Some Remarks on the Convergence of Approximate Solutions of Nonlinear Evolution Equations in Hilbert Spaces

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Abstract. Let \( \partial \Phi \) be the subdifferential of some lower semicontinuous convex function \( \Phi \) of a real Hilbert space \( H \), \( f \in L^2(0, T; H) \) and \( u_n \) a continuous piecewise linear approximate solution of \( du/dt + \partial \Phi(u) \ni f \), obtained by an implicit scheme. If \( u_0 \in \text{Dom}(\partial \Phi) \), then \( du_n/dt \) converges to \( du/dt \) in \( L^2(0, T; H) \). Moreover, if \( u_0 \in \text{Dom}(\partial \Phi) \), we construct a step function \( \eta_n(t) \) approximating \( f \) such that \( \lim_{n \to \infty} \int_0^T \eta_n |du_n/dt - du/dt|^2 \, dt = 0 \). When \( \Phi \) is inf-compact and when the sequence of approximation of \( f \) is weakly convergent to \( f \), then \( u_n \) converges to \( u \) in \( C([0, T]; H) \) and \( \eta_n du_n/dt \) is weakly convergent to \( t du/dt \).

Introduction. Set \( H \) a real Hilbert space with scalar product \( \langle \cdot, \cdot \rangle \) and norm \( |\cdot| \) and \( \Phi \) a lower semicontinuous proper convex function from \( H \) into \((-\infty, +\infty]\). The subdifferential \( \partial \Phi \) of \( \Phi \) is the multivalued operator from \( H \) into \( 2^H \) defined by

\[
y \in \partial \Phi(x) \text{ if and only if}, \quad \forall u \in H, \quad \Phi(u) - \Phi(x) \geq \langle y, u - x \rangle.
\]

The operator \( \partial \Phi \) is a maximal monotone operator on \( H \) (cf. [4]) and the semigroup generated by \( -\partial \Phi \) has strong regularizing properties which have been discovered by Brezis [6]. In particular, if \( f \in L^2(0, T; H) \) and \( u_0 \in \overline{\text{Dom}(\partial \Phi)} \), then the weak solution of

\[
\begin{align*}
du/dt + \partial \Phi(u) & \ni f \quad \text{on} \ (0, +\infty), \\
u(0) & = u_0,
\end{align*}
\]

is in fact a strong one. If we assume \( \text{Min} \Phi = 0 \) (which is always possible), then the following estimate holds

\[
\left( \int_0^T \left| \frac{du}{dt} \right|^2 \, dt \right)^{1/2} \leq \left( \int_0^T |f|^2 \, dt \right)^{1/2} + \frac{1}{\sqrt{2}} \int_0^T |f| \, dt + \frac{1}{\sqrt{2}} \text{dist}(u_0, K),
\]

where \( K = \{ x \in H: \Phi(x) = 0 \} \). Moreover if \( u_n \) is the solution of

\[
\begin{align*}
du_n/dt + \partial \Phi(u_n) & \ni f_n \quad \text{on} \ (0, +\infty), \\
u_n(0) & = u_{0,n},
\end{align*}
\]

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and if \( \lim_{n \to +\infty} u_{0,n} = u_0 \) and \( \lim_{n \to +\infty} \int_0^T |f - f_n|^2 \, dt = 0 \), then

\[
\begin{align*}
\text{(i)} & \quad \lim_{n \to +\infty} \|u - u_n\|_{C([0,T]; H)} = 0, \\
\text{(ii)} & \quad \lim_{n \to +\infty} \int_0^T \left| \frac{du_n}{dt} - \frac{du}{dt} \right|^2 \, dt = 0.
\end{align*}
\]

However, it can be noticed that if \( \text{(4)(i)} \) is an obvious consequence of the properties of weak solutions and remains valid in more general cases, \( \text{(4)(ii)} \) was only discovered in 1977 (cf. [5] and [1] for a similar result).

In this paper we introduce the approximate solutions of \( (1) \) through the implicit scheme

\[
\begin{align*}
\text{(5)} & \quad \begin{cases}
\frac{u_n^k - u_{n-1}^k}{t_n^k - t_{n-1}^k} + \partial \Phi(u_n^k) \ni f_n^k, & k = 1, 2, \ldots, N(n), \\
u_n^0 = u_0, & 0 = t_n^0 < t_n^1 < \cdots < t_n^{N(n)} = T.
\end{cases}
\end{align*}
\]

If \( f_n \) is the step function taking the value \( f_n^k \) on \( (t_n^{k-1}, t_n^k) \), \( u_n \) the continuous piecewise linear function taking the value \( u_n^k \) at \( t_n^k \), and \( \eta_n \) the step function taking the value \( t_n^{k-1} \) on \( (t_n^{k-1}, t_n^k) \), then

\[
\begin{align*}
\text{(6)} & \quad \left( \int_0^T \left| \frac{du_n}{dt} \right| \, dt \right)^{1/2} \leq \left( \int_0^T |f_n|^2 \, \eta_n \, dt \right)^{1/2} + \sqrt{2} \int_0^T |f_n| \, dt + \frac{1}{\sqrt{2}} \, \text{dist}(u_0, K).
\end{align*}
\]

Moreover if \( \lim_{n \to +\infty} \max_{0 \leq k \leq N(n)} (t_n^k - t_n^{k-1}) = 0 \) and \( \lim_{n \to +\infty} \int_0^T |f_n - f|^2 \, dt = 0 \), then

\[
\begin{align*}
\text{(7)} & \quad \begin{cases}
\text{(i)} & \quad \lim_{n \to +\infty} \|u_n - u\|_{C([0,T]; H)} = 0, \\
\text{(ii)} & \quad \lim_{n \to +\infty} \int_0^T \left| \frac{du_n}{dt} - \frac{du}{dt} \right|^2 \, \eta_n \, dt = 0.
\end{cases}
\end{align*}
\]

It must be noticed that if \( \text{(7)(i)} \) is already well known in more general cases (cf. [7]) and is more or less a consequence of the theorem of Crandall and Liggett, \( \text{(7)(ii)} \) is new and could be of some use in numerical analysis. However, in the general case it appears that it is not possible to obtain error estimates for the convergence of \( \frac{du_n}{dt} \).

When \( \Phi \) is inf-compact and \( f_n \) weakly convergent to \( f \) in \( L^2(0, T; H) \), then \( \text{(7)(i)} \) remains valid and \( \eta_n \frac{du_n}{dt} \) is weakly convergent to \( t \frac{du}{dt} \) as \( n \to +\infty \).

We give also an extension of that type of result for more general operator \( A \) generating a semigroup \( (S(t))_{t \geq 0} \) which is compact for \( t > 0 \).

**The Main Estimates.** Set \( \Phi \) a lower continuous proper convex function from \( H \) into \( (-\infty, +\infty) \) such that \( \min \Phi = 0 \) and \( K = \{ x \in H : \Phi(x) = 0 \} \). For \( T > 0 \) we set \( P_n = \{ 0 = t_n^0 < t_n^1 < \cdots < t_n^{N(n)} = T \} \) a partition of \( [0, T] \), \( e_n^k = t_n^k - t_n^{k-1} \) and \( \|P_n\| = \max_{0 \leq k \leq N(n)} e_n^k \). We define the two step functions \( \eta_n \) and \( f_n \) from \( [0, T] \) into \( \mathbb{R} \) and \( H \), respectively, as the functions taking the values \( t_n^{k-1} \) and \( f_n^k \) on \( (t_n^{k-1}, t_n^k) \).
Thanks to the maximal monotonicity of \( \partial \Phi \), the sequence \( \{u_n^k\}_{k=0}^{N} \) is well defined by the following relations

\[
\begin{aligned}
\frac{u_n^k - u_n^{k-1}}{\varepsilon_n^k} + \partial \Phi(u_n^k) &\ni f_n^k, \quad k = 1, \ldots, N, \\
u_n^0 &= u_0.
\end{aligned}
\]  

(8)

If \( u_n \) is the continuous piecewise linear function taking the value \( u_n^k \) at \( t_n^k \), then

\[
u_n(t) = \frac{t_n^k - t}{\varepsilon_n^k} u_n^{k-1} + \frac{t - t_n^{k-1}}{\varepsilon_n^k} u_n^k \quad \text{and} \quad \frac{du_n}{dt} = \frac{u_n^k - u_n^{k-1}}{\varepsilon_n^k}
\]
on \((t_n^{k-1}, t_n^k)\).

**Theorem 1.** Suppose \( u_0 \in \text{Dom}(\partial \Phi) \). Then the following estimates hold:

\[
\left( \int_0^T \left| \frac{du_n}{dt} \right|^2 \eta_n \, dt \right)^{1/2} \leq \left( \int_0^T |f_n|^2 \eta_n \, dt \right)^{1/2} + \sqrt{2} \int_0^T |f_n| \, dt
\]

\[+ \frac{1}{\sqrt{2}} \text{dist}(u_0, K),
\]

(9)

\[
\left( \int_{\delta}^T \left| \frac{du_n}{dt} \right|^2 \, dt \right)^{1/2} \leq \left( \int_0^T |f_n|^2 \, dt \right)^{1/2} + \frac{1}{\sqrt{2} \delta} \text{dist}(u_0, K),
\]

(10)

for all \( \delta \) such that \( \delta = t_n^p \), \( 1 \leq p \leq N(n) \). Moreover if \( u_0 \in \text{Dom}(\Phi) \), we have

\[
\left( \int_0^T \left| \frac{du_n}{dt} \right|^2 \eta_n \, dt \right)^{1/2} \leq \left( \int_0^T |f_n|^2 \eta_n \, dt \right)^{1/2} + \sqrt{\Phi(u_0)}.
\]

(11)

**Remark 1.** If we no longer assume \( \min \Phi = 0 \), we obtain inequalities similar to those of Theorem 1 in changing \( \Phi \) and \( f_n \). To get this we consider \( x_0 \in \text{Dom}(\partial \Phi) \) and \( \xi \in \partial \Phi(x_0) \), and we set \( \tilde{\Phi}(x) = \Phi(x) - \Phi(x_0) - (\xi, x - x_0) \) for \( x \in \text{Dom}(\Phi) \) and \( \tilde{f}_n = f_n - \xi \). As \( \partial \Phi = \partial \Phi - \xi \), \( \{u_n^k\} \) satisfies (8) with \( \Phi \) and \( f_n \) replaced by \( \tilde{\Phi} \) and \( \tilde{f} \) and \( \min \Phi = 0 \).

**Remark 2.** If we let \( f_n = 0 \) in Theorem 1, we deduce from (9) that

\[
\left( \int_0^T \left| \frac{du_n}{dt} \right|^2 \eta_n \, dt \right)^{1/2} \leq \frac{1}{\sqrt{2}} \text{dist}(u_0, K),
\]

(12)

for any \( 0 \leq t \leq T \). If we want to have a pointwise estimate on \( \frac{du_n}{dt} \), we have to suppose \( \varepsilon_n^k = T/N \), for \( k = 0, 1, \ldots, N \). We then deduce from the monotonicity of \( \partial \Phi \) that \( |\frac{du_n}{dt}| \) is nonincreasing. Hence

\[
\left| \frac{du_n}{dt}(t) \right| = \frac{N}{T} |u_n^k - u_n^{k-1}| \leq \frac{1}{\sqrt{t_n^k t_n^{k-1}}} \text{dist}(u_0, K),
\]

(13)

for any \( t_n^{k-1} < t < t_n^k, k > 1 \).
Before proving our estimates we need the following result which is somewhat analogous to Lemma A.5 of [4].

**Lemma 1.** Suppose \( \{a_n\} \) and \( \{b_n\} \) are two sequences of nonnegative numbers such that

\[
\frac{1}{2} a_n^2 \leq \frac{1}{2} a_0^2 + \sum_{j=1}^{n} a_j b_j \quad \forall n \geq 0.
\]

Then

\[
a_n \leq a_0 + 2 \sum_{j=1}^{n} b_j \quad \forall n \geq 0.
\]

**Proof.** By induction we define the nonnegative sequence \( \{\alpha_n\}_{n \geq 1} \) by

\[
\frac{1}{2} \alpha_0^2 = \frac{1}{2} a_0^2 + \sum_{j=1}^{n} \alpha_j b_j \quad \forall n \geq 1.
\]

Then \( \alpha_n^2 - 2 \alpha_n b_n - \left( \alpha_0^2 + 2 \sum_{j=1}^{n-1} \alpha_j b_j \right) = 0 \). But \( \alpha_{n-1}^2 = \alpha_0^2 + 2 \sum_{j=1}^{n-1} \alpha_j b_j \), so \( \alpha_n^2 - 2 \alpha_n b_n - \alpha_{n-1}^2 = 0 \) and \( \alpha_n = b_n + \sqrt{b_n^2 + \alpha_{n-1}^2} \). Hence \( \alpha_n \leq 2 b_n + \alpha_{n-1} \), and \( \alpha_n \leq a_0 + 2 \sum_{j=1}^{n} b_j \). As it is easy to see by induction that \( \alpha_n \leq \alpha_n \) for \( n \geq 1 \), we get (15).

**Proof of Theorem 1.** First we assume \( u_0 \in \text{Dom}(\Phi) \), and we prove (11). From the definition of \( \{u_k^1\} \), we have \( f_n^{k} - (u_k^1 - u_k^{1-1})/\varepsilon_k^1 \in \partial \Phi(u_n^k) \). Then

\[
\left( u_n^k - u_n^{k-1}, f_n^k - \frac{u_n^k - u_n^{k-1}}{\varepsilon_n^k} \right) \geq \Phi(u_n^k) - \Phi(u_n^{k-1});
\]

so we get

\[
\frac{|u_n^k - u_n^{k-1}|^2}{\varepsilon_n^k} + \Phi(u_n^k) - \Phi(u_n^{k-1}) \leq \left( f_n^k, u_n^k - u_n^{k-1} \right) \quad \forall 1 \leq k \leq N.
\]

Summing all those inequalities, we have

\[
\sum_{k=1}^{N} \frac{|u_n^k - u_n^{k-1}|^2}{(\varepsilon_n^k)^2} \varepsilon_n^k + \Phi(u_n^N) - \Phi(u_0) \leq \sum_{k=1}^{N} \left( f_n^k, \frac{u_n^k - u_n^{k-1}}{\varepsilon_n^k} \right) \varepsilon_n^k,
\]

that is,

\[
\int_0^T \left| \frac{du_n}{dt} \right|^2 dt + \Phi(u_n^N) \leq \Phi(u_0) + \int_0^T \left( f_n, \frac{du_n}{dt} \right) dt.
\]

So we deduce (11).

From (17) we get for \( 1 \leq k \leq N \),

\[
\frac{|u_n^k - u_n^{k-1}|}{(\varepsilon_n^k)^2} \varepsilon_n^k t_n^{k-1} + \left( \Phi(u_n^k) - \Phi(u_n^{k-1}) \right) t_n^{k-1} \leq \left( f_n^k, \frac{u_n^k - u_n^{k-1}}{\varepsilon_n^k} \right) \varepsilon_n^k t_n^{k-1}.
\]

Summing those inequalities and using the fact that

\[
\sum_{k=1}^{N} \left( \Phi(u_n^k) - \Phi(u_n^{k-1}) \right) t_n^{k-1} = \Phi(u_n^N) t_n^{N-1} - \sum_{k=1}^{N-1} \Phi(u_n^k) (t_n^k - t_n^{k-1})
\]

\[
= \Phi(u_n^N) t_n^{N-1} - \sum_{k=1}^{N-1} \Phi(u_n^k) \varepsilon_n^k,
\]

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we deduce that
\[
\sum_{k=1}^{N} \frac{|u_k^n - u_k^{n-1}|^2}{e_k^n} e_k^{n-1} + \Phi(u_n^n) t^{n-1} 
\leq \sum_{k=1}^{N} \left(f_k^n, \frac{u_k^n - u_k^{n-1}}{e_k^n}\right) e_k^{n-1} + \sum_{k=1}^{N-1} \Phi(u_k^n) e_n^n,
\]
and so, as \( \Phi \geq 0 \) in \( H \),
\[
\int_0^T \left| \frac{d u_n}{d t} \right|^2 \eta_n dt \leq \int_0^T \left(f_n^n, \frac{d u_n}{d t}\right) \eta_n dt + \int_0^T \Phi(\tilde{u}_n) dt,
\]
where \( \tilde{u}_n \) is the step function taking the value \( u_k^n \) on \( (t_k^{n-1}, t_k^n) \). By Schwarz inequality we get
\[
\left(\int_0^T \left| \frac{d u_n}{d t} \right|^2 \eta_n dt \right)^{1/2} \leq \left(\int_0^T \Phi(\tilde{u}_n) dt \right)^{1/2} + \left(\int_0^T |f_n|^2 \eta_n dt \right)^{1/2}.
\]
On the other hand, we have for any \( \nu \in \text{Dom}(\Phi) \)
\[
\Phi(\nu) - \Phi(u_n^n) = \left(\int f_n^n, u_n^n - \nu\right).
\]
In particular, if \( \nu \in K \), \( \Phi(u_n^n) - \Phi(\nu) = \Phi(u_n^n) \) and
\[
\Phi(u_n^n) + \left(\frac{u_k^n - u_k^{n-1}}{e_k^n}, u_n^n - \nu\right) \leq \left(f_k^n, u_k^n - \nu\right) \forall 1 \leq k \leq N.
\]
But \((u_k^n - u_k^{n-1}, u_n^n - \nu) = ((u_k^n - \nu) - (u_k^{n-1} - \nu), u_k^n - \nu) \geq \frac{1}{2} |u_k^n - \nu| - \frac{1}{2} |u_k^{n-1} - \nu|^2 \). So we get from (22)
\[
\sum_{k=1}^{N} \Phi(u_k^n)e_k^n + \frac{1}{2} |u_n^n - \nu|^2 \leq \sum_{k=1}^{N} \left(f_k^n, u_k^n - \nu\right)e_k^n + \frac{1}{2} |u_0^n - \nu|^2,
\]
that is,
\[
\int_0^T \Phi(\tilde{u}_n) dt + \frac{1}{2} |u_n^n - \nu|^2 \leq \frac{1}{2} |u_0^n - \nu|^2 + \int_0^T \left(f_n^n, \tilde{u}_n - \nu\right) dt.
\]
Moreover, that last inequality remains true if we replace \( T = t_n^n \) by \( t_k^n \) and \( u_n^n \) by \( u_k^n \), so
\[
\frac{1}{2} |u_k^n - \nu|^2 + \sum_{j=1}^{k} \Phi(u_j^n) e_j^n
\leq \frac{1}{2} |u_0^n - \nu|^2 + \sum_{j=1}^{k} \left(f_j^n, u_j^n - \nu\right)\left(|u_j^n - \nu|^2 + 2 \sum_{i=1}^{j} \Phi(u_i^n) e_i^n\right)^{1/2}.
\]
By applying Lemma 1, we deduce
\[
\left(|u_k^n - \nu|^2 + 2 \sum_{j=1}^{k} \Phi(u_j^n) e_j^n\right)^{1/2} \leq |u_0^n - \nu| + 2 \sum_{j=1}^{k} |f_j^n| e_j^n.
\]
In particular, for \( k = N \), we have

\[
(26) \quad \left( \int_0^T \Phi(\tilde{u}_n) \, dt \right)^{1/2} \leq \frac{1}{\sqrt{2}} |u_0 - v| + \sqrt{2} \int_0^T |f_n| \, dt.
\]

From that inequality and (21) we get (9) when \( u_0 \in \text{Dom}(\Phi) \). For \( u_0 \in \overline{\text{Dom}(\partial \Phi)} = \overline{\text{Dom}(\Phi)} \), we consider \( \{u_{0,m}\} \in \text{Dom}(\Phi) \) such that \( u_{0,m} \to_{m \to \infty} u_0 \). If \( \{u_{n,m}\} \) is the sequence of continuous piecewise linear approximate solutions defined by the same implicit scheme as \( u_n \), with initial data \( u_{0,m} \) instead of \( u_0 \), then \( \|u_n - u_{n,m}\|_{C([0,T]; H)} \leq |u_0 - u_{0,m}| \). Hence we get (9) as \( \lim_{m \to +\infty} du_{n,m}/dt = du_n/dt \), uniformly on \([0,T]\).

In order to get (10), set \( \delta = t_n^p \) for some \( p \geq 1 \). We have

\[
\frac{1}{\delta} \int_0^\delta \Phi(\tilde{u}_n) \, dt \in \left[ \min \left\{ \Phi(u_n^i) : 1 \leq i \leq p \right\} ; \max \left\{ \Phi(u_n^i) : 1 \leq i \leq p \right\} \right].
\]

If \( \min \{\Phi(u_n^i) : 1 \leq i \leq p\} = \Phi(u_n^k) \), then \( \Phi(u_n^k) = (1/\delta) \int_0^\delta \Phi(\tilde{u}_n) \, dt \). In summing the inequalities (17) for \( k = k_0 + 1, \ldots, N \), we get

\[
\int_{t_n^k}^T \left| \frac{du_n}{dt} \right|^2 \, dt + \Phi(u_n^N) - \Phi(u_0) \leq \int_{t_n^k}^T \left( f_n, \frac{du_n}{dt} \right) \, dt,
\]

so we have

\[
\left( \int_0^\delta \left| \frac{du_n}{dt} \right|^2 \, dt \right)^{1/2} \leq \left( \int_0^T |f_n|^2 \, dt \right)^{1/2} + \left( \frac{1}{\delta} \int_0^\delta \Phi(\tilde{u}_n) \, dt \right)^{1/2}.
\]

But, from (26) with \( T \) replaced by \( \delta \), we have

\[
\left( \frac{1}{\delta} \int_0^\delta \Phi(\tilde{u}_n) \, dt \right)^{1/2} \leq \frac{1}{\sqrt{2\delta}} |u_0 - v| + \sqrt{\frac{2}{\delta}} \int_0^\delta |f_n| \, dt.
\]

With those two last inequalities we deduce (10).

Remark 3. By changing slightly the proof of (9) we can also obtain the following inequality valid for any \( v \in \text{Dom}(\Phi) \):

\[
(27) \quad \left( \int_0^T \left| \frac{du_n}{dt} \right|^2 \eta_n \, dt \right)^{1/2} \leq \left( \int_0^T |f_n|^2 \eta_n \, dt \right)^{1/2} + \sqrt{\frac{T}{2}} \int_0^T |f_n| \, dt
\]

\[+ \frac{1}{\sqrt{2}} |u_0 - v| + \sqrt{T \Phi(v)}. \]

To get this, we start from

\[
\Phi(v) - \Phi(u_n^k) \geq \left( f_n^k - \frac{u_n^k - u_n^{k-1}}{\varepsilon_n^k}, v - u_n^k \right) \quad \forall v \in \text{Dom}(\Phi),
\]

and then we get

\[
\int_0^T \Phi(\tilde{u}_n) \, dt + \frac{1}{2} |u_n^N - v|^2 \leq \frac{1}{2} |u_0 - v|^2 + T \Phi(v) + \int_0^T \left( f_n, \tilde{u}_n - v \right) \, dt.
\]

Using (24) (with \( \frac{1}{2} |u_0 - v|^2 \) replaced by \( \frac{1}{2} |u_0 - v|^2 + T \Phi(v) \)) and Lemma 1 we deduce (27).
The Convergence Theorems. We still keep the notations of the first section and we suppose that \( u \) is the strong solution of

\[
\begin{aligned}
\frac{du}{dt} + \partial \Phi(u) &\ni f \quad \text{on } (0, +\infty), \\
\left. u \right|_{t=0} &= u_0,
\end{aligned}
\]

with \( u_0 \in \text{Dom}(\partial \Phi) \) and \( f \in L^2(0, T; H) \). Our first result is the following

**Theorem 2.** If \( u_0 \in \text{Dom}(\partial \Phi) \) and \( \lim_{n \to \infty} \| P_n \| = \lim_{n \to \infty} \int_0^T |f - f_n|^2 \, dt = 0 \), then

\[(29) \quad \lim_{n \to \infty} \| u_n - u \|_{C([0,T], H)} = 0, \]

\[(29) \quad \lim_{n \to \infty} \left( \int_0^T \left| \frac{du_n}{dt} - \frac{du}{dt} \right|^2 \right) dt = 0. \]

Moreover, if \( u_0 \in \text{Dom}(\Phi) \), then

\[(30) \quad \lim_{n \to \infty} \int_0^T \left| \frac{du_n}{dt} - \frac{du}{dt} \right|^2 dt = 0. \]

Before proving Theorem 2, first notice that (29)(i) is known in quite more general cases (cf. [7]). We first need the following result.

**Lemma 2.** Suppose \( u_0 \in \text{Dom}(\partial \Phi) \) and \( \lim_{n \to \infty} \| P_n \| = \lim_{n \to \infty} \int_0^T |f - f_n|^2 \, dt = 0 \). Then \( u_n \to u \) in weakly in \( L^2(0, T; H) \).

**Proof.** As \( \| P_n \| \to 0 \), for every \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( \eta_n(t) > \varepsilon / 2 \) on \((\varepsilon, T] \) for any \( n > n_0 \). By Theorem 1 and classical results on maximal monotone operators (cf. [4]) \( u_n \) converges weakly to \( u \) in \( L^2(0, T; H) \).

Set \( \psi \in L^2(0, T; H) \). For any \( \alpha > 0 \), we have

\[
\int_0^T \left( \eta_n \frac{du_n}{dt} - t \frac{du}{dt}, \psi \right) dt = \int_0^T \left( \eta_n \frac{du_n}{dt} - t \frac{du}{dt}, \psi \right) dt + \int_0^T \left( \eta_n \frac{du_n}{dt} - t \frac{du}{dt}, \psi \right) dt.
\]

When \( n \) tends to \( +\infty \), \( \eta_n(t) - t \) tends to \( 0 \), uniformly with respect to \( t \), so

\[
\lim_{n \to \infty} \int_0^T \left( \eta_n \frac{du_n}{dt} - t \frac{du}{dt}, \psi \right) dt = 0.
\]

Set \( \delta > 0 \). There exists \( \psi \in \text{Dom}(\Phi) \) such that \( \| u_0 - \psi \| < \delta \). For such a \( \psi \) there exists \( \alpha_0 > 0 \) such that for \( 0 < \alpha < \alpha_0 \) and \( n \in \mathbb{N} \) we have (see Remark 3)

\[
\left( \int_0^\alpha |f_n|^2 \eta_n \, dt \right)^{1/2} + \sqrt{2} \int_0^\alpha \| f_n \| dt < \sqrt{2} \alpha \Phi(\psi) < \delta,
\]

as \( |f_n| \) is uniformly equi-integrable on \((0, T)\). Hence \( \left( \int_0^\alpha |du_n/dt|^2 \eta_n \, dt \right)^{1/2} < 2\delta \). If \( \alpha \) is also fixed such that \( \left( \int_0^\alpha |du/dt|^2 \eta_n \, dt \right)^{1/2} < \delta \), we deduce

\[
\limsup_{n \to \infty} \left| \int_0^T \left( \eta_n \frac{du_n}{dt} - t \frac{du}{dt}, \psi \right) dt \right| < 3\delta \| \psi \|_{L^2(0,T,H)}.
\]

Hence \( \eta_n \frac{du_n}{dt} - t \frac{du}{dt} \) converges weakly to \( 0 \) in \( L^2(0,T; H) \) and similarly \( \eta_n (\frac{du_n}{dt} - \frac{du}{dt}) \), as \( \lim_{n \to \infty} \int_0^T (t - \eta_n) |du/dt|^2 \, dt = 0 \).
LEMMA 3. Under the hypotheses of Lemma 2, we have
\[
\limsup_{n \to +\infty} \int_0^T \Phi(\tilde{u}_n) \, dt \leq \int_0^T \Phi(u) \, dt,
\]
\(\tilde{u}_n\) being the step function taking the value \(u^k_n\) on \((t^{k-1}_n, t^k_n)\).

Proof. We start from the following inequalities valid for \(v \in \text{Dom}(\Phi)\) and \(1 \leq k \leq N\).
\[
\Phi(u(t)) - \Phi(u^k_n) \geq \left( f_n^k - \frac{u^k_n - u^{k-1}_n}{\epsilon^k_n}, u(t) - u^k_n \right),
\]
\[
\Phi(v) - \Phi(u^k_n) \geq \left( f_n^k - \frac{u^k_n - u^{k-1}_n}{\epsilon^k_n}, v - u^k_n \right).
\]
From (33) we deduce
\[
t_n^k \Phi(v) - \int_0^{t_n^k} \Phi(\tilde{u}_n) \, dt \geq \int_0^{t_n^k} (f_n, v - \tilde{u}_n) \, dt - \sum_{j=1}^k (u^j_n - u^{j-1}_n, v - u^j_n).
\]
But \(-(u^j_n - u^{j-1}_n, v - u^j_n) \geq \frac{1}{2} |u^j_n - v|^2 - \frac{1}{2} |u^{j-1}_n - v|^2\), so we get
\[
\int_0^{t_n^k} \Phi(\tilde{u}_n) \, dt \leq t_n^k \Phi(v) - \int_0^{t_n^k} (f_n, v - \tilde{u}_n) \, dt - \frac{1}{2} |v - u^k_n|^2 + \frac{1}{2} |v - u_0|^2.
\]
On the other hand we have from (32) (integrating over \((t^j, t^{j+1})\) and summing for \(j = k, \ldots, N\))
\[
\int_{t^k_n}^{t^j_n} \Phi(\tilde{u}_n) \, dt \leq \int_{t^k_n}^{t^j_n} (f_n, u - \tilde{u}_n) \, dt + \int_{t^k_n}^{t^j_n} \left( \frac{du_n}{dt}, u - \tilde{u}_n \right) dt.
\]
From (34) and (35) we get for \(v \in \text{Dom}(\Phi)\)
\[
\int_0^T \Phi(\tilde{u}_n) \, dt \leq \int_0^T \Phi(u) \, dt - \int_0^T (f_n, u - \tilde{u}_n) \, dt + \int_0^T \left( \frac{du_n}{dt}, u - \tilde{u}_n \right) dt.
\]
For \(\epsilon > 0\), as \(\|P_n\| \to n \to +\infty\) 0, there exists a sequence \(\{t^k_n\}\) such that \(t^k_n \to \epsilon\). As \(\{f_n\}\) and \(\{du_n/dt\}\) remain bounded in \(L^2(t^k_n, T; H)\) independently of \(n\) and as \(u - \tilde{u}_n\) goes to 0 in \(L^\infty(0, T; H)\) when \(n\) tends to +\(\infty\), we deduce from (36)
\[
\limsup_{n \to +\infty} \int_0^T \Phi(\tilde{u}_n) \, dt \leq \epsilon \Phi(v) - \int_0^T (f, v - u) \, dt - \frac{1}{2} |v - u(\epsilon)|^2
\]
\[
\cdots + \frac{1}{2} |v - u_0|^2 + \int_0^\epsilon \Phi(u) \, dt.
\]
As \(u\) is continuous, we deduce (31) in letting \(\epsilon \to 0\).

Proof of Theorem 2. First we prove (29)(ii) by supposing \(u_0 \in \overline{\text{Dom}(\partial \Phi)}\).

From (19), as in the proof of Theorem 1, we get
\[
\int_0^T \left| \frac{du_n}{dt} \right|^2 \eta_n \, dt + t^{N-1}_n \Phi(u^N_n) \leq \int_0^T \Phi(\tilde{u}_n) \, dt + \int_0^T \left( f_n, \frac{du_n}{dt} \right) \eta_n \, dt.
\]
But we also have, obviously,

\[(39) \quad \int_0^T \left| \frac{du}{dt} \right|^2 t \, dt + T \Phi(u(T)) = \int_0^T \Phi(u) \, dt + \int_0^T \left( f, \frac{du}{dt} \right) t \, dt. \]

Moreover as \( \lim_{n \to +\infty} t_n^{N-1} = T \), \( T \Phi(u(T)) \leq \lim \inf_{n \to +\infty} t_n^{N-1} \Phi(u_n^{N-1}) \). Using the fact that \( \lim_{n \to +\infty} \int_0^T (f, du/dt) \eta_n \, dt = \int_0^T (f, du/dt) t \, dt \) and Lemma 2, we deduce

\[(40) \quad \limsup_{n \to +\infty} \int_0^T \left| \frac{du_n}{dt} \right|^2 \eta_n \, dt \leq \int_0^T \left| \frac{du}{dt} \right|^2 t \, dt. \]

As \( \{\sqrt{\eta_n} du_n/dt\} \) converges weakly in \( L^2(0, T; H) \) to \( \sqrt{t} \, du/dt \), we conclude that

\[\lim_{n \to +\infty} \int_0^T \left| \frac{du_n}{dt} \sqrt{\eta_n} - \frac{du}{dt} \sqrt{t} \right|^2 \, dt = 0.\]

But

\[
\left\{ \int_0^T \left| \frac{du_n}{dt} - \frac{du}{dt} \right|^2 \eta_n \, dt \right\}^{1/2} \leq \left\{ \int_0^T \left| \frac{du_n}{dt} \sqrt{\eta_n} - \frac{du}{dt} \sqrt{t} \right|^2 \right\}^{1/2} + \left\{ \int_0^T \left| \frac{du}{dt} \right|^2 \left( \sqrt{t} - \sqrt{\eta_n} \right)^2 \right\}^{1/2}.
\]

From Lebesgue's theorem and the estimate (2),

\[
\lim_{n \to +\infty} \int_0^T \left| \frac{du}{dt} \right|^2 \left( \sqrt{t} - \sqrt{\eta_n} \right)^2 \, dt = 0,
\]

so we get (29)(ii).

To prove (30), first we notice that \( \{du_n/dt\} \) remains bounded in \( L^2(0, T; H) \) when \( u_0 \in \text{Dom}(\Phi) \). Hence, from the maximality of \( \partial \Phi \), it converges weakly in \( L^2(0, T; H) \) to \( du/dt \). As we have already seen in the proof of Theorem 1, we have

\[
\int_0^T \left| \frac{du_n}{dt} \right|^2 \, dt \leq \Phi(u_0) + \int_0^T \left( f, \frac{du_n}{dt} \right) \, dt - \Phi(u_n^{N-1}),
\]

so we get

\[(41) \quad \limsup_{n \to +\infty} \int_0^T \left| \frac{du_n}{dt} \right|^2 \, dt \leq \Phi(u_0) + \int_0^T \left( f, \frac{du}{dt} \right) \, dt - \Phi(u(T)). \]

But as \( |du/dt|^2 = d(\Phi(u))/dt + (f, du/dt) \) a.e. on \( (0, T) \) (cf. [4]), integrating on \( (0, T) \) we obtain

\[
\limsup_{n \to +\infty} \int_0^T \left| \frac{du_n}{dt} \right|^2 \, dt \leq \int_0^T \left| \frac{du}{dt} \right|^2 \, dt.
\]

So we have (30), which ends our proof.

**Remark 4.** Using similar devices, we can obtain the convergence result (29) in replacing the initial data \( u_0 \) of \( \{u_n\} \) by a sequence \( \{u_{0,n}\} \) in \( \text{Dom}(\partial \Phi) \) such that \( u_{0,n} \to +\infty u_0 \). Moreover if \( u_{0,n} \in \text{Dom}(\Phi) \) and \( \Phi(u_{0,n}) \to \Phi(u_0) \), we have (30).

**Remark 5.** We can also obtain some results concerning the convergence of \( \{\Phi(u_n)\} \) from Theorem 2. For example, if \( u_0 \in \text{Dom}(\Phi) \), we have

\[(42) \quad \lim_{n \to +\infty} \|\Phi(u_n) - \Phi(u)\|_{C([0, T])} = 0.\]
In order to prove this, we start from (17) and we have for $1 \leq k \leq N$,

$$\Phi(u_n^k) \leq \Phi(u_0) + \int_0^{t_n^k} \left( f_n, \frac{du_n}{dt} \right) dt - \int_0^{t_n^k} \left| \frac{du_n}{dt} \right|^2 dt. \tag{43}$$

For $t \in [0, T]$, as $\| P_n \| \to 0$, there exists a sequence $\{ t_n^k \}$ such that $\lim t_n^k = t$ and $t_n^k > t$ so $\Phi(u_n^k) = \Phi(\tilde{u}_n(t))$ and

$$\lim_{n \to +\infty} \sup_{t \in \Omega} \Phi(\tilde{u}_n(t)) \leq \Phi(u_0) + \int_0^t \left( f, \frac{du}{dt} \right) dt - \int_0^t \left| \frac{du}{dt} \right|^2 dt. \tag{44}$$

But from [4]

$$\Phi(u_0) + \int_0^t \left( f, \frac{du}{dt} \right) dt - \int_0^t \left| \frac{du}{dt} \right|^2 dt = \Phi(u(t)) \leq \liminf_{n \to +\infty} \Phi(\tilde{u}_n(t));$$

hence $\lim_{n \to +\infty} \Phi(\tilde{u}_n(t)) = \Phi(u(t))$, and it is not difficult to see that this limit is uniform on $[0, T]$ and that $\tilde{u}_n$ can be replaced by $u_n$.

For $k \geq 0$ we set $\Omega_k = \{ v \in H: \Phi(v) + |v|^2 \leq k \}$. Our second result is the following

**THEOREM 3.** Suppose $\Omega_k$ is compact for any $k \geq 0$ and $u_0 \in \overline{\text{Dom}(\partial\Phi)}$. If $\lim_{n \to +\infty} \| P_n \| = 0$ and if $u_n$ converges weakly to $f$ in $L^2(0, T; H)$, then

$$\begin{align*}
(\text{i}) & \quad \lim_{n \to +\infty} \| u_n - u \|_{C([0, T]; H)} = 0, \\
(\text{ii}) & \quad \eta_n \frac{du_n}{dt} - t \frac{du}{dt} \text{ weakly in } L^2(0, T; H) \text{ as } n \to +\infty. \tag{45}
\end{align*}$$

Moreover if $u_0 \in \text{Dom}(\Phi)$, then $du_n/dt$ converges weakly to $du/dt$ in $L^2(0, T; H)$.

**Proof.** First we suppose that $u_0 \in \text{Dom}(\Phi)$. From (10) $\{ du_n/dt \}$ remains bounded in $L^2(0, T; H)$, so we deduce from (18), with $N$ replaced by $k$, that $\{ \Phi(\tilde{u}_n(t)) \}$ is bounded, uniformly with respect to $t$ and $n$. From the convexity of the function $\Phi$, it is the same with $\{ \Phi(u_n(t)) \}$, and, as $\{ u_n(t) \}$ is bounded, the set $\{ u_n(t) \}$ is relatively compact in $H$ for any $t > 0$. From (10) the set $\{ u_n \}$ is equicontinuous, so by Ascoli’s theorem, it is relatively compact in $C([0, T]; H)$. Hence there exist a subsequence $\{ u_{n,m} \}$ and $\tilde{u} \in H^1(0, T; H)$ such that $u_n \to u_m \to \tilde{u}$ in $C([0, T]; H)$ and $du_n/dt \to \tilde{u} \to du/dt$ weakly in $L^2(0, T; H)$. Hence $\tilde{u}$ satisfies (28) and is equal to $u$, so instead of the subsequence $\{ u_{n,m} \}$ we can take $\{ u_n \}$ in the previous convergences.

We suppose now that $u_0 \in \overline{\text{Dom}(\partial\Phi)}$. There exists a sequence $\{ u_{0,m} \}$ in $\text{Dom}(\Phi)$ converging to $u_0$ as $m \to +\infty$. If $u_{n,m}$ is the continuous piecewise linear solution of (8) with initial data $u_{0,m}$ instead of $u_0$, then $\| u_n - u_{n,m} \|_{C([0, T]; H)} \leq \| u_0 - u_{0,m} \|$. Hence $\{ u_n \}$ converges to $u$ in $C([0, T]; H)$. But as Lemma 2 remains valid we have (45)(ii).

**Remark 6.** Using classical estimates on convex functions, it is easy to check that $\Phi((I + \lambda \partial\Phi)^{-1}x) (\lambda > 0)$ remains bounded when $x$ is bounded. If $\Omega_k$ is compact for any $k \geq 0$, the resolvents $(I + \lambda \partial\Phi)^{-1}$ are compact operators, and then the semigroup $(S(t))_{t \geq 0}$ generated by $-\partial\Phi$ is compact for $t > 0$. In the following section we give an extension of Theorem 3 to a more general situation.

**Remark 7.** As the convergence of $\{ du_n/dt \}$ in both Theorems 2 and 3 is obtained via weak compactness and lower semicontinuity arguments, it is clear that in the
general case it is not possible to obtain any error estimate of interest (see (55), Lemma 4). However, such error estimates should exist in many applications.

**Extension to More General Operators.** Set $A$ a maximal monotone operator of $H$ and $(S(t))_{t \geq 0}$ the semigroup of contractions of $\overline{D(A)}$ generated by $-A$. Thanks to the maximal monotonicity of $A$ we can construct a sequence $\{u^k_n\}$ with the relations

$$\left\{ \begin{array}{l}
    \frac{u^k_n - u^{k-1}_n}{t^n_k - t^{k-1}_n} + Au^k_n \equiv f^n_k, \\
    u^0_n = u_0 \in D(A) \quad \text{and} \quad 0 = t^0_n < t^n_1 < \cdots < t^{n(n)}_n = T.
\end{array} \right.$$  

We define the functions $f_n$ and $u_n$ as before. If $f$ is a given function of $L^1(0, T; H)$, we set $u$ the weak solution (cf. [4]) of

$$\left\{ \begin{array}{l}
    \frac{dz}{dt} + Au \equiv f \quad \text{on } [0, T], \\
    u(0) = u_0 \in D(A),
\end{array} \right.$$  

which means that $u \in C([0, T]; H)$ satisfies $u(0) = u_0$ and

$$\int_0^t \int_0^s (u(t) - x, f(t) - y) \, dt \, ds \leq \frac{1}{2} |u(t) - x|^2 - \frac{1}{2} |u(s) - x|^2 \leq \int_s^t (u(\tau) - x, f(\tau) - y) \, d\tau,$$  

for any $x \in D(A), y \in Ax$ and $0 \leq s \leq t \leq T$.

If $h$ is defined on $(0, T) \times (0, T)$, we set

$$\|h\|^* = \inf \{ \|g\|_{L^1(0, T)} + \|f\|_{L^1(0, T)}, f \text{ and } g \in L^1(0, T),$$  

| \int_0^t \int_0^s |h(\sigma, \tau)| \leq f(\sigma) + g(\tau) \text{ a.e. on } (0, T) \times (0, T) \},$$  

and we call $W$ the completion of $C([0, T] \times [0, T])$ for $\| \|^*$ (cf. [7]).

**Theorem 4.** Suppose the operators $S(t)$ are compact for $t > 0$, the set of real valued functions $h_n(t, s) = |f_n(t) - f_n(s)|$ is relatively compact in $W$, $\lim_{n \to +\infty} \|P_n\| = 0$ and $\{f_n\}$ converges weakly to $f$ in $L^1(0, T; H)$. Then $\{u_n\}$ converges to the solution $u$ of (47) in $C([0, T]; H)$.

**Proof.** We set $u_n$ the weak solution of

$$\left\{ \begin{array}{l}
    \frac{dz_n}{dt} + Au \equiv f_n \quad \text{on } (0, T), \\
    v_n(0) = u_0 \in D(A).
\end{array} \right.$$  

As the set $\{|f_n|\}$ is uniformly equi-integrable on $(0, T)$, we deduce with a slight modification of the proof of Theorem 2 of [2] that $\{v_n\}$ is relatively compact in $C([0, T], H)$, so there exist $\tilde{u} \in C([0, T]; H)$ and a subsequence $\{v_{n'}\}$ of $\{v_n\}$ such that $v_{n'} \to \tilde{u}$ in $C([0, T]; H)$. For any $x \in D(A), y \in Ax$, and $0 \leq s \leq t \leq T$, we have

$$\frac{1}{2} |v_{n'}(t) - x|^2 - \frac{1}{2} |v_{n'}(s) - x|^2 \leq \int_s^t (v_{n'}(\tau) - x, f_{n'}(\tau) - y) \, d\tau.$$  

Going to the limit in (51), we see that $\tilde{u}$ is a weak solution of (47), so $\tilde{u} = u$ and $\lim_{n \to +\infty} v_n = u$ in $C([0, T]; H)$. 


At this point of the proof we need the following result:

**Lemma 4.** Under the hypotheses of Theorem 4, \( u_n - v_n \) converges to 0 in \( C([0, T]; H) \) as \( n \to +\infty \).

**Proof.** That result is a consequence of Theorem 2.8 of [7]. We first suppose that \( u_0 \in D(A) \), and we consider a partition \( \tilde{P}_m \) which is a refinement of \( P_n : \tilde{P}_m = \{0 = s_m^0 < s_m^1 < \ldots < s_m^M = T\} \), and we construct the sequence \( \{\tilde{v}_m^j\} \) as follows

\[
\begin{align*}
\tilde{v}_m^j - \tilde{v}_m^{j-1} &= s_m^j - s_m^{j-1} \quad j = 1, \ldots, M, \\
\tilde{v}_m^0 &= u_0, \\
\tilde{f}_m^j &= f_n^k \quad \text{if } (s_m^{j-1}, s_m^j) \subset (t_n^{k-1}, t_n^k).
\end{align*}
\]

For \( y \in Au_0 \) we set \( \omega_n(\tau - s) = \int_0^{\tau - s} |f_n(\alpha)| + |y| \, d\alpha \) and

\[
G(\omega_n, h_n)(s, \tau) = \omega_n(\tau - s) + \begin{cases} 
\int_0^s h_n(\alpha, \tau - s + \alpha) \, d\alpha & \text{if } \tau \geq s, \\
\int_0^\tau h_n(s - \tau + \alpha, \alpha) \, d\alpha & \text{if } s \geq \tau.
\end{cases}
\]

Set \( H_m^{n,m} \) the piecewise constant function on \( (0, T) \times (0, T) \) taking the value \( |v_n^j - u_n^k| \) on \( (s_m^{j-1}, s_m^j) \times (t_n^{k-1}, t_n^k) \). For a function \( h \) defined on \( (0, T) \times (0, T) \), we call \( h_\Delta \) the piecewise constant function taking the value \( h(s_m^j, t_n^k) \) on \( (s_m^{j-1}, s_m^j) \times (t_n^{k-1}, t_n^k) \).

If \( \tilde{\omega} \in C^2([-T, T]) \) and \( \tilde{h} \in C^2([0, T] \times [0, T]) \) with \( \tilde{h}(0, 0) = 0 \), Theorem 2.8 of [7] gives

\[
\begin{align*}
\left\| H_m^{n,m} - G(\omega_n, h_n) \right\|_{L^\infty((0,T) \times (0,T))} &\leq 2\|\omega_n - \tilde{\omega}\|_{L^\infty(-T,T)} + 2\|h_n - \tilde{h}\| + \|h_\Delta - \tilde{h}\|^* \\
&+ 2\|P_n\| \left\{ \|\omega_n''\|_{L^\infty(-T,T)} + \|\omega_n'\|_{L^\infty(-T,T)} + (1 - 2T)^2\|\tilde{h}\|_{C^2([0,T] \times [0,T])} \right\}.
\end{align*}
\]

Moreover \( \|h_\Delta - \tilde{h}\|^* \leq T\|h_\Delta - \tilde{h}\|_{L^\infty((0,T) \times (0,T))} \) and as \( \tilde{h} \) is continuous, that last quantity goes to 0 as \( n \to +\infty \). When \( m \to +\infty \), the step function \( v_n \) defined from the sequence \( \{\tilde{v}_n^j\} \) converges to \( v_n \) uniformly on \( [0, T] \). As \( G(\omega_n, h_n)(s, s) = 0 \), we get

\[
\begin{align*}
\left\| u_n - v_n \right\|_{C([0,T]; H)} &\leq 2\|\omega_n - \tilde{\omega}\|_{L^\infty(-T,T)} + 2\|h_n - \tilde{h}\| + \limsup_{m \to +\infty} \|h_\Delta - \tilde{h}\| \\
&+ 2\|P_n\| \left\{ \|\omega_n''\|_{L^\infty(-T,T)} + \|\omega_n'\|_{L^\infty(-T,T)} \\
&+ (1 - 2T)^2\|\tilde{h}\|_{C^2([0,T] \times [0,T])} \right\}.
\end{align*}
\]

As \( \{f_n\} \) is uniformly equi-integrable on \( (0, T) \), the functions \( \omega_n \) are equi-continuous and uniformly bounded on \([- T, T] \), hence relatively compact in \( C([- T, T]) \). By hypothesis the functions \( h_n \) are relatively compact in \( W \). So there exist \( \tilde{\omega} \in C([-T, T]) \) and \( \tilde{h} \in W \) such that

\[
\begin{align*}
\lim_{n \to +\infty} \sup \left\| u_n - v_n \right\|_{C([0,T]; H)} &\leq 2\|\tilde{\omega} - \tilde{\omega}\|_{L^\infty(-T,T)} + 2\|\tilde{h} - \tilde{h}\|^*.
\end{align*}
\]

That last quantity can be made as small as we want, the density of the test functions \( (\tilde{\omega}, \tilde{h}) \) in \( C([-T, T]) \times W \) being easy to prove; so \( \lim_{n \to +\infty} \left\| u_n - v_n \right\|_{C([0,T]; H)} = 0 \).
If \( u_0 \in \overline{D(A)} \), we consider a sequence \( u_{0,m} \in D(A) \) such that \( u_{0,m} \to u_0 \), and we get as previously

\[
\limsup_{n \to +\infty} \|u_n - v_n\|_{C([0,T]; H)} \leq 2\|u_0 - u_{0,m}\|,
\]

which ends the proof.

**End of the Proof of Theorem 4.** We have

\[
\|u_n - u\|_{C([0,T]; H)} \leq \|u_n - v_n\|_{C([0,T]; H)} + \|v_n - u\|_{C([0,T]; H)}.
\]

From Lemma 4, \( \lim_{n \to +\infty} \|v_n - u_n\|_{C([0,T]; H)} = 0 \) and, from the first part of the proof of Theorem 4, \( \lim_{n \to +\infty} \|v_n - u\|_{C([0,T]; H)} = 0 \), which ends the proof.

**Remark 8.** Our result remains true if \( A \) is an \( m \)-accretive operator of some general Banach space \( X \) when we replace a weak solution by an integral solution (cf. [3]), under the assumption that the set \( \{(u, f) : u \in C([0,T]; X), f \in L^1(0,T; X), u \text{ is an integral solution of (47)} \} \) is closed in \( C([0,T]; X) \times L^1(0,T; X) \)-weak, in particular if \( X \) is uniformly convex. Without that assumption we just obtain the relative compactness of the \( \{u_n\} \) in \( C([0,T]; X) \).

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