On the Integral $\int_0^{\pi/2} \log^n \cos x \log^p \sin x \, dx$

By K. S. Kölblg

Abstract. A formula is derived for the integral in the title which allows easy evaluation by formula manipulation on a computer.

1. Introduction. In a monograph on generalized polylogarithms, and in a paper on series of reciprocal powers, Nielsen [7, 8] remarked that

$$r_{np} = \int_0^{\pi/2} \log^n \cos x \log^p \sin x \, dx \quad (n \geq 0, \ p \geq 0)$$

can be expressed as $\pi$ times a polynomial in $\eta(q)$ $(1 \leq q \leq n + p)$, with rational coefficients, where

$$\eta(q) = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k^q} = \begin{cases} \log 2 & \text{(q = 1)}, \\ (1 - 2^{1-q}) \zeta(q) & \text{(q > 1)}. \end{cases}$$

This polynomial is homogeneous of degree $n + p$ if one considers $\eta(q)$ to be of degree $q$, and if the degree of a product is the sum of the degrees of its factors. $\zeta(q)$ is the Riemann zeta function for integer arguments. Since $r_{np} = r_{pn}$, it is sufficient to consider $n \geq p \geq 0$.

In an earlier paper [4], the following complicated closed expression for $r_{np}$ was derived:

$$r_{np} = \frac{\pi n! p!}{2^{n+p+1}} \sum_{k=1}^{n+p} \sum_{\{p_i\}, \{n_i\}} f(p_1, n_1) \cdots f(p_k, n_k),$$

where the innermost sums run over all sets $\{p_i\}, \{n_i\}$, which satisfy the conditions

$$p_i \geq 0, \ \sum_{i=1}^k p_i = p; \quad n_i \geq 0, \ \sum_{i=1}^k n_i = n,$$

and where the function $f(r, s)$ is given by

$$f(r, s) = (1 - \delta_{0r})(1 - \delta_{0s}) \frac{(-1)^{r+s+1}}{r+s} \left( r + s \right) \xi(r+s) - \delta_{0r} - \delta_{0s} | \xi(r+s),$$

where

$$\xi(q) = \begin{cases} -2 \log 2, & q = 1, \\ (-1)^q \frac{2^q - 2}{q} \xi(q), & q > 1, \end{cases}$$

and $\delta_{ij}$ is the Kronecker symbol.

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The actual computation of \( r_{np} \) from (3), even for small \( n \) and \( p \), is quite complicated. Explicit expressions for \( r_{np} \) (\( 1 \leq n \leq 4, 0 \leq p \leq n \)) in terms of \( \eta(q) \), computed from (3), have been given in [4]. It may be noted that, in the relevant handbooks, only expressions for \( r_{n0} \) or \( r_{0n} \) with \( n = 1, 2 \) are listed.*

2. An Expression for \( r_{np} \). It is the purpose of this note to derive another expression for \( r_{np} \), which is well-adapted to evaluation by a formula-manipulation system such as REDUCE [2]. As in the derivation of (3), we start from the definition (1) and, after substituting \( t = \cos^2 x \) in (1), express \( r_{np} \) as a derivative of Euler's beta function:

\[
\begin{align*}
    r_{np} &= \frac{1}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial \alpha \partial \beta} \int_0^1 t^{\beta-1/2} (1 - t)^{\alpha-1/2} \, dt \bigg|_{\alpha=\beta=0} \\
    &= \frac{1}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial \alpha \partial \beta} \Gamma\left(\frac{1}{2} + \alpha\right) \Gamma\left(\frac{1}{2} + \beta\right) \bigg/ \Gamma(1 + \alpha + \beta) \bigg|_{\alpha=\beta=0}
\end{align*}
\]

We introduce the power series [6]

\[
\begin{align*}
    \Gamma(1 + x) &= \sum_{k=0}^{\infty} b_k x^k \quad (|x| < 1), \\
    \frac{1}{\Gamma(1 + x)} &= \sum_{k=0}^{\infty} a_k x^k \quad (|x| < \infty),
\end{align*}
\]

where \( a_0 = b_0 = 1 \),

\[
\begin{align*}
    a_k &= -\frac{1}{k} \sum_{m=1}^{k} (-1)^m \zeta^\prime(m) a_{k-m}, \\
    b_k &= \frac{1}{k} \sum_{m=1}^{k} (-1)^m \zeta^\prime(m) b_{k-m} \quad (k > 0),
\end{align*}
\]

and \( \zeta(1) = \gamma \) (Euler's constant), \( \zeta(m) = \xi(m) \) for \( m \geq 2 \). A direct approach by setting \( x = -\frac{1}{2} + \alpha \) and \( x = -\frac{1}{2} + \beta \) in (8) and then carrying out the differentiations in (7) does not lead to a satisfactory result; the resulting expression would contain infinite series, and the expected quantities \( \pi \) and \( \log 2 \) would appear only implicitly. We therefore take a different route and apply the duplication formula [5]

\[
\begin{align*}
    \Gamma(2x) &= \frac{1}{\sqrt{2\pi}} 2^{x-1/2} \Gamma(x) \Gamma\left(\frac{1}{2} + x\right),
\end{align*}
\]

so that from (7)

\[
\begin{align*}
    r_{np} &= \frac{\pi}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial \alpha \partial \beta} \frac{2^{-2\alpha-2\beta}}{\Gamma(1 + 2\alpha) \Gamma(1 + 2\beta)} \bigg|_{\alpha=\beta=0} \\
    &= \frac{\pi}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial \alpha \partial \beta} \frac{\Gamma(1 + 2\alpha) \Gamma(1 + 2\beta)}{\Gamma(1 + \alpha + \beta)} \bigg|_{\alpha=\beta=0}
\end{align*}
\]

We have separated out the factor \( \pi \), and the terms involving \( \log 2 \) will arise from the differentiation of \( 2^{-2\alpha-2\beta} \).

The differentiations with respect to \( \alpha \) and \( \beta \) may now be carried out in (11). The result is a complicated six-fold sum in a form suitable for evaluation by formula

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*Nielsen [8] made a similar comment.
A much simpler expression can, however, be obtained as follows. We rewrite (11) as

\[ r_{np} = \frac{\pi}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial \beta^n \partial \alpha^p} \frac{G(\alpha)G(\beta)}{\Gamma(1 + \alpha + \beta)} \bigg|_{\alpha = \beta = 0}, \]

where

\[ G(x) = 2^{-2x} \frac{\Gamma(1 + 2x)}{\Gamma(1 + x)}, \]

and develop \( G(x) \) as a power series. Using the series [6]

\[ \log \Gamma(1 + x) = -x + \sum_{k=2}^{\infty} \frac{(-1)^k}{k} x^k / k \quad (|x| < 1), \]

we find

\[ \log G(x) = -(\gamma + 2 \log 2) x + \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \frac{2^k - 1}{k} x^k \quad (|x| < \frac{1}{2}). \]

In order to obtain a series for \( G(x) \) from (15), we make use of the theorem that if

\[ f(x) = \sum_{k=1}^{\infty} a_k x^k \]

is a formal power series with \( a_0 = 0 \), then

\[ e^{f(x)} = \sum_{k=0}^{\infty} c_k x^k, \]

where the coefficients \( c_k \) are given recursively by

\[ c_k = \frac{1}{k} \sum_{m=1}^{k} ma_m c_{k-m} \quad (k > 0). \]

This theorem can be proved analogously to Theorem 1.6c of Henrici [3, p. 42]. Applying this result to the series (15), we find

\[ G(x) = \sum_{k=0}^{\infty} b^*_k x^k \quad (|x| < \frac{1}{2}), \]

where \( b^*_0 = 1 \),

\[ b^*_k = \frac{1}{k} \sum_{m=1}^{k} (-1)^m \xi^*(m) b^*_{k-m} \]

and

\[ \xi^*(m) = \begin{cases} \gamma + 2 \log 2, & m = 1, \\ (2^m - 1) \xi(m), & m > 1. \end{cases} \]

We now differentiate (12) with respect to \( \alpha \) and \( \beta \) and obtain

\[ H(\beta) = \frac{\partial^p}{\partial \alpha^p} \frac{G(\alpha)}{\Gamma(1 + \alpha + \beta)} \bigg|_{\alpha = 0} = \sum_{p=0}^{\infty} \binom{p}{\rho} \Gamma^{-1}(1 + \alpha + \beta)^{(\rho)} G(\alpha)^{(\rho - \rho)} \bigg|_{\alpha = 0} \]

\[ = p ! \sum_{\rho=0}^{p} b^*_{p-\rho} \sum_{k=\rho}^{\infty} a_k \left( \frac{k}{\rho} \right) \beta^{k-\rho}. \]
Similarly,

\begin{equation}
\frac{\partial^n}{\partial \beta^n} H(\beta) G(\beta) \bigg|_{\beta=0} = \sum_{\nu=0}^{n} \binom{n}{\nu} H(\beta)^{\nu} G(\beta)^{n-\nu},
\end{equation}

and therefore finally

\begin{equation}
r_{np} = \frac{\pi n!}{2^{n+p+1}} \sum_{\nu=0}^{n} b_{n-\nu}^{*} \sum_{\rho=0}^{p} \left( \nu + \rho \right) b_{\rho}^{*} a_{\nu+p}.
\end{equation}

This expression, although still complicated and revealing less of the structure of $r_{np}$ than formula (3), is much more suitable for actual computation. Using a formula manipulation system, the evaluation of (22) is in fact straightforward once the expressions for $a_k$ ($0 < k < n + p$) and $b_k^{*}$ ($0 < k < \max(n, p)$) have been initially established. It follows from (5) that, at least, all terms involving $\zeta(1) = \gamma$ will cancel in the final expression for (22). For the case $n \geqslant 0, p = 0$, (22) reduces to

\begin{equation}
r_{n0} = \frac{\pi n!}{2^{n+1}} \sum_{\nu=0}^{n} b_{n-\nu}^{*} a_{\nu},
\end{equation}

which is another form of Bowman’s determinant [1] for $r_{n0}$. The result of Bowman for $r_{n0}$ can also be found in the book by Lewin [5].

3. A Table for $r_{np}$. We give here, as examples, the expressions for $r_{np}$ for $n = 1, 2$ and $0 \leqslant p \leqslant n$. In order to complete the table of $r_{np}$ given in [4], we also present expressions for $r_{np}$ for $n = 5$ and $0 \leqslant p \leqslant 5$ in Table 1. Note that Nielsen has already given the expressions for $r_{20}$, $r_{11}$, $r_{30}$ in [9], for $r_{02}$, $r_{11}$, $r_{03}$ in [10], and for $r_{12}$, $r_{22}$, $r_{13}$ in [11], derived by different methods.

\begin{align*}
    r_{10} &= -\frac{1}{2} \pi \log 2, \\
    r_{11} &= \frac{\pi}{8} \left( -\zeta(2) + 4 \log^2 2 \right) = \frac{\pi}{2} \left( -\frac{1}{24} \pi^2 + \log^2 2 \right), \\
    r_{20} &= \frac{\pi}{4} \left( \zeta(2) + 2 \log^2 2 \right) = \frac{\pi}{2} \left( -\frac{1}{12} \pi^2 + \log^2 2 \right), \\
    r_{21} &= \frac{\pi}{8} \left( \zeta(3) - 4 \log^3 2 \right) = \frac{\pi}{2} \left( -\log^3 2 + \frac{1}{4} \zeta(3) \right), \\
    r_{22} &= \frac{\pi}{16} \left( -3 \zeta(4) - 8 \zeta(3) \log 2 + 3 \zeta^2(2) + 8 \log^4 2 \right) \\
    &= \frac{\pi}{2} \left( \frac{1}{160} \pi^4 + \log^4 2 - \zeta(3) \log 2 \right).
\end{align*}

Numerical values of $r_{np}$ for $0 \leqslant n \leqslant 5$, $0 \leqslant n \leqslant p$, with 21 digits are given in Table 2.
ON THE INTEGRAL $\int_0^{\pi/2} \log^a \cos x \log^b \sin x \, dx$

TABLE 1

\begin{align*}
r_{50} &= -\frac{\pi}{8} \left( 90\xi(5) + 105\xi(4) \log 2 + 30\xi(2)^2 \xi(3) + 60\xi(3) \log^2 2 \\
&\quad + 15\xi^2(2) \log 2 + 20\xi(2) \log^3 2 + 4 \log^5 2 \right) \\
r_{51} &= \frac{\pi}{32} \left( -30\xi(6) + 300\xi(5) \log 2 - 135\xi(2)^2 \xi(4) - 60\xi^2(3) + 360\xi(4) \log^2 2 \\
&\quad - 60\xi(2)^2 \xi(3) \log 2 - 15\xi^3(2) + 200\xi(3) \log^3 2 + 60\xi(2) \log^4 2 + 16 \log^6 2 \right) \\
r_{52} &= \frac{\pi}{32} \left( 90\xi(7) + 210\xi(6) \log 2 - 150\xi(2)^2 \xi(5) + 165\xi(3) \xi(4) \\
&\quad - 120\xi(5) \log^2 2 + 90\xi(2)^2 \xi(4) \log 2 + 210\xi^2(3) \log 2 - 105\xi^3(2) \xi(3) \\
&\quad + 240\xi(4) \log^3 2 + 120\xi(2)^2 \xi(3) \log^2 2 - 30\xi^4(2) \log 2 \\
&\quad - 140\xi(3) \log^4 2 - 48\xi(2) \log^5 2 - 16 \log^7 2 \right) \\
r_{53} &= \frac{\pi}{64} \left( -630\xi(8) - 1440\xi(7) \log 2 - 165\xi(2)^2 \xi(6) + 990\xi(3) \xi(5) - 180\xi^2(4) \\
&\quad - 1680\xi(6) \log^2 2 + 720\xi(2)^2 \xi(5) \log 2 + 360\xi(3) \xi(4) \log 2 \\
&\quad - 360\xi^2(2) \xi(4) + 405\xi(2)^2 \xi(3) - 600\xi(5) \log^3 2 + 180\xi(2) \xi(4) \log^2 2 \\
&\quad + 360\xi^2(2) \xi(3) \log 2 - 60 \xi^4(2) + 60 \xi(4) \log^4 2 + 120\xi(2) \xi(3) \log^3 2 \\
&\quad + 60\xi^3(2) \log^2 2 + 168\xi(3) \log^5 2 + 60\xi^2(2) \log^4 2 + 88\xi(2) \log^6 2 + 32 \log^8 2 \right) \\
r_{54} &= \frac{\pi}{64} \left( 2520\xi(9) + 5670\xi(8) \log 2 + 540\xi(2)^2 \xi(7) + 210\xi(3) \xi(6) \\
&\quad - 4230\xi(4) \xi(5) + 6480\xi(7) \log^2 2 + 1260\xi(2)^2 \xi(6) \log 2 \\
&\quad - 3960\xi(3) \xi(5) \log 2 - 4455\xi^2(4) \log 2 - 450\xi^2(2) \xi(5) + 990\xi(2) \xi(3) \xi(4) \\
&\quad + 630\xi^3(3) + 5040\xi(6) \log^3 2 - 720\xi(2)^2 \xi(5) \log^2 2 \\
&\quad - 6120\xi(3) \xi(4) \log^2 2 + 270\xi^2(2) \xi(4) \log 2 + 1260\xi(2) \xi^3(3) \log 2 \\
&\quad - 210\xi^3(2) \xi(3) + 2280\xi(5) \log^4 2 - 1440\xi(2) \xi(4) \log^3 2 \\
&\quad - 1680\xi^2(3) \log^3 2 + 360\xi^2(2) \xi(3) \log^2 2 - 45 \xi^4(2) \log 2 \\
&\quad + 432\xi(4) \log^4 2 - 840\xi(2)^2 \xi(3) \log^4 2 - 112 \xi^3(3) \log^3 2 \\
&\quad + 144\xi^2(2) \log^3 2 - 96 \xi(2) \log^5 2 - 32 \log^7 2 \right) \\
r_{55} &= \frac{\pi}{256} \left( -45360\xi(10) - 100800\xi(9) \log 2 - 9450\xi(2)^2 \xi(8) - 3600\xi(3) \xi(7) \\
&\quad + 5400\xi(4) \xi(6) + 73800\xi^2(5) - 113400\xi(8) \log^2 2 - 21600\xi(2)^2 \xi(7) \log 2 \\
&\quad - 8400\xi(3) \xi(6) \log 2 + 169200\xi(4) \xi(5) \log 2 - 1800\xi^2(2) \xi(6) \\
&\quad + 3960\xi(2)^2 \xi(3) \xi(5) - 33075\xi(2)^2 \xi(4)^2 - 33300\xi^2(3) \xi(4)^2 \\
&\quad - 86400\xi(7) \log^3 2 - 25200\xi(2)^2 \xi(6) \log^2 2 + 79200\xi(3) \xi(5) \log^2 2 \\
&\quad + 89100\xi^2(4) \log^2 2 + 1800\xi^2(2) \xi(5) \log 2 - 39600\xi(2)^2 \xi(3) \xi(4) \log 2 \\
&\quad - 25200\xi^3(3) \log 2 - 6750\xi^2(2) \xi(4) + 15300\xi^2(2) \xi^2(3) \xi(3) \\
&\quad - 50400\xi(6) \log^4 2 + 9600\xi(2)^2 \xi(5) \log^3 2 + 81600\xi(3) \xi(4) \log^3 2 \\
&\quad - 5400\xi^2(2) \xi(4) \log^4 2 - 25200\xi(2)^2 \xi^2(3) \log^2 2 + 8400\xi^2(2) \xi(3) \log 2 \\
&\quad - 765\xi^4(2) - 18240\xi(5) \log^4 2 + 14400\xi(2)^2 \xi(4) \log^4 2 \\
&\quad + 16800\xi^2(2) \xi(3) \log^4 2 + 4800\xi^2(2) \xi(3) \log^3 2 + 900\xi^2(2) \xi(2) \log^3 2 \\
&\quad - 2880\xi(4) \log^6 2 + 6720\xi(2)^2 \xi(3) \log^5 2 + 640\xi(3) \log^7 2 \\
&\quad + 960\xi^2(2) \log^6 2 + 480\xi(2) \log^8 2 + 128 \log^{10} 2 \right)
\end{align*}
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11. N. Nielsen, “Théorème sur les intégrales $\int_0^\phi \left( \phi \log \sin \phi \right)^n d\phi$ et $\int_0^\phi \left( \phi \log \sin \phi \right)^n d\phi$,” *Oversigt Danske Vid. Selsk. Forh.*, 1897, pp. 197–206.