Proposition 3.2. This completes the proof.

**4. Reduction to Time Zero and the Proof of Consistency.** Here we give the proof of the main consistency estimate. Besides the facts in Sections 2 and 3, we use several other technical lemmas which are proved in the Appendix. Before beginning the proof, we pick a function $\rho_1(z) \in C_0^\infty(\mathbf{R}^N)$ with $\rho_1(z) \equiv 1$ for $|z| \leqslant 1$ and $\rho_1(z) \equiv 0$ for $|z| > 2$. Below, $\rho_R(z)$ is the smooth cutoff function $\rho_R(z) = \rho_1(z/R)$.

We begin the proof by applying the Sobolev lemma,

$$\max_{\substack{|z| \leqslant R \\ 0 \leqslant t \leqslant T}} |u^h(z,t) - u(z,t)| \leqslant C \max_{0 \leqslant t \leqslant T} \|\rho_R(z)(u^h(z,t) - u(z,t))\|_{s_0} \tag{4.1}$$

with $s_0 = [N/2] + 1$, i.e., $s_0 = 2$ for $N = 2, 3$. Since the initial vorticity has compact support, we assume that

$$\operatorname{supp} \omega(z,t) \subseteq \{z \mid |z| \leqslant R_0\}, \qquad 0 \leqslant t \leqslant T,$$

for some fixed $R_0$ and that $R$ above satisfies $R \geqslant 2R_0$. We write $\omega_j(t) = \omega(z_j(t), t)$ and

$$\begin{aligned} &\rho_R(z)(u^h(z,t) - u(z,t)) \\ &\quad = \left\{ \rho_R(z) \int K(z - \tilde{z}) \rho_{5R}(\tilde{z}) \Big( \sum_{jh \in \Lambda_0^h} \psi_\delta(\tilde{z} - z_j) \omega_j h^N - \omega(\tilde{z}, t) \Big) d\tilde{z} \right\} \\ &\qquad + \left\{ \rho_R(z) \int K(z - \tilde{z})(1 - \rho_{5R}(\tilde{z})) \Big( \sum_{jh \in \Lambda_0^h} \psi_\delta(\tilde{z} - z_j) \omega_j(t) h^N \Big) \right\} \\ &\quad \equiv \{A_\delta\} + \{B_\delta\}. \end{aligned} \tag{4.2}$$

We claim that the second term $\{B_\delta\}$ defined in (4.2) is a negligible error term, i.e.,

$$\|B_\delta\|_{s_0} \leqslant C_r \delta^r \quad \text{for any positive integer } r > 0. \tag{4.3}$$

On the support of $\rho_{2R}(z)(1 - \rho_{5R}(\tilde{z}))$, necessarily $|z - \tilde{z}| \geqslant R$; because $K$ is smooth in $\mathbf{R}^N \sim \{0\}$ and homogeneous of degree $1 - N$, we have

$$\sum_{|\beta| \leqslant s_0} |D^\beta K(z - \tilde{z})| \leqslant C \quad \text{for } |z - \tilde{z}| \geqslant R. \tag{4.4}$$

Also, since $|z_j| \leqslant R$ and $\psi_\delta(\tilde{z} - z_j) = \delta^{-N}\psi((\tilde{z} - z_j)/\delta)$, from Lemma 2.1,

$$
\begin{aligned}
&(1 - \rho_{5R}(\tilde{z}))\Big(\sum_{jh \in \Lambda_0^h} \psi_\delta(\tilde{z} - z_j)\omega_j(t)h^N\Big) \\
&\qquad \leqslant C \sup|\omega|\,|1 - \rho_{5R}(\tilde{z})| \sup_{jh \in \Lambda_0^h}\left|\delta^{-N}\psi\left(\frac{\tilde{z} - z_j}{\delta}\right)\right| \\
&\qquad \leqslant C\delta^{-N} \max_{jh \in \Lambda_0^h}\left|\frac{\tilde{z} - z_j}{\delta}\right|^{-s} |1 - \rho_{5R}(\tilde{z})| \\
&\qquad \leqslant C\delta^{-N+s}(1 + |\tilde{z}|)^{-s}
\end{aligned}
\tag{4.5}
$$

for any positive integer $s > 0$. Thus, by (4.4) and (4.5),

$$
\begin{aligned}
&\sum_{|\alpha| \leqslant s_0} \sup\left|\rho_{2R}(z)\int D^\alpha K(z - \tilde{z})(1 - \rho_{5R}(\tilde{z}))\Big(\sum_{jh \in \Lambda_0^h} \psi_\delta(\tilde{z} - z_j)\omega_j h^N\Big)\right| \\
&\qquad\qquad \leqslant \delta^{-N+s}C\int(1 + |\tilde{z}|)^{-s}\,d\tilde{z} \leqslant C\delta^{-N+s}
\end{aligned}
\tag{4.6}
$$

provided that $s$ satisfies $s > N$ but otherwise $s$ is arbitrarily large. The estimate in (4.6) easily implies the estimate claimed in (4.3) by setting $r = -N + s$.

From the above argument, we only need to concentrate on the principal error term $A_\delta$ defined in (4.2). By applying Lemma 2.2, we observe that

$$
\|A_\delta\|_{s_0} \leqslant C\left\|\rho_{5R}(z)\Big(\sum_{jh \in \Lambda_0^h} \psi_\delta(z - z_j)\omega_j h^N - \omega(z, t)\Big)\right\|_{s_0 - 1}. \tag{4.7}
$$

The following straightforward estimate involves the moment condition on $\psi$, as mentioned in the Introduction, and is proved in the Appendix.

LEMMA 4.1. *If $\psi$ belongs to $FeS_{1,0}^{-L,p}$, $L \geqslant 0$, and $\omega$ is sufficiently smooth, then*

$$
\max_{0 \leqslant t \leqslant T}\left\|\int \psi_\delta(z - \tilde{z})\omega(\tilde{z}, t)\,d\tilde{z} - \omega(z, t)\right\|_{s_0 - 1} \leqslant C\delta^P.
$$

Because of Lemma 4.1, the right-hand side of (4.7) is estimated by

$$
C\delta^P + C\|G_\delta\|_{s_0 - 1}, \tag{4.8}
$$

with

$$
G_\delta(z, t) = \rho_{5R}(z)\left(\sum_{jh \in \Lambda_0^h} \psi_\delta(z - z_j)\omega_j(t)h^N - \int \psi_\delta(z - \tilde{z})\omega(\tilde{z}, t)\,d\tilde{z}\right). \tag{4.9}
$$

Since $\Phi^t(\alpha)$ is a one-parameter family of smooth diffeomorphisms, there is a fixed constant $C$ so that

$$
\|G_\delta(z, t)\|_{s_0 - 1} \leqslant C\|G_\delta(\Phi^t(\alpha), t)\|_{s_0 - 1}, \qquad 0 \leqslant t \leqslant T. \tag{4.10}
$$

The continuous contribution to $G_\delta(\Phi^t(\alpha), t)$ is given by

$$
\rho_{5R}(\Phi^t(\alpha))\int \psi_\delta(\Phi^t(\alpha) - \tilde{z})\omega(\tilde{z}, t)\,d\tilde{z}. \tag{4.11}
$$

If we change variables inside the integral in (4.11) using $\tilde{z} = \Phi^t(\tilde{\alpha})$ and $\det \nabla_\alpha \Phi^t \equiv 1$, the integral is equal to

$$\int \psi_\delta(\Phi^t(\alpha) - \Phi^t(\tilde{\alpha}))\omega(\Phi^t(\tilde{\alpha}), t)\, d\tilde{\alpha}. \tag{4.12}$$

Similarly, the pullback under the map $\Phi^t$ of the Dirac measure centered at $z_j$ is the Dirac measure $\delta(\tilde{\alpha} - jh)$. Therefore, following (3.2), we introduce the measure $f_h^t(\alpha)$ defined by

$$f_h^t(\alpha) = \sum_{jh \in \Lambda_0^h} \delta(\alpha - jh)\omega(\Phi^t(jh), t)h^N - \omega(\Phi^t(\alpha), t). \tag{4.13}$$

From (4.11)–(4.13), we deduce the important identity

$$G_\delta(\Phi^t(\alpha), t) = \rho_{5R}(\Phi^t(\alpha)) \int_{\mathbf{R}^N} \psi_\delta(\Phi^t(\alpha) - \Phi^t(\tilde{\alpha}))f_h^t(\tilde{\alpha})\, d\tilde{\alpha}. \tag{4.14}$$

To estimate the term on the right-hand side of (4.14), we use the following elementary lemma discussed in the Appendix and due to Kuranishi.

LEMMA 4.2. *There is an $\varepsilon_0 > 0$ and a smoothly varying invertible $N \times N$ matrix function, $e^t(\alpha, \tilde{\alpha})$ defined for $|\alpha - \tilde{\alpha}| \leqslant 6\varepsilon_0$ and $0 \leqslant t \leqslant T$ so that, for any vector $\zeta \in \mathbf{R}^N$,*

(1) $(\Phi^t(\alpha) - \Phi^t(\tilde{\alpha})) \cdot \zeta = (\alpha - \tilde{\alpha}) \cdot ((e^t(\alpha, \tilde{\alpha}))^{-1}\zeta)$;

(2) $C^{-1}|\zeta| \leqslant |e^t(\alpha, \tilde{\alpha})\zeta| \leqslant C|\zeta|$ *for* $0 \leqslant t \leqslant T$, $|\alpha - \tilde{\alpha}| \leqslant 4\varepsilon_0$, *and for a fixed positive constant $C$.*

We introduce the cutoff function $\rho_{\varepsilon_0}(\alpha - \tilde{\alpha})$ and write

$$\begin{aligned} &\int \psi_\delta(\Phi^t(\alpha) - \Phi^t(\tilde{\alpha}))f_h^t(\tilde{\alpha})\, d\tilde{\alpha} \\ &\quad = \left\{\int \rho_{\varepsilon_0}(\alpha - \tilde{\alpha})\psi_\delta(\Phi^t(\alpha) - \Phi^t(\tilde{\alpha}))f_h^t(\tilde{\alpha})\, d\tilde{a}\right\} \\ &\qquad + \left\{\int (1 - \rho_{\varepsilon_0}(\alpha - \tilde{\alpha})\psi_\delta(\Phi^t(\alpha) - \Phi^t(\tilde{\alpha}))f_h^t(\tilde{\alpha})\, d\tilde{\alpha}\right\} \\ &\quad \equiv \{E_\delta\} + \{F_\delta\}. \end{aligned} \tag{4.15}$$

The contribution to $\|G(\Phi^t(\alpha), t)\|_{s_0-1}$ from $\{F_\delta\}$ is negligible and satisfies an error estimate like $B_\delta$ in (4.3). To emphasize the main points of our argument, we state this fact in the lemma below and postpone the proof until the Appendix.

LEMMA 4.3. *For any positive integer $r$ and $0 \leqslant t \leqslant T$*

$$\|\rho_{5R}(\Phi^t(\alpha))F_\delta\|_{s_0-1} \leqslant C_r\delta^r.$$

Thus, by Lemma 4.3,

$$\begin{aligned} \|G_\delta(\Phi^t(\alpha), t)\|_{s_0-1} &\leqslant C\|\rho_{5R}E_\delta\|_{s_0-1} + C_r\delta^r \\ &\leqslant C \sum_{0 \leqslant |j| \leqslant s_0-1} \Bigg\| \iint e^{i(\Phi^t(\alpha) - \Phi^t(\tilde{\alpha}))\cdot\zeta} \\ &\qquad \times \rho_{5R}(\Phi^t(\alpha))\rho_{2\varepsilon_0}(\alpha - \tilde{\alpha})\zeta^j\hat{\psi}(\delta\zeta)f_h^t(\tilde{\alpha})\, d\tilde{\alpha}\, d\zeta\Bigg\|_0 + C_r\delta^r. \end{aligned} \tag{4.16}$$

In the last line in (4.16) above, we have used the Fourier integral representation,

$$\psi_\delta(\Phi^t(\alpha) - \Phi^t(\tilde{\alpha})) = \int e^{i(\Phi^t(\alpha)-\Phi^t(\tilde{\alpha}))\cdot\zeta}\hat{\psi}(\delta\zeta)\,d\zeta.$$

To study the terms on the right-hand side of (4.16) we will successively apply Lemma 4.2 and then Lemma 2.3. First, we introduce the change of variables $\zeta = e^t(\alpha, \tilde{\alpha})\tilde{\zeta}$ for the operators defined on the right-hand side of (4.16). Then the $j$th integral above becomes $P_\delta f_h^t$, defined as

$$\begin{aligned}\iint e^{i(\alpha-\tilde{\alpha})\cdot\tilde{\zeta}}\rho_{5R}(\Phi^t(\alpha))\rho_{2\varepsilon_0}(\alpha-\tilde{\alpha})(e^t(\alpha,\tilde{\alpha})\tilde{\zeta})^j \\ \times\det e^t(\alpha,\tilde{\alpha})\hat{\psi}(\delta e^t(\alpha,\tilde{\alpha})\tilde{\zeta})f_h^t(\tilde{\alpha})\,d\tilde{\alpha}\,d\tilde{\zeta},\end{aligned} \tag{4.17}$$

where $P_\delta$ is the properly supported operator associated with the multiple symbol

$$\begin{aligned}p_\delta(\alpha,\zeta,\tilde{\alpha}) = \rho_{2\varepsilon_0}(\alpha-\tilde{\alpha})\rho_{5R}(\Phi^t(\alpha)) \\ \times(e^t(\alpha,\tilde{\alpha})\zeta)^j\hat{\psi}(\delta e^t(\alpha,\tilde{\alpha})\zeta)\det e^t(\alpha,\tilde{\alpha}).\end{aligned} \tag{4.18}$$

To apply Lemmas 2.3 and 2.4, we need the estimates in Lemma 4.4 below which are proved by direct computations in the Appendix.

LEMMA 4.4. *Assume that* $\hat{\psi} \in S_{1,0}^{-L}$; *then there are fixed constants* $C_{\beta,\gamma}$ *so that, with* $p_\delta$ *from* (4.18),

$$|D_\zeta^\gamma D_\alpha^{\beta_1} D_{\tilde{\alpha}}^{\beta_2} p_\delta(\alpha,\zeta,\tilde{\alpha})| \leqslant C_{\beta,\gamma}(1+|\delta\zeta|)^{-L}(1+|\zeta|)^{|j|-|\gamma|}.$$

Now, by Lemma 4.4, $p_\delta$ is a uniformly bounded family of multiple symbols in $S_{1,0}^{|j|}$ so, by Lemma 2.3,

$$P_\delta f_h^t = \sum_{-k\leqslant\gamma\leqslant|j|} p_\delta^\gamma(\alpha, D_x)f_h^t + \mathcal{R}_\delta f_h^t, \tag{4.19}$$

where given any $r > 0$, $K(r)$ is chosen so that

$$\|\mathcal{R}_\delta f_h^t\|_0 \leqslant C\|f_h^t\|_{-r}. \tag{4.20}$$

We next apply Proposition 3.2 to $f_h^t(\alpha)$ as defined in (4.13), but with $\omega(\alpha)$ of Section 3 replaced by $\omega(\Phi^t(\alpha), t)$; thus

$$\max_{0\leqslant t\leqslant T}\|f_h^t\|_{-r} \leqslant Ch^{r-N/2} \tag{4.21}$$

for any fixed arbitrarily large $r$. From (4.7), (4.8), and (4.15)–(4.21), it remains to analyze the principal error contribution to $\|E_\delta\|_{s_0-1}$ arising from the finite sum of terms with the form

$$\|p_\delta^\gamma(\alpha, D_x)f_h^t\|_0, \qquad -K(r) \leqslant \gamma \leqslant |j| \leqslant s_0 - 1, \tag{4.22}$$

where the symbol $p_\delta^\gamma(\alpha,\zeta)$ is computed from $p_\delta(\alpha,\zeta,\tilde{\alpha})$ in (4.18) by the formula in (2.9). We recall that $p_\delta^\gamma(\alpha, D_x)f_h^t$ is defined by the formula

$$p_\delta^\gamma(\alpha, D_x)f_h^t = \int e^{i\alpha\cdot\zeta}p_\delta^\gamma(\alpha,\zeta)f_h^t(\zeta)\,d\zeta. \tag{4.23}$$

Next, we will apply Lemma 2.4 with $Q_\delta(\alpha,\zeta) = p_\delta^\gamma(\alpha,\zeta)\hat{f}_h^t(\zeta)$, and this will reduce the principal error at time $t$ to the quantity estimated in Proposition 3.2 at time

$t = 0$. From Lemma 4.4 and the fact that $\rho_{5R}(\Phi^t(\alpha))$ has compact support, it follows immediately that

$$\sum_{|\beta| \leq s_0 - 1} \int | D_\alpha^\beta Q_\delta(\alpha, \zeta) | \, d\alpha$$
$$\leq C_\beta (1 + |\delta\zeta|)^{-L} (1 + |\zeta|)^{s_0 - 1} |\hat{f}_h^t(\zeta)| \equiv g_\delta(\zeta).$$

Thus, we apply Lemma 2.4 to estimate

$$\max_{0 \leq t \leq T} \| p_\delta^\gamma(\alpha, D_x) f_h^t \|_0^2 \leq C \int (1 + |\zeta|)^{2(s_0 - 1)} (1 + |\delta\zeta|)^{-2L} |\hat{f}_h^t(\zeta)|^2 \, d\zeta. \tag{4.24}$$

By applying Proposition 3.1 of Section 3 with $M \equiv s_0 - 1 = 1$ for $N = 2, 3$, we find that

$$\max_{0 \leq t \leq T} \| p_\delta^\gamma(\alpha, D_x) f_h^t \|_0 \leq C \delta^{-L} h^{L - 1 - \varepsilon}. \tag{4.25}$$

By adding together the error terms in (4.25) contributing to $E_\delta$ to those obtained earlier in this argument (especially see Lemma 4.1), we deduce from (4.1), (4.3), (4.8), and (4.16) that

$$\max_{\substack{0 \leq t \leq T \\ |z| \leq R}} | u(z, t) - u^h(z, t) | \leq C\delta^p + C\delta^{-L} h^{L - 1 - \varepsilon} + C_r \delta^r, \tag{4.26}$$

where $r$ can be arbitrarily large. We choose $\delta$ so that the first two terms in (4.26) have equal strength; this requires

$$\delta^p = \delta^{-L} h^{L - 1 - \varepsilon} \quad \text{or} \quad \delta = h^{(L - 1 - \varepsilon)/(p + L)}.$$

By choosing $r$ to be larger than $p$, we have finished the required error bound in the main consistency lemma.

**5. Proof of Convergence in Two Dimensions.** The proof of Theorem 1 is based on the consistency lemma whose proof was completed in Section 4, and on the following stability estimate for the discrete velocity approximation:

STABILITY LEMMA. *Assume the hypothesis of Theorem* 1. *Provided that*

$$\max_{\substack{jh \in \Lambda_0^h \\ 0 \leq t \leq T_*}} | \tilde{z}_j(t) - z_j(t) | \leq \delta \tag{5.1}$$

*for some* $T_* \leq T$, *we have for* $0 \leq t \leq T_*$ *the estimates*

$$| \tilde{u}_j^h(t) - u_j^h(t) |_{L_h^\mu} \leq C | \tilde{z}_j(t) - z_j(t) |_{L_h^\mu}, \tag{5.2}$$

$$| \tilde{u}^h(\cdot, t) - u^h(\cdot, t) |_{L^\mu(B(R_0))} \leq C | \tilde{z}_j(t) - z_j(t) |_{L_h^\mu}. \tag{5.3}$$

The constant $C$ in (5.2) and (5.3) has the same dependence as in Theorem 1; it is independent of $T_*$. We will first present the proof of Theorem 1, and then prove the Stability Lemma. To derive (1.10) we set $e_j = \tilde{z}_j - z_j$ and obtain, as in [1, Section 2] or in [7], a differential inequality for $| e_j |_{L_h^\mu}$. From the ordinary differential equations (1.6), (1.3), we have

$$\dot{e}_j(t) = \tilde{u}_j^h(t) - u(z_j, t) = \left[ \tilde{u}_j^h(t) - u_j^h(t) \right] + \left[ u^h(z_j, t) - u(z_j, t) \right].$$

The consistency estimate (1.15) asserts that the second term on the right is $O(\delta^p) = O(h^{pq})$, uniformly in $j$, and thus also in $L_h^\mu$ on the bounded set $\Lambda_0^h$. Assuming (5.1) holds, we can apply the stability estimate (5.2) to the first term and obtain

$$|\dot{e}_j(t)|_{L_h^\mu} \leqslant C_0\big(|e_j(t)|_{L_h^\mu} + h^{pq}\big) \tag{5.4}$$

for $0 \leqslant t \leqslant T_*$. Since $e_j(0) = 0$, it follows from this differential inequality that

$$|e_j(t)|_{L_h^\mu} \leqslant y(t), \qquad 0 \leqslant t \leqslant T_*,$$

where $y$ is the solution of $y' = C_0(y + h^{pq})$, $y(0) = 0$. (E.g., see [9, Section 1.6].) Therefore,

$$|e_j(t)|_{L_h^\mu} \leqslant C_1 h^{pq}, \tag{5.5}$$

as long as (5.1) holds. Here $C_1$ depends on $T$ but not on $T_*$.

To complete the proof of (1.10), we need to remove the restriction (5.1). To do this, we estimate $|e_j|$ uniformly in terms of the $L^\mu$ norm. We find

$$\max_j |e_j|^\mu h^2 \leqslant \big(|e_j|_{L_h^\mu}\big)^\mu \quad \text{or} \quad \max |e_j| \leqslant h^{-2/\mu} |e_j|_{L_h^\mu},$$

so that for $t \leqslant T_*$, by (5.5),

$$\max |e_j| \leqslant C_1 h^{pq-2/\mu}.$$

Now since the estimates (1.10)–(1.12) are over a fixed bounded set, there is no loss of generality in making $\mu$ as large as we wish. We will assume $\mu$ is large enough so that $pq - 2/\mu = q + \theta$ with $\theta > 0$. (Since $p \geqslant 2$, this is always possible. For $p \geqslant 4$, we could take $\mu = 2$ provided $q > 1/3$.) Then from above,

$$\max |e_j| \leqslant C_1 h^\theta h^q = C_1 h^\theta \delta \leqslant \delta/2,$$

for $h \leqslant (2C_1)^{1/\theta}$, as long as (5.1) holds. But this means that $\max |e_j|$ can never reach $\delta$, and it follows that (5.5) holds with $T_* = T$.

The first estimate (1.10) of Theorem 1 is now verified. The remaining two estimates are immediate consequences: for (1.11) we write

$$\tilde{u}_j^h(t) - u_j(t) = \big[\tilde{u}_j^h(t) - u_j^h(t)\big] + \big[u^h\big(z_j(t), t\big) - u\big(z_j(t), t\big)\big]$$

and apply the stability estimate (5.2) to the first term and the consistency estimate (1.15) to the second. Similarly

$$\tilde{u}^h(z, t) - u(z, t) = \big[\tilde{u}^h(z, t) - u^h(z, t)\big] + \big[u^h(z, t) - u(z, t)\big].$$

Using the continuous stability estimate (5.3) for the first term and the consistency for the second, we obtain (1.12). The completes the proof of convergence.

In proving the Stability Lemma, we will need a few estimates for the kernel $K_\delta$. Since $K * f = (\partial_2, -\partial_1)\Delta^{-1} f$, the convolution with $\nabla K$ is a second derivative of $\Delta^{-1}$, and we have, according to the Calderón-Zygmund inequality, $|\nabla K * f|_{L^\mu} \leqslant C|f|_{L^\mu}$ for $1 < \mu < \infty$. (E.g., see [4, pp. 224, 245–250].) Therefore we can estimate $\nabla(K_\delta * f) = \nabla(K * (\psi_\delta * f))$ by

$$|\nabla(K_\delta * f)|_{L^\mu} \leqslant C|\psi_\delta * f|_{L^\mu} \leqslant C'|\psi_\delta|_{L^1}|f|_{L^\mu}$$

or

$$|(\nabla K_\delta) * f|_{L^\mu} \leqslant C''|f|_{L^\mu}, \tag{5.6}$$

since $|\psi_\delta|_{L^1}$ is bounded independent of $\delta$. (For $\mu = 2$, (5.6) can be verified in a more elementary way by expressing the convolutions in the transform; see Lemma 3.7 of [1] or Lemma 8 of [7].)

Pointwise estimates for $K_\delta$ can be found directly, as in Lemma 5.1 of [1]. Making only dimensional changes in the argument given there, we have

$$|D^\beta K_\delta(z)| \leqslant C\delta^{1-|\beta|}, \qquad \text{all } z, \tag{5.7}$$

$$|D^\beta K_\delta(z)| \leqslant C|z|^{1-|\beta|}, \qquad |z| \geqslant \delta. \tag{5.8}$$

These pointwise bounds lead to integral estimates. If we apply (5.7) for $|z| \leqslant \delta$ and (5.8) for $|z| \geqslant \delta$, with $\beta = 0$, we obtain

$$\int_{|z| \leqslant R} |K_\delta(z)|\, dz \leqslant CR. \tag{5.9}$$

We will also need estimates for discrete approximations to the $L^1$ norm of $D^\beta K_\delta$ on a bounded set. With time $t$ fixed and $z_j = \Phi^t(jh)$, let

$$M_{jk}^{(l)} = \max_{\substack{|y| \leqslant C_0\delta \\ |\beta| = l}} |D^\beta K_\delta(z_j - z_k + y)|. \tag{5.10}$$

Then for $|z_j| \leqslant R$ we have

$$\sum_{|z_k| \leqslant R} M_{jk}^{(l)} h^2 \leqslant \begin{cases} C|\log \delta|, & l = 1, \\ C\delta^{-1}, & l = 2. \end{cases} \tag{5.11}$$

Here $C$ depends only on $C_0$, $R$, and bounds for the flow. The order of $\delta$ in (5.11) is the same in two or three space dimensions. For the derivation of (5.11), based on (5.7) and (5.8), see [1, Lemma 3.2] or [7, Lemma 5].

To verify (5.2), we estimate the difference $\tilde{u}_j^h - u_j^h$ in terms of $e_k = \tilde{z}_k - z_k$ under the assumption (5.1). We can write $\tilde{u}_j^h - u_j^h = v_j^{(1)} + v_j^{(2)}$, where

$$v_j^{(1)} = \sum_k \left[K_\delta(z_j - \tilde{z}_k) - K_\delta(z_j - z_k)\right]\omega_k h^2, \tag{5.12}$$

$$v_j^{(2)} = \sum_k \left[K_\delta(\tilde{z}_j - \tilde{z}_k) - K_\delta(z_j - \tilde{z}_k)\right]\omega_k h^2. \tag{5.13}$$

We begin by estimating (5.12). Applying the Mean Value Theorem along the line from $z_k$ to $\tilde{z}_k$, we have

$$v_j^{(1)} = \sum_k \nabla K_\delta(z_j - z_k + y_{jk}) \cdot e_k \omega_k h^2. \tag{5.14}$$

(We ignore the fact that $y_{jk}$ might depend on the component.) According to (5.1),

$$|y_{jk}| \leqslant \delta \tag{5.15}$$

for each $j$, $k$.

It will be convenient to regard $v_j^{(1)}$ and $e_k\omega_k$ as step functions on a partition naturally associated with $\{z_j\}$. With $j = (j_1, j_2)$ a pair of integers, let $Q_j$ be the square of side $h$ centered at $jh$. Then $\{Q_j : j \in \Lambda\}$ partitions the plane, and at later time $t$, the fluid particles beginning in $Q_j$ have evolved to a cell $B_j = \Phi^t(Q_j)$ containing $z_j$. Since the flow is smooth and area-preserving, each cell has area $h^2$ and diameter uniformly of order $h$, and $\{B_j : j \in \Lambda\}$ again partitions the plane.

Now let $S = \bigcup \{B_j : |jh| \leq R_0\}$, where $R_0$ is the radius of support of the initial vorticity. We define functions $v^{(1)}$ and $f$ on $S$ by setting $v^{(1)}(z) = v_j^{(1)}$ for $z \in B_j$, $f(z') = e_k \omega_k$ for $z' \in B_k$. Also let

$$\mathcal{K}(z, z') = DK_\delta(z_j - z_k + y_{jk}), \qquad z \in B_j, z' \in B_k.$$

Then (5.14) is equivalent to

$$v^{(1)}(z) = \int_S \mathcal{K}(z, z') f(z')\, dz', \qquad z \in S. \tag{5.16}$$

Moreover, since the $B_j$'s have area $h^2$,

$$|v^{(1)}|_{L^\mu(S)} = |v_j^{(1)}|_{L_h^\mu}, \qquad |f|_{L^\mu(S)} = |e_k \omega_k|_{L_h^\mu}.$$

Thus estimating (5.16) is equivalent to estimating (5.14).

It is natural to rewrite $\mathcal{K}$ as $\mathcal{K}_1(z, z') + \mathcal{K}_2(z, z')$, with

$$\mathcal{K}_1(z, z') = \nabla K_\delta(z - z'),$$
$$\mathcal{K}_2(z, z') = \nabla K_\delta(z_j - z_k + y_{jk}) - \nabla K_\delta(z - z').$$

Then the $\mathcal{K}_1$-term in (5.16) is just $(\nabla K_\delta * f)(z)$, and we have from (5.6), $|\mathcal{K}_1 f|_{L^\mu} \leq C|f|_{L^\mu}$ or, since $\omega_k$ is uniformly bounded, $|\mathcal{K}_1 f|_{L^\mu} \leq C|e_k|_{L_h^\mu}$. Using the Mean Value Theorem again, along with (5.15), we can estimate

$$|\mathcal{K}_2(z, z')| \leq M_{jk}^{(2)} \delta, \qquad z \in B_j, z' \in B_k, \tag{5.17}$$

with $M$ as in (5.10). It is well known that

$$|\mathcal{K}_2 f|_{L^\mu(S)} \leq \|\mathcal{K}_2\| \, |f|_{L^\mu(S)},$$

where $\|\mathcal{K}\|$ is the smallest number such that

$$\int_S |\mathcal{K}(z, y')|\, dy' \leq \|\mathcal{K}\|, \qquad \int_S |\mathcal{K}(y, z')|\, dy \leq \|\mathcal{K}\|, \tag{5.18}$$

for all $z, z' \in S$. (For a proof, see, e.g., [6, Section 0.C].) Using (5.17), we see that the integrals of (5.18) are estimated by the sums in (5.11) with $l = 2$. Thus

$$|\mathcal{K}_2 f|_{L^\mu} \leq C\delta \cdot \delta^{-1} |f|_{L^\mu} \leq C' |e_k|_{L_h^\mu}.$$

This finishes the estimation of $v_j^{(1)}$.

We may apply the Mean Value Theorem to write (5.13) as

$$v_j^{(2)} = \sum_k e_j \cdot \nabla K_\delta(z_j - z_k + y_{jk}) \omega_k h^2$$

with $y_{jk}$ again satisfying (5.15). Since $e_j$ factors out of the sum, we will know that $|v_j^{(2)}|_{L_h^\mu} \leq C|e_k|_{L_h^\mu}$, provided we check that

$$\left| \sum_k \nabla K_\delta(z_j - z_k + y_{jk}) \omega_k h^2 \right| \leq C$$

uniformly in $j$. With $j$ fixed, let $g(z') = \omega_k$ and $\mathcal{K}(z') = \nabla K_\delta(z_j - z_k + y_{jk})$ for $z' \in B_k$. Then the sum above is the same as

$$\int \mathcal{K}(z') g(z')\, dz'.$$

Now

$$|\nabla K_\delta(z_j - z_k + y_{jk}) - \nabla K_\delta(z_j - z')| \leqslant M_{jk}^{(2)}\delta$$

for $z' \in B_k$. Thus if we replace $\mathcal{K}(z')$ by $\nabla K_\delta(z_j - z')$ in the integral, we commit an error which, according to (5.11), is bounded by $C\delta \cdot \delta^{-1} \cdot \max|\omega_j| \leqslant C'$. Furthermore, in the same way we can replace $g(z')$ by $\omega(z')$ with an error bounded by $Ch|\log\delta|\max|D\omega| \leqslant C'$. The integral is now replaced by

$$\int_S \nabla K_\delta(z_j - z')\omega(z')\,dz' = \int_S K_\delta(z_j - z')\nabla\omega(z')\,dz'.$$

We estimate this by

$$\max|\nabla\omega| \int_S |K_\delta(z_j - z')|\,dz',$$

which, in view of (5.9), is bounded by a constant. This completes the proof of (5.2). The inequality (5.3) can be established by an argument very similar to the estimation of $v_j^{(1)}$ above.

**Appendix: Proofs of the Lemmas.** Here we give the proof of Lemmas 4.1–4.4, which we used in Section 4.

*Proof of Lemma* 4.1. By Plancherel's Theorem, we have

$$\text{(A-1)}\qquad \begin{aligned} &\max_{0\leqslant t\leqslant T}\left\|\int \psi_\delta(z-\tilde{z})\omega(\tilde{z},t)\,d\tilde{z} - \omega(z,t)\right\|_{s_0-1}^2 \\ &\qquad\leqslant \max_{0\leqslant t\leqslant T}\int\left(1+|\zeta|^2\right)^{s_0-1}|1-\hat{\psi}(\delta\zeta)|^2|\hat{\omega}(\zeta,t)|^2\,d\zeta. \end{aligned}$$

Since $\hat{\psi}$ belongs to $S_{1,0}^{-L,p}$, it follows that

$$\text{(A-2a)}\qquad \hat{\psi}(0) = \int \psi(z)\,dz = 1,$$

$$\text{(A-2b)}\qquad D_\zeta^\beta\hat{\psi}(0) = C_\beta\int z^\beta\psi(z)\,dz = 0 \quad \text{for } 1 \leqslant |\beta| \leqslant |p|-1,$$

$$\text{(A-2c)}\qquad \sum_{|\alpha|=p}|D^\alpha\hat{\psi}(\zeta)| \leqslant C_p(1+|\zeta|)^{-L-|p|}.$$

From (A-2) and Taylor's theorem with remainder,

$$\text{(A-3)}\qquad \begin{aligned} &|1-\hat{\psi}(\delta\zeta)| = |\hat{\psi}(0) - \hat{\psi}(\delta\zeta)| \\ &\qquad\leqslant C|\delta\zeta|^p \sup_{0\leqslant\theta\leqslant 1}\left|\sum_{|\beta|=p} D_\zeta^\beta\hat{\psi}(\theta\delta\zeta)\right| \leqslant C|\delta\zeta|^p. \end{aligned}$$

Using (A-3), we estimate the right-hand side of (A-1) by

$$\text{(A-4)}\qquad \delta^{2p}C\max_{0\leqslant t\leqslant T}\int\left(1+|\zeta|^2\right)^{s_0-1+p}|\hat{\omega}(\zeta,t)|^2\,d\zeta \leqslant C\delta^{2p}.$$

We have used the fact that the vorticity is sufficiently smooth for $0 \leqslant t \leqslant T$ in the last inequality. This finishes the proof of Lemma 4.1.

*Proof of Lemma* 4.2. We include the proof for completeness; it is given in a more general setting on pp. 102–109 of [10]. By Taylor's Theorem with remainder,

$$\text{(A-5)} \qquad \left(\Phi^t(\alpha) - \Phi^t(\tilde{\alpha})\right)_i = \sum_{j=1}^{N} D^t_{ij}(\alpha, \tilde{\alpha})(\alpha_j - \tilde{\alpha}_j),$$

where the $N \times N$ matrix $(D^t_{ij})$ satisfies $D^t_{ij}(\alpha, \tilde{\alpha})|_{\tilde{\alpha}=\alpha} = \nabla_\alpha \Phi^t(\alpha)$. Since $\det(\nabla_\alpha \Phi^t(\alpha)) = 1$, it follows by continuity that there is an $\varepsilon_0 > 0$ so that

$$\text{(A-6)} \qquad \det\left(D^t_{ij}(\alpha, \tilde{\alpha})\right) \geqslant \tfrac{1}{2} \quad \text{for } |\alpha - \tilde{\alpha}| \leqslant 6\varepsilon_0, 0 \leqslant t \leqslant T.$$

Call $e^t(\alpha, \tilde{\alpha})$ the transpose matrix of $(D^t_{ij}(\alpha, \tilde{\alpha}))$. Then, from (A-6), $e^t(\alpha, \tilde{\alpha})$ is uniformly invertible for $0 \leqslant t \leqslant T$, $|\alpha - \tilde{\alpha}| \leqslant 4\varepsilon_0$, and satisfies $C^{-1}|\zeta| \leqslant |e^t(\alpha, \tilde{\alpha})\zeta| \leqslant C|\zeta|$ as required in (2) of Lemma 4.2. Also, from (A-5) and the definition of $e^t(\alpha, \tilde{\alpha})$, for any $\zeta \in \mathbf{R}^n$,

$$\left(\Phi^t(\alpha) - \Phi^t(\tilde{\alpha})\right) \cdot \zeta = (\alpha, \tilde{\alpha}) \cdot \left({}^tD^t(\alpha, \tilde{\alpha})\zeta\right) \equiv (\alpha, \tilde{\alpha}) \cdot \left(e^t(\alpha, \tilde{\alpha})\zeta\right)$$

as required in (1) of Lemma 4.2.

*Proof of Lemma* 4.3. We recall that the term $F_\delta$ is given by

$$F_\delta = \int \left(1 - \rho_{\varepsilon_0}(\alpha - \tilde{\alpha})\right)\psi_\delta\left(\Phi^t(\alpha) - \Phi^t(\tilde{\alpha})\right) f^t_h(\tilde{\alpha})\, d\tilde{\alpha},$$

where $f^t_h(\tilde{\alpha})$ is defined in (4.13) above. In proving Lemma 4.3, our objective is to estimate

$$\text{(A-7)} \qquad \max_{0 \leqslant t \leqslant T} \|\rho_{5R}\left(\Phi^t(\alpha)\right) F_\delta\|_{s_0 - 1}.$$

First, we concentrate on the contribution to $F_\delta$ from the continuous portion of $f^t_h(\tilde{\alpha})$, more precisely,

$$\text{(A-8)} \qquad F^1_\delta = \int \left(1 - \rho_{\varepsilon_0}(\alpha, \tilde{\alpha})\right)\psi_\delta\left(\Phi^t(\alpha) - \Phi^t(\tilde{\alpha})\right)\omega^t(\tilde{\alpha})\, d\tilde{\alpha}$$

with $\omega^t(\alpha) = \omega(\Phi^t(\alpha), t)$. Because $\rho_{5R}(\Phi^t(\alpha))$ has compact support in $|\alpha| < R'$ for $0 \leqslant t \leqslant T$, we estimate

$$\text{(A-9)} \qquad \begin{aligned} &\max_{0 \leqslant t \leqslant T} \|\rho_{5R}\left(\Phi^t(\alpha)\right) F^1_\delta\|_{s_0 - 1} \\ &\leqslant C \sum_{|\beta| \leqslant s_0 - 1} \sup_{|\alpha| \leqslant 2R'} \left| \int \left(1 - \rho_{\varepsilon_0/2}\right) \delta^{-|\beta| - N} \psi^{(\beta)}\left(\frac{\Phi^t(\alpha) - \Phi^t(\tilde{\alpha})}{\delta}\right) \omega^t(\tilde{\alpha})\, d\tilde{\alpha} \right| \end{aligned}$$

with $\psi^{(\beta)}(z) = (D^\beta_z \psi)(z)$. Since the fluid velocity vanishes at infinity, $\Phi^t(\alpha)$ is a uniformly Lipschitz family of diffeomorphisms; therefore,

$$\text{(A-10)} \qquad C^{-1}|\alpha - \tilde{\alpha}| \leqslant |\Phi^t(\alpha) - \Phi^t(\tilde{\alpha})| \leqslant C|\alpha - \tilde{\alpha}|$$

for $0 \leqslant t \leqslant T$ and some positive constant $C$. From (A-10), we deduce that

$$\left|\frac{\Phi^t(\alpha) - \Phi^t(\tilde{\alpha})}{\delta}\right| \geqslant C^{-1}\varepsilon_0 \quad \text{for } |\alpha - \tilde{\alpha}| \geqslant \tfrac{1}{2}\varepsilon_0 \text{ and } \delta \leqslant 1,$$

so by applying Lemma 2.1 we have the key estimate, for any $s > 0$,

$$\begin{aligned}\delta^{-N-|\beta|}|1-\rho_{\varepsilon_0/2}(\alpha,\tilde\alpha)|\left|\psi^\beta\left(\frac{\Phi^t(\alpha)-\Phi^t(\tilde\alpha)}{\delta}\right)\right| \\ \leqslant C_{\beta,s}\delta^{-N-|\beta|+s}|1-\rho_{\varepsilon_0/2}(\alpha,\tilde\alpha)||\Phi^t(\alpha)-\Phi^t(\tilde\alpha)|^{-s} \\ \leqslant C_{\beta,s}\delta^{-N-|\beta|+2}(1+|\tilde\alpha-\alpha|)^{-s}.\end{aligned} \tag{A-11}$$

Here we have used (A-10) in the last inequality. Thus, the right-hand side of (A-9) is estimated by

$$\begin{aligned}C\delta^{-N-|\beta|+s}\sup_{0<t<T,\alpha}\int(1+|\alpha-\tilde\alpha|)^{-s}|\omega^t(\tilde\alpha)|\,d\tilde\alpha \\ \leqslant C\delta^{-N-|\beta|+s}\max_{0\leqslant t\leqslant T}|\omega^t(\alpha)|\leqslant C\delta^{-N-|\beta|+2},\end{aligned} \tag{A-12}$$

since $\omega^t$ has bounded support in $\alpha$. Therefore, the term on the left-hand side of (A-9) is dominated by

$$C\delta^{-N-|\beta|+s}\quad\text{for any } s>0,$$

so that by choosing $r = -N - \beta + s$, we have verified Lemma 4.3 for the contribution to $F_\delta$ from $F_\delta^1$. Given the inequalities (A-10) and (A-11), we need only repeat a simple discrete variant of the above argument (as we have already done in (4.5) and (4.6) above) to complete the proof of Lemma 4.3—we leave these straightforward details to the reader.

*Proof of Lemma* 4.4. From the form of $p_\delta$ in (4.18), it is evident that we only need to verify the bounds

$$|\rho_{2\varepsilon_0}(\alpha,\tilde\alpha)||D_\zeta^\gamma D_\alpha^{\beta_1}D_{\tilde\alpha}^{\beta_2}\hat\psi(\delta e^t(\alpha,\tilde\alpha)\zeta)|\leqslant C_{\beta,r}(1+|\delta\zeta|)^{-L}(1+|\zeta|)^{-|\gamma|}, \tag{A-13}$$

provided $\hat\psi(\zeta)\in S_{1,0}^{-L}$ to prove Lemma 4.4. The reader can verify the following identity by induction:

$$\begin{aligned}\rho_{2\varepsilon_0}(\alpha,\tilde\alpha)D_\zeta^\gamma D_\alpha^{\beta_1}D_{\tilde\alpha}^{\beta_2}\hat\psi(\delta e^t(\alpha,\tilde\alpha)\zeta) \\ \equiv\sum_{\substack{|\theta|=|\gamma|+|\Omega|\\|\Omega|\leqslant|\beta_1|+|\beta_2|}}\delta^{|\gamma|+|\Omega|}\zeta^\Omega\rho_{\gamma,\Omega}(\alpha,\tilde\alpha)\hat\psi^{(\theta)}(\delta e^t(\alpha,\tilde\alpha)\zeta),\end{aligned} \tag{A-14}$$

where $\rho_{\gamma,\Omega}(\alpha,\tilde\alpha)$ are smooth functions vanishing for $|\alpha-\tilde\alpha|\leqslant 4\varepsilon_0$ and $\hat\psi^{(\theta)}(\zeta) = D_\zeta^\theta\tilde\psi(\zeta)$. Since $\hat\psi(\zeta)\in S_{1,0}^{-L}$, we have

$$|\hat\psi^{(\theta)}(\zeta)|\leqslant C_\theta(1+|\zeta|)^{-L-|\theta|}. \tag{A-15}$$

Also, by (2) of Lemma 4.2,

$$C^{-1}|\zeta|\leqslant|e^t(\alpha,\tilde\alpha)\zeta|\leqslant C|\zeta|\quad\text{for } |\alpha-\tilde\alpha|\leqslant 4\varepsilon_0.$$

This fact and (A-15) imply that

$$\begin{aligned}|\rho_{\gamma,\Omega}(\alpha,\tilde\alpha)\hat\psi^{(\theta)}(\delta e^t(\alpha,\tilde\alpha)\zeta)| \\ \leqslant C_\theta(1+|\delta\zeta|)^{-L-|\theta|}=C_\theta(1+|\delta\zeta|)^{-L-|\gamma|-|\Omega|}.\end{aligned} \tag{A-16}$$

From (A-14) and (A-16), we have

$$
\begin{aligned}
&(1+|\zeta|)^{|\gamma|}\left|\rho_{2\varepsilon_0}(\alpha,\tilde{\alpha})D_\zeta^\gamma D_\alpha^{\beta_1}D_{\tilde{\alpha}}^{\beta_2}\hat{\psi}\left(\delta e^t(\alpha,\tilde{\alpha})\zeta\right)\right| \\
&\qquad \leqslant C \sum_{|\Omega|\leqslant|\beta_1|+|\beta_2|} (1+|\zeta|)^{|\gamma|}\delta^{|\gamma|+|\Omega|}|\zeta|^{|\Omega|}(1+|\delta\zeta|)^{-L-|\gamma|-|\Omega|} \qquad \text{(A-17)} \\
&\qquad \leqslant C \sum_{|\Omega|\leqslant|\beta_1|+|\beta_2|} \left[\frac{(\delta+\delta|\zeta|)^{|\gamma|}}{(1+\delta|\zeta|)^{|\gamma|}}\,\frac{|\delta\zeta|^{|\Omega|}}{(1+\delta|\zeta|)^{|\Omega|}}\right](1+|\delta\zeta|)^{-L}.
\end{aligned}
$$

Since each quotient in brackets is less than one, the last term in (A-17) is dominated by

$$C(1+|\delta\zeta|)^{-L}. \qquad \text{(A-18)}$$

From (A-17) and (A-18), we see that the assertion in (A-13) is verified, as required to prove Lemma 4.4.

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