Closed Expressions for \( \int_0^1 t^{-1} \log^{n-1} t \log^p (1 - t) \, dt \)

By K. S. Kölbig

Abstract. Closed expressions for the integral \( \int_0^1 t^{-1} \log^{n-1} t \log^p (1 - t) \, dt \), whose general form is given elsewhere, are listed for \( n = 1(1)9, p = 1(1)9 \). A formula is derived which allows an easy evaluation of these expressions by formula manipulation on a computer.

1. Introduction. At the beginning of this century, Nielsen discussed, in a little-known monograph [9], properties of a family of functions

\[ S_{n,p}(x) = \int_0^1 t^{-1} \log^{n-1} t \log^p (1 - xt) \, dt \]

for positive integers \( n, p \), and complex \( x \). These functions include many special cases such as Euler's dilogarithm, Kummer's trilogarithm, the Spence functions and polylogarithms. As already proposed [4], it seems appropriate to call the family (1) Nielsen's generalized polylogarithms.

Although the monograph [9] contains quite a number of misprints and a few erroneous results, it does present a considerable amount of useful information, in particular transformation formulae relating \( S_{n,p}(x) \) to \( S_{n,p}(1/x) \) and \( S_{n,p}(1 - x) \). It is remarkable that these formulae, and consequently also those for \( S_{n,p}(1/(1 - x)) \), \( S_{n,p}((x - 1)/x) \), and \( S_{n,p}(x/(x - 1)) \) contain, apart from logarithms and constants, only functions \( S_{n,p}(x) \). However, as far as the author knows, the important formulae of [9] have never found their way into any of the relevant handbooks.

Interest in these functions revived some time ago, at least for the case \( p = 1 \), in the context of multi-dimensional integration of rational functions in quantum electrodynamics (see, for example, [1], [8]). Their properties are also of interest in group theory and geometry [7]. The book of Lewin [6] gives many formulae and properties of \( S_{n,1}(x) \). A general discussion of Nielsen's monograph is given in [4].

2. The Values \( s_{n,p} = S_{n,p}(1) \). The purpose of this note is to give explicit expressions for the special values

\[ s_{n,p} = S_{n,p}(1) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 t^{-1} \log^{n-1} t \log^p (1 - t) \, dt, \]

at least for some \( n \) and \( p \). It is easy to show that \( s_{n,p} = s_{p,n} \), and hence we can restrict \( p \) to \( n \geq p \). A closed expression for \( s_{n,p} \) is given in [4] (in implicit form also in

Received September 15, 1981; revised April 7, 1982.
1980 Mathematics Subject Classification. Primary 33A70.

©1982 American Mathematical Society

0025-5718/82/0000-0381/010.25

647
(3) \[ s_{n,p} = \sum_{k=1}^{p} \frac{(-1)^{k+1}}{k!} \sum_{m_i} H_p(m_1, \ldots, m_k) \frac{\zeta(m_1) \cdots \zeta(m_k)}{m_1 \cdots m_k}, \]

where

(4) \[ H_p(m_1, \ldots, m_k) = \sum_{p_i} \left( \frac{m_1}{p_1} \right) \cdots \left( \frac{m_k}{p_k} \right). \]

The sum over \( m_i \) is to be taken over all sets of integers \( \{m_i\} \) \((i = 1, \ldots, k)\) which satisfy

(5) \[ m_i \geq 2, \quad \sum_{i=1}^{k} m_i = n + p, \]

and the sum over \( p_i \) over all sets of integers \( \{p_i\} \) \((i = 1, \ldots, k)\) which satisfy

(6) \[ 1 \leq p_i \leq m_i - 1, \quad \sum_{i=1}^{k} p_i = p. \]

The function

(7) \[ \zeta(m) = \sum_{k=1}^{\infty} k^{-m} \]

is the Riemann zeta function for integer argument. Nielsen remarked that the functions \( S_{n,p}(x) \) are probably the simplest analytic functions which coincide with \( \zeta(m) \) for special values of its arguments. He added that he was not able to use his theory of \( S_{n,p}(x) \) to find expressions for \( \zeta(2\mu + 1) \) analogous to the known expressions for \( \zeta(2\mu) \).

Nielsen [9] formulated a theorem about the structure of \( s_{n,p} \) and gave the principle of the proof. He also calculated the cases \( p \leq 3 \). The case \( p = 1 \) is trivial, giving

(8) \[ \int_{0}^{1} t^{n-1} \log^{n-1} t \log(1 - t) \, dt = (-1)^{n} (n - 1)! s_{n,1} = (-1)^{n} (n - 1)! \zeta(n + 1). \]

The case \( p = 2 \) can also be handled easily, but the \( k = 3 \) term in (3) for \( p = 3 \) is somewhat more involved, and Nielsen's final expression [9, Section 18 (19), (20)] is incorrect. However, the expression for \( s_{73} \) given as an example in [9] differs from the correct expression only by a difference in the coefficient of \( \zeta(2) \zeta^2(4) \) (\( \frac{1}{2} \) instead of \( \frac{1}{3} \)), and this could be due to a misprint.

Writing (3) as

(9) \[ s_{n,p} = \sum_{k=1}^{p} \frac{(-1)^{k+1}}{k!} \alpha_k(n, p), \]

it is easy to find from (4) the following expressions for \( \alpha_k(n, p) \) in the case of some special values of \( p \) and \( k \):

(10) \[ \alpha_1(n, p) = \frac{(n + p - 1)!}{n! p!} \zeta(n + p), \]
(11) \[ \alpha_2(n, 2) = \sum_{\nu=2}^{n} \xi(\nu)\xi(n - \nu + 2), \]

(12) \[ \alpha_n(n, n) = \xi^n(2). \]

For \( k = 2, p = 3 \), we have from (4), for \( \nu = 2, \ldots, n + 1 \),

\[ H_3(n - \nu + 3, \nu) = \xi_\nu(2) \left( \begin{array}{c} n - \nu + 3 \\ 1 \end{array} \right) + \xi_{\nu+1}(n - \nu + 3) \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \]

and

(13) \[ \frac{H_3(n - \nu + 3, \nu)}{(n - \nu + 3)\nu} = \begin{cases} \frac{3}{2}n & \text{if } \nu = 2, \nu = n + 1, \\ \frac{1}{2}(n + 1) & \text{if } \nu = 3, \ldots, n, \end{cases} \]

where \( \xi_{\nu, \mu} = 0 \) for \( \nu = \mu \) and \( \xi_{\nu, \mu} = 1 \) for \( \nu \neq \mu \), so that

(14) \[ \alpha_2(n, 3) = \frac{n\xi(2)\xi(n + 1) + \frac{1}{2}(n + 1) \sum_{\nu=3}^{n} \xi(\nu)\xi(n - \nu + 3)}{\sum_{\nu=2}^{n} (n - \nu + 2)\xi(\nu)\xi(n - \nu + 3)}. \]

In the case \( k = 2, p = 4 \), one finds for \( \nu = 2, \ldots, n + 2 \),

\[ H_4(n - \nu + 4, \nu) = \xi_\nu(2)\xi_\nu(3) \left( \begin{array}{c} n - \nu + 4 \\ 1 \end{array} \right) + \xi_{\nu+2}(n - \nu + 4) \left( \begin{array}{c} 2 \\ 2 \end{array} \right) \]

Thus

(15) \[ \frac{H_4(n - \nu + 4, \nu)}{(n - \nu + 4)\nu} = \begin{cases} \frac{1}{6}n(n + 1) & \text{if } \nu = 2, \nu = n + 2, \\ \frac{1}{6}n(n + 2) & \text{if } \nu = 3, \nu = n + 1, \\ \frac{1}{12} \left[ \nu^2 - (n + 4)\nu + 2n^2 + 7n + 7 \right] & \text{if } \nu = 4, \ldots, n, \end{cases} \]

and therefore

(16) \[ \alpha_2(n, 4) = \frac{1}{3}n(n + 1)\xi(2)\xi(n + 2) + \frac{1}{3}n(n + 2)\xi(3)\xi(n + 1) + \frac{1}{12} \sum_{\nu=4}^{n} \left[ \nu^2 - (n + 4)\nu + 2n^2 + 7n + 7 \right]\xi(\nu)\xi(n - \nu + 4). \]

For larger values of \( p \), \( \alpha_2(n, p) \) becomes more and more complicated.

For \( k = p = 3 \), we see that \( p_1 = p_2 = p_3 = 1 \) and \( H_3(m_1, m_2, m_3) = m_1m_2m_3 \).

The sum over \( m_i \) in (3) therefore equals the sum over the products \( \xi(m_1)\xi(m_2)\xi(m_3) \) for all partitions \( \{m_1, m_2, m_3\} \) of \( n + 3 \) satisfying \( 2 \leq m_i \leq \lfloor (n + 3)/3 \rfloor \), with a weight for possible permutations, where \( \lfloor \xi \rfloor \) denotes the integer part of \( \xi \). This leads to

(17) \[ \alpha_3(n, 3) = \sum_{\mu=2}^{n} \xi(\mu) \sum_{\nu=\mu}^{n} \omega(n; \mu, \nu)\xi(\nu)\xi(n + 3 - \nu - \mu), \]
where $\mu^* = [(n + 3)/3]$, $\nu^* = [(n - \mu + 3)/2]$, and
\begin{equation}
(18) \quad \omega(n; \mu, \nu) = \begin{cases} 
1 & \text{if } \mu = \nu \text{ and } 3\mu = n + 3, \\
3 & \text{if } \mu = \nu \text{ and } 3\mu \neq n + 3 \text{ or } \\
1 & \text{if } \mu \neq \nu \text{ and } 2\mu + \nu = n + 3 \text{ or } \\
6 & \text{otherwise.}
\end{cases}
\end{equation}

From (1), (10), and (11) it follows that
\begin{equation}
\int_0^1 t^{-1} \log^{n-1} t \log^2(1 - t) \, dt = 2(-1)^{n-1} (n - 1)! s_{n,2}
\end{equation}
\begin{equation}
= (-1)^{n-1} (n - 1)! \left[(n + 1)\xi(n + 2) - \sum_{\nu = 2}^n \xi(\nu)\xi(n - \nu + 2)\right],
\end{equation}

and from (10), (14), and (17),
\begin{equation}
\int_0^1 t^{-1} \log^{n-1} t \log^3(1 - t) \, dt = 6(-1)^n (n - 1)! s_{n,3}
\end{equation}
\begin{equation}
= (-1)^n (n - 1)! \left[(n + 1)(n + 2)\xi(n + 3)
\end{equation}
\begin{equation*}
- 3 \sum_{\nu = 2}^n (n - \nu + 2)\xi(\nu)\xi(n - \nu + 3)
\end{equation*}
\begin{equation*}
+ \sum_{\mu = 2}^{\mu^*} \xi(\mu) \sum_{\nu = \mu}^{\nu^*} \omega(n; \mu, \nu)\xi(\nu)\xi(n + 3 - \nu - \mu)\right].
\end{equation*}

This last formula corrects formula [9, Section 18 (19)] of Nielsen.

For arbitrary $n$ and $p$, it is obvious that (3) can, in practice, be evaluated only by means of a computer. Even then, the problem is complicated. The main task consists in constructing the sets $\{m_i\}$ and $\{p_i\}$. Because of the fact that all permutations have to be taken into account, the number of these sets grows rapidly with increasing values of $n + p$. We have constructed these sets up to $n = p = 9$ by means of a FORTRAN program. As an example, their number is shown for $n = p = 9$ in Table 1. Therefore $\sum \{m_i\}\{p_i\} = 85376$ sets would have to be analyzed in this case. Because of the condition $1 \leq p_i \leq m_i - 1$, only 12870 of these would contribute to the 88 different terms in the result (3) for $s_{9,9}$.

**Table 1**

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>${m_i}$</td>
<td>1</td>
<td>15</td>
<td>91</td>
<td>286</td>
<td>495</td>
<td>462</td>
<td>210</td>
<td>36</td>
<td>1</td>
</tr>
<tr>
<td>${p_i}$</td>
<td>1</td>
<td>8</td>
<td>28</td>
<td>56</td>
<td>70</td>
<td>56</td>
<td>28</td>
<td>8</td>
<td>1</td>
</tr>
</tbody>
</table>

The complicated calculations required for the evaluation of (3) may be avoided by using an alternative expression, well-adapted to evaluation by formula-manipulation.
systems such as REDUCE [8]. As in the derivation of (3), we start from the relation [4, 9]:

\[ s_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \frac{\partial^{n+p-1}}{\partial \beta^{n-1} \partial \alpha^p} \left( \frac{\gamma(1 + \alpha)\gamma(1 + \beta)}{\Gamma(1 + \alpha + \beta)} \right) \bigg|_{\alpha=\beta=0} . \]

We now introduce the power series [2, No. 8.321]

\[ \Gamma(1 + x) = \sum_{k=0}^{\infty} b_k x^k \quad (|x| < 1), \]

\[ \frac{1}{\Gamma(1 + x)} = \sum_{k=0}^{\infty} a_k x^k, \]

where \( a_0 = b_0 = 1 \), and

\[ a_k = \frac{1}{k} \sum_{m=1}^{k} (-1)^{m+1} \xi(m) a_{k-m}, \]

\[ b_k = -\frac{1}{k} \sum_{m=1}^{k} (-1)^{m+1} \xi(m) b_{k-m} \quad (k > 0), \]

with the definition \( \xi(1) = \gamma \) (Euler's constant). Then, performing the differentiations with respect to \( \alpha \) in (21), and using the relation

\[ \sum_{\rho=0}^{p} b_{p-\rho} a_{\rho} = 0 \quad (p > 0), \]

we obtain

\[ \frac{\partial^p}{\partial \alpha^p} \left( \frac{\gamma(1 + \alpha)}{\Gamma(1 + \alpha + \beta)} \right) \bigg|_{\alpha=0} \]

\[ = \sum_{\rho=0}^{p} \binom{p}{\rho} \left( \sum_{k=0}^{\infty} a_k \sum_{\kappa=0}^{k} \binom{k}{\kappa} \alpha^\kappa \beta^{k-\kappa} \right)^{(p)} \left( \sum_{k=0}^{\infty} b_k \alpha^k \right)^{(p-\rho)} \bigg|_{\alpha=0} \]

\[ = p! \sum_{\rho=0}^{p} b_{p-\rho} \sum_{k=\rho+1}^{\infty} a_k \binom{k}{\rho} \beta^{k-\rho} = H(\beta). \]

Similarly

\[ \frac{\partial^{n-1}}{\partial \beta^{n-1}} \frac{1}{\beta} H(\beta) \Gamma(1 + \beta) \]

\[ = p! \sum_{\nu=0}^{n-1} \binom{n-1}{\nu} \left[ H(\beta)/\beta \right]^{(\nu)} \left( \sum_{k=0}^{\infty} b_k \beta^k \right)^{(n-\nu-1)} \bigg|_{\beta=0}, \]

and therefore, finally,

\[ s_{n,p} = (-1)^{n+p-1} \sum_{\nu=0}^{n-1} b_{n-\nu-1} \sum_{\rho=0}^{p} \binom{\nu + \rho + 1}{\rho} b_{p-\rho} a_{\nu+\rho+1}. \]

This expression, although revealing less of the structure (already inferred by Nielsen [9]) of \( s_{n,p} \) than formula (3), namely that \( s_{n,p} \) can be expressed as a homogeneous polynomial of "degree" \( n + p \) in the terms \( \xi(m) \), \((2 \leq m \leq n + p)\), with rational
coefficients, is much more suitable for actual computation. Using a formula-manipulation system, the evaluation of (27) is in fact straightforward once the expressions (23) for \( a_k \) (0 \( \leq \) \( k \) \( \leq \) \( n + p \)) and \( b_k \) [0 \( \leq \) \( k \) \( \leq \) max(\( n - 1 \), \( p \))] in terms of \( \zeta(m) \) have been initially established. It follows from (5) that, at least, all terms involving \( \zeta(1) = \gamma \) will cancel in the final expression for (27). For example, the special cases \( s_{n,1} \) and \( s_{1,p} \) reduce to a single term:

\[
s_{n,1} = (-1)^n \sum_{\nu=0}^{n-1} b_{n-\nu-1} \left[ (\nu + 2) a_{\nu+2} + b_1 a_{\nu+1} \right] = \zeta(n + 1)
\]

and

\[
s_{1,p} = (-1)^p \sum_{\rho=0}^{p} (\rho + 1) b_{p-\rho} a_{\rho+1} = \zeta(\rho + 1).
\]

The results obtained with REDUCE have been checked by evaluating the definition integral (2) by numerical integration, replacing the limits 0 and 1 by \( \epsilon = 10^{-8} \) and \( 1 - \epsilon \), respectively, and using Stieltjes' 32 decimal table [10] of \( \zeta(m) \), \( m = 2(1)70 \), which is reproduced in [9], for the evaluation of \( s_{n,p} \).

We add here that the substitution \( t = \sin^2 \theta \) in (2) leads to the integral [6],

\[
s_{n,p} = -\frac{(-2)^{n+p}}{(n-1)!p!} \int_0^{\pi/2} \cot^n \theta \log^n \sin t \log^p \cos t \, d\theta.
\]

A closed expression for a similar integral,

\[
R_{n,p} = \int_0^{\pi/2} \log^n t \log^p t \, dt \quad (n > 0, p > 0),
\]

has been given in [5], with examples up to \( n = p = 4 \).

3. A Table of the Integral. We list the expressions for \( s_{n,p} \), \( n = 1(1)9, p = 1(1)n \). The values for the integral in (2) itself,

\[
r_{n,p} = \int_0^1 t^{n-1} \log^n t \log^p (1 - t) \, dt = (-1)^{n+p-1} (n - 1)! p! s_{n,p},
\]

would lead for higher \( n \) or \( p \) to rather large coefficients. The reference work [2, No. 4.2912] lists only the case \( n = p = 1 \), whereas Lewin [6] gives (31) for \( n = 2, 3, 4 \), and \( p = 2 \).

Using the well-known relation [2, No. 9.5421],

\[
\zeta(2\mu) = \frac{2^{2\mu-1} \pi^{2\mu} | B_{2\mu} |}{(2\mu)!},
\]

where \( B_{2\mu} \) are the Bernoulli numbers, the expressions for \( r_{n,p} \) simplify to some extent. We also give these values for \( n = 1(1)7, p = 1(1)n \).

\[
s_{11} = \zeta(2),
\]
\[
s_{21} = \zeta(3),
\]
\[
s_{22} = -\frac{1}{2} \zeta^2(2) + \frac{1}{3} \zeta(4),
\]
\[
s_{31} = \zeta(4),
\]
\[
s_{32} = -\zeta(2) \zeta(3) + 2 \zeta(5),
\]
\[ s_{33} = \frac{1}{6} \zeta^3(2) - \frac{1}{4} \zeta(2) \zeta(4) - \zeta^2(3) + \frac{19}{3} \zeta(6), \]
\[ s_{41} = \zeta(5), \]
\[ s_{42} = -\zeta(2) \zeta(4) - \frac{1}{2} \zeta^2(3) + \frac{3}{5} \zeta(6), \]
\[ s_{43} = \frac{1}{3} \zeta^2(2) \zeta(3) - 2 \zeta(2) \zeta(5) - \frac{1}{3} \zeta(3) \zeta(4) + 5 \zeta(7), \]
\[ s_{44} = -\frac{1}{36}\zeta^4(2) + \frac{7}{36} \zeta^2(2) \zeta(4) + \zeta(2) \zeta^2(3) - \frac{19}{10} \zeta(2) \zeta(6) - 4 \zeta(3) \zeta(5) \]
\[ - \frac{1}{12} \zeta^2(4) + \frac{33}{4} \zeta(8), \]
\[ r_{11} = -\frac{1}{6} \pi^2, \]
\[ r_{21} = \zeta(3), \]
\[ r_{22} = -\frac{1}{180} \pi^4, \]
\[ r_{31} = -\frac{1}{45} \pi^4, \]
\[ r_{32} = -\frac{2}{3} \pi^2 \zeta(3) + 8 \zeta(5), \]
\[ r_{33} = -\frac{23}{1260} \pi^6 + 12 \zeta^2(3), \]
\[ r_{41} = 6 \zeta(5), \]
\[ r_{42} = -\frac{1}{105} \pi^6 + 6 \zeta^2(3), \]
\[ r_{43} = -\frac{1}{5} \pi^4 \zeta(3) - 12 \pi^2 \zeta(5) + 180 \zeta(7), \]
\[ r_{44} = -\frac{1}{1800} \pi^8 - 24 \pi^2 \zeta^2(3) + 576 \zeta(3) \zeta(5). \]

The remaining expressions for \( s_{n,p}, n = 5(1)9, p = 1(1)n, \) and \( r_{n,p}, n = 5(1)7, p = 1(1)n, \) are given in the microfiche section at the end of this issue. Numerical values of \( s_{n,p} \) with 21 digits are presented in Table 2.

**Table 2**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( p )</th>
<th>( S_{n,p} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1 \times 10^9</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.20205 \times 10^{10}</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2.70580 \times 10^9</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1.08232 \times 10^{10}</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>9.65511 \times 10^9</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1.74898 \times 10^9</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1.03692 \times 10^9</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>4.05368 \times 10^9</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>4.12316 \times 10^8</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>6.02891 \times 10^8</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1.01734 \times 10^8</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>1.83559 \times 10^8</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>1.10762 \times 10^8</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>1.06090 \times 10^8</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>1.29078 \times 10^8</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1.00834 \times 10^8</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>8.65052 \times 10^7</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>3.20419 \times 10^7</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>2.08107 \times 10^7</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>1.81177 \times 10^7</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>1.88257 \times 10^7</td>
</tr>
</tbody>
</table>

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Table 2 (continued)

<table>
<thead>
<tr>
<th>n</th>
<th>p</th>
<th>$s_{n,p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1</td>
<td>1.00407</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>4.17024</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>9.70014</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>4.37446</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>2.79046</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>2.19761</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>1.99035</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>1.00200</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>2.03771</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>3.02392</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>9.63193</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>4.58067</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>2.78108</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>1.98864</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>1.59526</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>1.00099</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>1.00397</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>9.61339</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>2.19049</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>7.86919</td>
</tr>
<tr>
<td>9</td>
<td>6</td>
<td>3.73432</td>
</tr>
<tr>
<td>9</td>
<td>7</td>
<td>2.13492</td>
</tr>
<tr>
<td>9</td>
<td>8</td>
<td>1.39313</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>1.00261</td>
</tr>
</tbody>
</table>

European Organization for Nuclear Research
CERN
CH–1211 Geneva 23, Switzerland

10. T. J. Stieltjes, “Table des valeurs des sommes $S_n = \sum_{i=1}^{\infty} n^{-s}$,” *Acta Math.*, v. 10, 1887, pp. 299–302.