On the Integral $\int_0^{\pi/2} \log^n \cos x \log^p \sin x \, dx$

By K. S. Kölblig

Abstract. A formula is derived for the integral in the title which allows easy evaluation by formula manipulation on a computer.

1. Introduction. In a monograph on generalized polylogarithms, and in a paper on series of reciprocal powers, Nielsen [7], [8] remarked that

\[(1) \quad r_{np} = \int_0^{\pi/2} \log^n \cos x \log^p \sin x \, dx \quad (n \geq 0, \ p \geq 0)\]

can be expressed as $\pi$ times a polynomial in $\eta(q)$ ($1 \leq q \leq n + p$), with rational coefficients, where

\[(2) \quad \eta(q) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^q} = \begin{cases} \log 2 & (q = 1), \\ (1 - 2^{1-q})\zeta(q) & (q > 1). \end{cases}\]

This polynomial is homogeneous of degree $n + p$ if one considers $\eta(q)$ to be of degree $q$, and if the degree of a product is the sum of the degrees of its factors. $\zeta(q)$ is the Riemann zeta function for integer arguments. Since $r_{np} = r_{pn}$, it is sufficient to consider $n \geq p \geq 0$.

In an earlier paper [4], the following complicated closed expression for $r_{np}$ was derived:

\[(3) \quad r_{np} = \frac{\pi n! p!}{2^{n+p+1}} \sum_{k=1}^{n+p} \frac{1}{k!} \sum_{\{p_i\}, \{n_i\}} \sum f(p_1, n_1) \cdots f(p_k, n_k),\]

where the innermost sums run over all sets $\{p_i\}, \{n_i\}$, which satisfy the conditions

\[(4) \quad p_i \geq 0, \quad \sum_{i=1}^k p_i = p; \quad n_i \geq 0, \quad \sum_{i=1}^k n_i = n,\]

and where the function $f(r, s)$ is given by

\[(5) \quad f(r, s) = (1 - \delta_{or})(1 - \delta_{os})\left(\frac{(-1)^{r+s+1}}{r+s}(r+s)\zeta(r+s)\right) + \delta_{or} - \delta_{os} \xi(r+s),\]

where

\[(6) \quad \xi(q) = \begin{cases} -2 \log 2, & q = 1, \\ (-1)^q \frac{2^q - 2}{2} \xi(q), & q > 1, \end{cases}\]

and $\delta_{ij}$ is the Kronecker symbol.
The actual computation of \( r_{np} \) from (3), even for small \( n \) and \( p \), is quite complicated. Explicit expressions for \( r_{np} \) \((1 \leq n \leq 4, 0 \leq p \leq n)\) in terms of \( \eta(q) \), computed from (3), have been given in [4]. It may be noted that, in the relevant handbooks, only expressions for \( r_{n0} \) or \( r_{0n} \) with \( n = 1, 2 \) are listed.*

2. An Expression for \( r_{np} \). It is the purpose of this note to derive another expression for \( r_{np} \) which is well-adapted to evaluation by a formula-manipulation system such as REDUCE [2]. As in the derivation of (3), we start from the definition (1) and, after substituting \( t = \cos^2 x \) in (1), express \( r_{np} \) as a derivative of Euler's beta function:

\[
\begin{align*}
\theta_{np} & = \frac{1}{2n+p+1} \frac{\partial^{n+p}}{\partial \beta^n \partial \alpha^p} \int_0^1 t^{\beta-1/2} (1-t)^{\alpha-1/2} dt \bigg|_{\alpha=\beta=0} \\
& = \frac{1}{2n+p+1} \frac{\partial^{n+p}}{\partial \beta^n \partial \alpha^p} \left. \frac{\Gamma(\frac{1}{2}+\alpha)\Gamma(\frac{1}{2}+\beta)}{\Gamma(1+\alpha+\beta)} \right|_{\alpha=\beta=0}
\end{align*}
\]

We introduce the power series [6]

\[
\theta(1+x) = \sum_{k=0}^{\infty} b_k x^k \quad (|x| < 1),
\]

\[
\frac{1}{\theta(1+x)} = \sum_{k=0}^{\infty} a_k x^k \quad (|x| < \infty),
\]

where \( a_0 = b_0 = 1 \),

\[
\begin{align*}
a_k &= \frac{-1}{k} \sum_{m=1}^{k} (-1)^m \xi(m) a_{k-m} \\
b_k &= \frac{1}{k} \sum_{m=1}^{k} (-1)^m \xi(m) b_{k-m} \quad (k > 0),
\end{align*}
\]

and \( \xi(1) = \gamma \) (Euler's constant), \( \xi(m) = \zeta(m) \) for \( m \geq 2 \). A direct approach by setting \( x = -\frac{1}{2} + \alpha \) and \( x = -\frac{1}{2} + \beta \) in (8) and then carrying out the differentiations in (7) does not lead to a satisfactory result; the resulting expression would contain infinite series, and the expected quantities \( \pi \) and \( \log 2 \) would appear only implicitly.

We therefore take a different route and apply the duplication formula [5]

\[
\theta(2x) = \frac{1}{\sqrt{2\pi}} 2^{2x-1/2} \theta(x) \Gamma\left(\frac{1}{2} + x\right),
\]

so that from (7)

\[
\begin{align*}
\theta_{np} &= \frac{\pi}{2n+p+1} \frac{\partial^{n+p}}{\partial \beta^n \partial \alpha^p} \left. 2^{-2\alpha-2\beta} \frac{\Gamma(1+2\alpha)\Gamma(1+2\beta)}{\Gamma(1+\alpha+\beta)\Gamma(1+\alpha)\Gamma(1+\beta)} \right|_{\alpha=\beta=0}
\end{align*}
\]

We have separated out the factor \( \pi \), and the terms involving \( \log 2 \) will arise from the differentiation of \( 2^{-2\alpha-2\beta} \).

The differentiations with respect to \( \alpha \) and \( \beta \) may now be carried out in (11). The result is a complicated six-fold sum in a form suitable for evaluation by formula

*Nielsen [8] made a similar comment.
manipulation. A much simpler expression can, however, be obtained as follows. We rewrite (11) as

\[ r_{n,p} = \frac{\pi}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial \beta^n \partial \alpha^p} \frac{G(\alpha)G(\beta)}{\Gamma(1 + \alpha + \beta)} \bigg|_{\alpha = \beta = 0}, \]

where

\[ G(x) = 2^{-2x} \frac{\Gamma(1 + 2x)}{\Gamma(1 + x)}, \]

and develop \( G(x) \) as a power series. Using the series [6]

\[ \log \Gamma(1 + x) = -\gamma x + \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k) x^k}{k} \quad (|x| < 1), \]

we find

\[ \log G(x) = -(\gamma + 2 \log 2)x + \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k} \frac{2^k - 1}{k} x^k \quad \left( |x| < \frac{1}{2} \right). \]

In order to obtain a series for \( G(x) \) from (15), we make use of the theorem that if

\[ f(x) = \sum_{k=0}^{\infty} a_k x^k \]

is a formal power series with \( a_0 = 0 \), then

\[ e^{f(x)} = \sum_{k=0}^{\infty} c_k x^k, \]

where the coefficients \( c_k \) are given recursively by

\[ c_k = \frac{1}{k} \sum_{m=1}^{k} ma_m c_{k-m} \quad (k > 0). \]

This theorem can be proved analogously to Theorem 1.6c of Henrici [3, p. 42]. Applying this result to the series (15), we find

\[ G(x) = \sum_{k=0}^{\infty} b_k^* x^k \quad \left( |x| < \frac{1}{2} \right), \]

where \( b_0^* = 1 \),

\[ b_k^* = \frac{1}{k} \sum_{m=1}^{k} (-1)^m \xi^*(m) b_{k-m}^* \]

and

\[ \xi^*(m) = \begin{cases} \gamma + 2 \log 2, & m = 1, \\ (2^m - 1) \xi(m), & m > 1. \end{cases} \]

We now differentiate (12) with respect to \( \alpha \) and obtain

\[ H(\beta) = \frac{\partial^p}{\partial \alpha^p} \left. \frac{G(\alpha)}{\Gamma(1 + \alpha + \beta)} \right|_{\alpha = \beta = 0} = \sum_{\rho=0}^{p} \binom{p}{\rho} \Gamma^{-1}(1 + \alpha + \beta)^{(\rho)} G(\alpha)^{(p-\rho)} \bigg|_{\alpha = \beta = 0} = p! \sum_{\rho=0}^{p} b_{p-\rho}^* \sum_{k=\rho}^{\infty} a_k \binom{k}{\rho} \beta^{k-\rho}. \]
Similarly,

\[ \frac{\partial^n}{\partial \beta^n} H(\beta) G(\beta) \bigg|_{\beta=0} = \sum_{\nu=0}^{n} \binom{n}{\nu} H(\beta)^{(\nu)} G(\beta)^{(n-\nu)}. \]

and therefore finally

\[ r_{np} = \frac{\pi n! p!}{2^{n+p+1}} \sum_{\nu=0}^{n} b^{*}_{n-\nu} \sum_{\rho=0}^{p} \binom{\nu+\rho}{\rho} b^{*}_{p-\rho} a_{\nu+\rho}. \]

This expression, although still complicated and revealing less of the structure of \( r_{np} \) than formula (3), is much more suitable for actual computation. Using a formula manipulation system, the evaluation of (22) is in fact straightforward once the expressions for \( a_k \) \((0 < k < n + p)\) and \( b^{*}_k \) \((0 < k < \text{max}(n, p))\) have been initially established. It follows from (5) that, at least, all terms involving \( \tilde{\zeta}(1) = \gamma \) will cancel in the final expression for (22). For the case \( n \geq 0, p = 0 \), (22) reduces to

\[ r_{n0} = \frac{\pi n!}{2^{n+1}} \sum_{\nu=0}^{n} b^{*}_{n-\nu} a_{\nu}, \]

which is another form of Bowman’s determinant [1] for \( r_{n0} \). The result of Bowman for \( r_{n0} \) can also be found in the book by Lewin [5].

\[ r_{n0} = -\frac{1}{2} \pi \log 2, \]

\[ r_{11} = \frac{\pi}{8} (-\zeta(2) + 4 \log^2 2) = \frac{\pi}{2} \left( -\frac{1}{24} \pi^2 + \log^2 2 \right), \]

\[ r_{20} = \frac{\pi}{4} (\zeta(2) + 2 \log^2 2) = \frac{\pi}{2} \left( \frac{1}{12} \pi^2 + \log^2 2 \right), \]

\[ r_{21} = \frac{\pi}{8} (\zeta(3) - 4 \log^3 2) = \frac{\pi}{2} \left( -\log^3 2 + \frac{1}{4} \zeta(3) \right), \]

\[ r_{22} = \frac{\pi}{16} (-3\zeta(4) - 8\zeta(3)\log 2 + 3\zeta^2(2) + 8 \log^4 2) \]

\[ = \frac{\pi}{2} \left( \frac{1}{160} \pi^4 + \log^4 2 - \zeta(3)\log 2 \right). \]

Numerical values of \( r_{np} \) for \( 0 \leq n \leq 5, 0 \leq n \leq p \), with 21 digits are given in Table 2.
ON THE INTEGRAL \(\int_0^{\pi/2} \log^a \cos x \log^b \sin x \, dx\)

**Table 1**

\[ r_{50} = -\frac{\pi}{8} \left( 90\xi(5) + 105\xi(4)\log 2 + 30\xi(2)\xi(3) + 60\xi(3)\log^2 2 \right) + 15\xi^2(2)\log 2 + 20\xi(2)\log^3 2 + 4 \log^5 2 \]

\[ r_{51} = \frac{\pi}{32} \left( -30\xi(6) + 300\xi(5)\log 2 - 135\xi(2)\xi(4) - 60\xi^2(3) + 360\xi(4)\log^2 2 \right) - 60\xi(2)\xi(3)\log 2 - 15\xi^3(2) + 200\xi(3)\log^3 2 + 60\xi(2)\log^4 2 + 16 \log^6 2 \]

\[ r_{52} = \frac{\pi}{32} \left( 90\xi(7) + 210\xi(6)\log 2 - 150\xi(2)\xi(5) + 165\xi(3)\xi(4) \right) - 120\xi(5)\log^2 2 + 90\xi(2)\xi(4)\log 2 + 210\xi^2(3)\log 2 - 105\xi^2(2)\xi(3) - 240\xi(4)\log^3 2 + 120\xi(2)\xi(3)\log^2 2 - 30\xi^4(2)\log 2 - 140\xi(3)\log^4 2 - 48\xi(2)\log^5 2 - 16 \log^7 2 \]

\[ r_{53} = \frac{\pi}{64} \left( -630\xi(8) - 1440\xi(7)\log 2 - 165\xi(2)\xi(6) + 990\xi(3)\xi(5) - 180\xi^2(4) \right) - 1680\xi(6)\log^2 2 + 720\xi(2)\xi(5)\log 2 + 360\xi(3)\xi(4)\log 2 \]

\[ r_{54} = \frac{\pi}{64} \left( 2520\xi(9) + 5670\xi(8)\log 2 + 540\xi(2)\xi(7) + 210\xi(3)\xi(6) \right) - 4230\xi(4)\xi(5) + 6480\xi(7)\log^2 2 + 1260\xi(2)\xi(6)\log 2 \]

\[ r_{55} = \frac{\pi}{256} \left( -45360\xi(10) - 100800\xi(9)\log 2 - 9450\xi(2)\xi(8) - 3600\xi(3)\xi(7) \right) + 5400\xi(4)\xi(6) + 73800\xi^2(5) - 113400\xi(8)\log^2 2 - 21600\xi(2)\xi(7)\log 2 \]

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Table 2

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1. F. Bowman, “Note on the integral \( \int_0^{\pi/2} (\log \sin \theta)^n d\theta \),” J. London Math. Soc., v. 22, 1947, pp. 172–173.
11. N. Nielsen, “Théorème sur les intégrales \( \int_0^{\pi/4} \log^p \sin 2\phi d\phi \) et \( \int_0^{\pi/4} \tan \phi \log^p \sin 2\phi d\phi \),” Oversigt Danske Vid. Selsk. Forh., 1897, pp. 197–206.