On the Integral $\int_0^\infty e^{-\mu t^{v-1}} \log^m t \, dt$

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Abstract. A recurrence relation is given for the integral in the title. Formulae which allow easy evaluation by formula manipulation are given for integer and half-integer values of $v$. Explicit expressions for these integrals for small values of $m$ and $v = n, v = n + \frac{1}{2}$, are also presented.

1. Introduction. The purpose of this note is to give expressions for the Laplace (or Mellin) integral

\begin{equation}
R_m(\mu, v) = \int_0^\infty e^{-\mu t^{v-1}} \log^m t \, dt \quad (\text{Re} \, \mu > 0, \text{Re} \, \nu > 0),
\end{equation}

in particular for integer and half-integer values of $v$. This integral, with integer values of $v$, occurs when establishing an asymptotic expansion, as $x \to \infty$, for the Landau density function [2], [10]

\begin{equation}
\phi(x) = \frac{1}{\pi} \int_0^\infty e^{-x t^{v-1}} \sin \pi t \, dt.
\end{equation}

A contour integral related to (1) appears in the theory of heat conduction [4], [12] and in other physical problems. For $m \leq 3$, $R_m(\mu, v)$ can be found, for example, in [6, No. 4.3521, 4.3582, 4.3583]:

\begin{alignat}{2}
R_1(\mu, v) &= \frac{\Gamma(v)}{\mu^v} \left[ \psi(v) - \log \mu \right],
\quad \quad \\
R_2(\mu, v) &= \frac{\Gamma(v)}{\mu^v} \left\{ \left[ \psi(v) - \log \mu \right]^2 + \zeta(2, v) \right\},
\quad \quad \\
R_3(\mu, v) &= \frac{\Gamma(v)}{\mu^v} \left\{ \left[ \psi(v) - \log \mu \right]^3 + 3 \left[ \psi(v) - \log \mu \right] \zeta(2, v) - 2 \zeta(3, v) \right\},
\end{alignat}

where

\begin{equation}
\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)},
\end{equation}

and where

\begin{equation}
\zeta(r, q) = \sum_{j=0}^\infty \frac{1}{(q+j)^r}
\end{equation}

Received July 8, 1982; revised October 20, 1982.
1980 Mathematics Subject Classification. Primary 33A70, 33A15.

* Note that the corresponding formula in [6], copied from McLachlan et al. [11], is incorrect, as has been pointed out in [3]. In addition, $\zeta(2, v)$ and $\zeta(3, v)$ must be replaced by $\zeta(2, v+1)$ and $\zeta(3, v+1)$, respectively, in the formulae corresponding to $m = 2$ and $m = 3$ in [11].
is the generalized zeta function related to $\psi(x)$ by [6, No. 8.3638]

$$
\psi^{(k)}(x) = (-1)^{k+1} k! \xi(k+1, x).
$$

2. A Recurrence Formula for $R_m(\mu, \nu)$. In order to give a recurrence formula for $R_m(\mu, \nu)$, we need the following

**Lemma.** Let $f(x)$ be $m$ times differentiable. Then

$$
\left( \frac{d}{dx} \right)^m e^{f(x)} = e^{f(x)} F_m(x),
$$

where $F_0(x) = 1$, $F_1(x) = f'(x)$, and

$$
F_k(x) = F_{k-1}(x) + F_1(x) F_{k-1}(x) \quad (k = 2, 3, \ldots, m).
$$

The proof by induction is easy.

We apply this lemma to $R_m(\mu, \nu)$ and obtain

$$
R_m(\mu, \nu) = \int_0^\infty e^{-\mu t} t^{\nu-1} \log^m t \ dt = \left( \frac{d}{dv} \right)^m e^{-\mu t} t^{\nu-1} dt = \left( \frac{d}{dv} \right)^m \frac{\Gamma(v)}{\mu^v} G_m(\mu, \nu),
$$

where $G_0(\mu, \nu) = 1$, $G_1(\mu, \nu) = \psi(\nu) - \log \mu$, and

$$
G_k(\mu, \nu) = \frac{d}{dv} G_{k-1}(\mu, \nu) + G_1(\mu, \nu) G_{k-1}(\mu, \nu).
$$

With a formula manipulation system, such as REDUCE [8], it is easy to compute $R_m(\mu, \nu)$ from (8) and (12) in terms of $\psi(\nu) - \log \mu$ and $\xi(k, \nu)$ $(2 \leqslant k \leqslant m)$. The resulting expressions for $R_m(\mu, \nu)$ for $m = 1(1)9$ are given in Table 1.

**Table 1**

Let

$$
y_m = \frac{\mu^v}{\Gamma(v)} \int_0^\infty e^{-\mu t} t^{\nu-1} \log^m t \ dt = \frac{\mu^v}{\Gamma(v)} R_m(\mu, \nu) \quad (\text{Re} \mu > 0, \text{Re} \nu > 0)
$$

$$
\phi = \psi(\nu) - \log \mu,
$$

and

$$
\beta_2 = \xi(2, \nu)
$$
$$
\beta_3 = \xi(3, \nu)
$$
$$
\beta_4 = \xi^2(2, \nu) + 2 \xi(4, \nu)
$$
$$
\beta_5 = 5 \xi(2, \nu) \xi(3, \nu) + 6 \xi(5, \nu)
$$
$$
\beta_6 = 35 \xi^3(2, \nu) + 180 \xi(4, \nu) + 8 \xi^2(3, \nu) + 24 \xi(6, \nu)
$$
$$
\beta_7 = 35 \xi^2(2, \nu) \xi(3, \nu) + 84 \xi(5, \nu) \xi(5, \nu) + 70 \xi(3, \nu) \xi(4, \nu) + 120 \xi(7, \nu)
$$
$$
\beta_8 = 15 \xi^4(2, \nu) + 180 \xi^2(2, \nu) \xi(4, \nu) + 160 \xi(2, \nu) \xi^2(3, \nu) + 480 \xi(2, \nu) \xi(6, \nu)
$$
$$
+ 384 \xi(3, \nu) \xi(5, \nu) + 180 \xi^2(4, \nu) + 720 \xi(8, \nu)
$$
$$
\beta_9 = 315 \xi^3(2, \nu) \xi(3, \nu) + 1134 \xi^2(2, \nu) \xi(5, \nu) + 1890 \xi(2, \nu) \xi(3, \nu) \xi(4, \nu)
$$
$$
+ 3240 \xi(2, \nu) \xi(7, \nu) + 280 \xi^3(3, \nu) + 2520 \xi(3, \nu) \xi(6, \nu)
$$
$$
+ 2268 \xi(4, \nu) \xi(5, \nu) + 5040 \xi(9, \nu).
$$
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Then

\begin{align*}
Y_0 &= 1 \\
Y_1 &= 0 \\
Y_2 &= \phi^2 + \beta_2 \\
Y_3 &= \phi^3 + 3\beta_2\phi - 2\beta_3 \\
Y_4 &= \phi^4 + 6\beta_2\phi^2 - 8\beta_3\phi + 3\beta_4 \\
Y_5 &= \phi^5 + 10\beta_2\phi^3 - 20\beta_3\phi^2 + 15\beta_4\phi - 4\beta_5 \\
Y_6 &= \phi^6 + 15\beta_2\phi^4 - 40\beta_3\phi^3 + 45\beta_4\phi^2 - 24\beta_5\phi + 5\beta_6 \\
Y_7 &= \phi^7 + 21\beta_2\phi^5 - 70\beta_3\phi^4 + 105\beta_4\phi^3 - 84\beta_5\phi^2 + 35\beta_6\phi - 6\beta_7 \\
Y_8 &= \phi^8 + 28\beta_2\phi^6 - 112\beta_3\phi^5 + 210\beta_4\phi^4 - 224\beta_5\phi^3 + 140\beta_6\phi^2 - 48\beta_7\phi + 7\beta_8 \\
Y_9 &= \phi^9 + 36\beta_2\phi^7 - 216\beta_3\phi^6 + 378\beta_4\phi^5 - 504\beta_5\phi^4 \\
&\quad + 420\beta_6\phi^3 - 216\beta_7\phi^2 + 63\beta_8\phi - 8\beta_9 \\
\end{align*}

Thus

\begin{equation}
Y_m = \phi^m + \sum_{j=2}^m c_{m,j}\beta_j\phi^{m-j} \quad (m = 2, \ldots, 9)
\end{equation}

\begin{equation}
c_{m,j} = (-1)^j (j - 1) \sum_{k=j}^m \binom{k-1}{j-1}.
\end{equation}

3. A Formula for $R_m(\mu, n + 1)$, $n$ an Integer. Expressions for $R_m(\mu, n + 1)$, where $n$ is an integer, can be obtained from (12) by setting $\nu = n + 1$ in $G_m(\mu, \nu)$ and using

\begin{align}
(13) & \quad \xi(k, n + 1) = \xi(k) - \sigma(n, k), \\
(14) & \quad \psi(n + 1) = -\gamma + \sigma(n, 1),
\end{align}

where

\begin{equation}
(15) \quad \sigma(n, k) = \sum_{j=1}^n \frac{1}{j^k}.
\end{equation}

$\gamma$ is Euler's constant and $\xi(k)$ is the Riemann zeta function.

In order to derive another formula for this case, we set $\tilde{R}_m(\mu, n) = R_m(\mu, n + 1)$ and write

\begin{equation}
\tilde{R}_m(\mu, n) = \int_0^\infty e^{-\mu t} t^n \log^m t \, dt \quad (n \geq 0)
\end{equation}

\begin{align*}
&= \left( \frac{d}{d\alpha} \right)^m \int_0^\infty e^{-\mu t} t^{n+\alpha} \log^m t \, dt \bigg|_{\alpha = 0} \\
&= \frac{1}{\mu^{n+1}} \left( \frac{d}{d\alpha} \right)^m \left[ \frac{1}{\mu^\alpha} \Gamma(1 + n + \alpha) \right]_{\alpha = 0} \\
&= \frac{1}{\mu^{n+1}} \left( \frac{d}{d\alpha} \right)^m \left[ \frac{1}{\alpha} (\alpha)_{n+1} \frac{1}{\mu^\alpha} \Gamma(1 + \alpha) \right]_{\alpha = 0},
\end{align*}

where [1, No. 24.1.3]

\begin{equation}
(\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1) = \sum_{j=1}^k (-1)^{j+k} S_k^j \alpha^j
\end{equation}
is the Pochhammer symbol. The \( S_{k}^{(j)} \) are the Stirling numbers of the first kind, defined by

\[
S_{k}^{(j)} = S_{k}^{(j-1)} - kS_{k}^{(j)} \quad (k \geq j \geq 1),
\]

\[
S_{k}^{(1)} = (-1)^{k-1}(k-1)!,
\]

\[
S_{k}^{(k)} = 1, \quad S_{k}^{(0)} = \delta_{0k},
\]

\[
S_{k}^{(j)} = 0 \quad \text{for } k < j,
\]

\[
S_{k}^{(j)} = \sum_{i=0}^{k-j} \sum_{\lambda=0}^{i} (-1)^{\lambda} \binom{i}{\lambda} \binom{k-1+i}{k-j+i} (2k-j)\lambda^{j-i}.
\]

Applying the Leibniz formula, we have

\[
R_{m}(\mu, n) = \frac{1}{\mu^{n+1}} \sum_{\rho=0}^{m} \binom{m}{\rho} \left[ \frac{1}{\alpha} \left( \alpha n + 1 \right) \right]^{(\rho)} \left[ \frac{1}{\mu^\alpha} \Gamma(1 + \alpha) \right]^{(m-\rho)} \bigg|_{\alpha=0}.
\]

We see from (17) that

\[
\left( \frac{d}{d\alpha} \right)^{\rho} \frac{1}{\alpha} (\alpha n + 1) \bigg|_{\alpha=0} = \begin{cases} (\rho + n)!S_{n+1}^{(\rho+1)} & (\rho \leq n), \\ 0 & (\rho > n). \end{cases}
\]

We note that [6, No. 8.3421]

\[
\log \mu^{-\alpha} \Gamma(1 + \alpha) = - (\gamma + \log \mu) \alpha + \sum_{k=2}^{\infty} (-1)^{k} \frac{\xi(k)}{k} \alpha^{k}/k \quad (|\alpha| < 1),
\]

and therefore, by exponentiation (see, for example, Henrici [9, Section 1.6]),

\[
\mu^{-\alpha} \Gamma(1 + \alpha) = \sum_{k=0}^{\infty} b_{k} \alpha^{k} \quad (|x| < 1),
\]

where \( b_{0} = 1 \),

\[
b_{k} = \frac{1}{k} \sum_{\kappa=1}^{k} (-1)^{\kappa} \xi(\kappa) b_{k-\kappa},
\]

and \( \xi(1) = \gamma + \log \mu, \xi(\kappa) = \xi(\kappa) \) for \( \kappa > 1 \).

Substituting (20) and (22) into (19), we finally have

\[
\tilde{R}_{m}(\mu, n) = \int_{0}^{\infty} e^{-\mu t} t^{n} \log^{m} t \, dt
\]

\[
= \frac{m!}{\mu^{n+1}} \sum_{\rho=0}^{\min(n,m)} (-1)^{\rho} S_{n+1}^{(\rho+1)} b_{m-\rho} \quad (n \geq 0).
\]

We consider a few special cases. For \( n = 0 \) we obtain

\[
\tilde{R}_{m}(\mu, 0) = \frac{m!}{\mu} b_{m}.
\]

For \( m = 1 \) we have

\[
\tilde{R}_{1}(\mu, n) = \frac{(-1)^{n}}{\mu^{n+1}} \left( S_{n+1}^{(1)} b_{1} - S_{n+1}^{(2)} \right) = \frac{n!}{\mu^{n+1}} [\sigma(n, 1) - \gamma - \log \mu]
\]
in accordance with (3), since from (18),

\[ S^{(1)}_{n+1} = (-1)^n n!, \quad S^{(2)}_{n+1} = (-1)^{n+1} n! \sigma(n, 1). \]

For \( m = 2 \) it follows from (24) and (4) that

\[ \tilde{R}_2(\mu, n) = 2 \left\{ \left( \gamma + \log \mu \right)^2 - 2(\gamma + \log \mu) \sigma(n, 1) \right. \]
\[ \left. + \frac{\pi^2}{6} + 2 \frac{(-1)^n}{n!} S^{(3)}_{n+1} \right\} \]
\[ = \frac{n!}{\mu^{n+1}} \left\{ \left( \gamma + \log \mu \right)^2 - 2(\gamma + \log \mu) \sigma(n, 1) \right. \]
\[ \left. + \sigma^2(n, 1) - \sigma(n, 2) + \frac{\pi^2}{6} \right\}, \]

where we have used \( \zeta(2) = \pi^2/6 \). Note that \( S^{(2)}_{n+1} = 0 \) for \( n = 0 \), and \( S^{(3)}_{n+1} = 0 \) for \( n = 0, 1 \). From (28), we obtain the relation

\[ S^{(3)}_{n+1} = \frac{1}{2} (-1)^{n+1} \left[ \sigma^2(n, 1) - \sigma(n, 2) \right], \]

which is not immediately obvious from (18). In general, by comparing the result obtained from the recurrence relation (12) with that given by (24), we find that

\[ S^{(k+1)}_{n+1} = (-1)^n \frac{n!}{k!} \Omega_n^{(k)}(\sigma(n, 1), \ldots, \sigma(n, k)), \]

where \( \Omega_n^{(k)} \) is a “homogeneous” polynomial with integer coefficients in the \( k \) variables \( \sigma(n, 1), \ldots, \sigma(n, k) \). By a different approach, Comtet [5] has shown that

\[ \Omega_n^{(k)}(\sigma(n, 1), \ldots, \sigma(n, k)) = Y_k(\sigma(n, 1), -1! \sigma(n, 2), 2! \sigma(n, 2), \ldots, (-1)^{k-1}(k-1)! \sigma(n, k)), \]

where \( Y_k(x_1, \ldots, x_k) \) is the exponential complete Bell polynomial in \( k \) variables defined by

\[ \exp \left( \sum_{j=1}^{\infty} x_j t^j j! \right) = 1 + \sum_{k=1}^{\infty} Y_k(x_1, \ldots, x_k) \frac{t^k}{k!}. \]

We give two more examples:

\[ S^{(4)}_{n+1} = \frac{1}{6} (-1)^{n+1} n! \left[ \sigma^3(n, 1) - 3 \sigma(n, 1) \sigma(n, 2) + 2 \sigma(n, 3) \right], \]
\[ S^{(5)}_{n+1} = \frac{1}{24} (-1)^n n! \left[ \sigma^4(n, 1) + 3 \sigma^2(n, 2) - 6 \sigma^2(n, 1) \sigma(n, 2) \right. \]
\[ \left. + 8 \sigma(n, 1) \sigma(n, 3) - 6 \sigma(n, 4) \right]. \]

Note that \( \Omega_n^{(k)} \equiv 0 \) for \( 0 \leq n < k \). This fact is not apparent when one derives \( R_m(\mu, \nu) \) by means of the recurrence (12), replacing \( \nu \) by \( n + 1 \) in \( R_m(\mu, \nu) \).

In general, formula (24) can easily be evaluated by formula manipulation. Expressions for \( \tilde{R}_m(\mu, n), m = 0(1)5, n = 0(1)5, \) are given in Table 2. For special
values of $m$ and $n$ (e.g. $n > 0$, $m = 1$, or $n = 0$, $m ≤ 3$), these expressions can also be found in the relevant handbooks or tables of integral transforms.

Table 2

Let

$$I_{mn} = \mu^{n-1} \int_0^\infty e^{-\mu t^n} \log^m t \, dt = \mu^{n-1} \tilde{R}_m(\mu, n). \quad C = \gamma + \log \mu \quad (\text{Re} \mu > 0)$$

Then

$$I_{0m} = n! \quad (n > 0)$$
$$I_{1m} = - C$$

$$I_{nm} = n! \left[ \sum_{j=1}^{n} j^{-1} - C \right] \quad (n > 0)$$

$$I_{20} = C^2 + \frac{\pi^2}{6}$$
$$I_{21} = C^2 - 2C + \frac{\pi^2}{6}$$
$$I_{22} = 2C^2 - 6C + \frac{\pi^2}{3} + 2$$
$$I_{23} = 6C^2 - 22C + \pi^2 + 12$$
$$I_{24} = 24C^2 - 100C + 4\pi^2 + 70$$
$$I_{25} = 120C^2 - 548C + 20\pi^2 + 450$$

$$I_{30} = - C^3 - \frac{\pi^2}{2} C - 2\zeta(3)$$
$$I_{31} = - C^3 + 3C^2 - \frac{\pi^2}{2} C - 2\zeta(3) + \frac{\pi^2}{2}$$
$$I_{32} = - 2C^3 + 9C^2 - (\pi^2 + 6)C - 4\zeta(3) + \frac{3}{2}\pi^2$$
$$I_{33} = - 6C^3 + 33C^2 - 3(\pi^2 + 12)C - 12\zeta(3) + \frac{11}{2}\pi^2 + 6$$
$$I_{34} = - 24C^3 + 150C^2 - 6(2\pi^2 + 35)C - 48\zeta(3) + 25\pi^2 + 60$$
$$I_{35} = - 120C^3 + 822C^2 - 30(2\pi^2 + 45)C - 240\zeta(3) + 137\pi^2 + 510$$

$$I_{40} = C^4 + \pi^2 C^2 + 8\zeta(3)C + \frac{3}{20}\pi^4$$
$$I_{41} = C^4 - 4C^3 + \pi^2 C^2 + 2(4\zeta(3) - \pi^2)C - 8\zeta(3) + \frac{3}{20}\pi^4$$
$$I_{42} = 2C^4 - 12C^3 + 2(\pi^2 + 6)C^2 + 2(8\zeta(3) - 3\pi^2)C - 24\zeta(3) + \frac{3}{10}\pi^4 + 2\pi^2$$
$$I_{43} = 6C^4 - 44C^3 + 6(\pi^2 + 12)C^2 + 2(24\zeta(3) - 11\pi^2 - 12)C - 88\zeta(3) + \frac{9}{10}\pi^4 + 12\pi^2$$
$$I_{44} = 24C^4 - 200C^3 + 12(2\pi^2 + 35)C^2 + 4(48\zeta(3) - 25\pi^2 - 60)C$$
$$- 400\zeta(3) + \frac{18}{5}\pi^4 + 70\pi^2 + 24$$
$$I_{45} = 120C^4 - 1096C^3 + 60(2\pi^2 + 45)C^2 + 4(240\zeta(3) - 137\pi^2 - 510)C$$
$$- 2192\zeta(3) + 18\pi^4 + 450\pi^2 + 360$$
\[ I_{50} = -C^5 - \frac{5}{3} \pi^2 C^3 - 20\xi(3) C^2 - \frac{3}{4} \pi^4 C - \frac{10}{3} \xi(3) \pi^2 - 24\xi(5) \]

\[ I_{51} = -C^5 - \frac{5}{3} \pi^2 C^3 - 5(4\xi(3) - \pi^2) C^2 + \left( 40\xi(3) - \frac{3}{4} \pi^4 \right) C \]

\[ - \frac{10}{3} \xi(3) \pi^2 - 24\xi(5) + \frac{3}{4} \pi^4 \]

\[ I_{52} = -2C^5 + 15C^4 - 10 \left( \frac{1}{3} \pi^2 + 2 \right) C^3 - 5(8\xi(3) - 3\pi^2) C^2 \]

\[ + \left( 120\xi(3) - \frac{3}{2} \pi^4 - 10\pi^2 \right) C - \frac{20}{3} \xi(3) \pi^2 - 40\xi(3) - 48\xi(5) + \frac{9}{4} \pi^4 \]

\[ I_{53} = -6C^5 + 55C^4 - 10(\pi^2 + 12) C^3 - 5(24\xi(3) - 11\pi^2 - 12) C^2 \]

\[ + \left( 440\xi(3) - \frac{9}{2} \pi^4 - 60\pi^2 \right) C - 20\xi(3) \pi^2 - 240\xi(3) - 144\xi(5) + \frac{33}{4} \pi^4 + 10\pi^2 \]

\[ I_{54} = -24C^5 + 250C^4 - 20(2\pi^2 + 35) C^3 - 10(48\xi(3) - 25\pi^2 - 60) C^2 \]

\[ + 2(1000\xi(3) - 9\pi^4 - 175\pi^2 - 60) C - 80\xi(3) \pi^2 \]

\[ - 1400\xi(3) - 576\xi(5) + \frac{75}{2} \pi^4 + 100\pi^2 \]

\[ I_{55} = -120C^5 + 1370C^4 - 100(2\pi^2 + 45) C^3 - 10(240\xi(3) - 13\pi^2 - 510) C^2 \]

\[ + 10(1096\xi(3) - 9\pi^4 - 225\pi^2 - 180) C - 400\xi(3) \pi^2 \]

\[ - 9000\xi(3) - 2880\xi(5) + \frac{411}{2} \pi^4 + 850\pi^2 + 120 \]

4. A Formula for \( R_m(\mu, n + 1/2) \), \( n \) an Integer. Setting \( \nu = n + 1/2 \) and [1, No. 6.3.4, 23.2.20]

\[ \xi(k, n + 1/2) = (2^k - 1)\xi(k) - 2^k \delta(n, k), \]

\[ \psi(n + 1/2) = -\gamma - \log 4 + 2\delta(n, 1). \]

where

\[ \delta(n, k) = \sum_{j=1}^{n} \frac{1}{(2j-1)^k}, \]

in the expression for \( R_m(\mu, \nu) \) obtained from (12), leads to expressions for \( R_m(\mu, n) = R_m(\mu, n + 1/2) \). In the same way as in Section 3, we may also derive another formula in this case. We start with

\[ R_m^*(\mu, n) = \int_{0}^{\infty} e^{-\mu t} n^{-1/2} \log^m t \, dt \quad (n \geq 0) \]

\[ \quad \left( \frac{d}{da} \right)^m \int_{0}^{\infty} e^{-\mu t} n^{-1/2 + a} \, dt \bigg|_{a=0} \]

\[ \quad = \frac{1}{\mu^{n+1/2}} \left( \frac{d}{da} \right)^m \left[ \frac{1}{\mu^a} \Gamma\left(n + \frac{1}{2} + a\right) \right]_{a=0}. \]

Using

\[ \Gamma\left(\frac{1}{2} + n + a\right) = \left( \alpha + \frac{1}{2} \right)_n \Gamma\left(\frac{1}{2} + a\right) = \sqrt{\pi} \left( \alpha + \frac{1}{2} \right)_n 2^{-2a} \Gamma(1 + 2a) \Gamma(1 + a), \]

we obtain

\[ R_m^*(\mu, \nu) = \frac{\sqrt{\pi}}{\mu^{n+1/2}} \left( \frac{d}{da} \right)^m \left[ \left( \alpha + \frac{1}{2} \right)_n H(\alpha, \mu) \right]_{a=0}, \]
where

\begin{equation}
H(\alpha, \mu) = \frac{1}{(4\mu)^\alpha} \frac{\Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)}.
\end{equation}

As for the series (22), we can write

\begin{equation}
H(\alpha, \mu) = \sum_{k=0}^{\infty} b_k^* \alpha^k \quad \left( |\alpha| < \frac{1}{2} \right),
\end{equation}

where \(b_0^* = 1\),

\begin{equation}
b_k^* = \frac{1}{k} \sum_{\kappa=1}^{k} (-1)^{\kappa} \xi^*(\kappa) b_{k-\kappa}^*,
\end{equation}

and

\begin{equation}
\xi^*(\kappa) = \begin{cases} 
\gamma + \log 4\mu & (\kappa = 1), \\
(2^{\kappa} - 1)\xi(\kappa) & (\kappa > 1).
\end{cases}
\end{equation}

Applying the Leibniz formula to (38), we use (17) and write

\begin{equation}
\left( \frac{d}{d\alpha} \right)^p \left( \alpha + \frac{1}{2} \right)\bigg|_{\alpha=0} = \left( \frac{d}{d\alpha} \right)^p \sum_{k=1}^{n} (-1)^{k+n} S_n^{(k)} \left( \alpha + \frac{1}{2} \right)^k
\end{equation}

\begin{equation*}
= \begin{cases} 
p! \sum_{k=\rho}^{n} (-1)^{k+n} S_n^{(k)} \left( \frac{k}{\rho} \right) 2^{\rho-k} & (\rho \leq n), \\
0 & (\rho > n),
\end{cases}
\end{equation*}

and therefore finally

\begin{equation}
R^*_m(\mu, n) = \int_0^\infty e^{-\mu t} n^{-1/2} \log^m t \, dt \quad (n \geq 0)
\end{equation}

\begin{equation*}
= \sqrt{\frac{\pi}{\mu}} \frac{(-1)^n m!}{\mu^n} \sum_{\rho=0}^{\min(m, n)} b_{m-\rho}^* \sum_{k=\rho}^{n} (-1)^{k} S_n^{(k)} \left( \frac{k}{\rho} \right) 2^{\rho-k}.
\end{equation*}

We consider two special cases. For \(n = 0\), we have

\begin{equation}
R^*_m(\mu, 0) = \sqrt{\frac{\pi}{\mu}} m! b^*_m,
\end{equation}

For \(m = 1\) we obtain from (3) and (44) the expression [6, No. 4.3523]

\begin{equation}
R^*_1(\mu, n) = \frac{\Gamma(n + \frac{1}{2})}{\mu^{n+1/2}} \left[ -\gamma - \log 4\mu + 2\delta(n, 1) \right]
\end{equation}

\begin{equation*}
= \sqrt{\frac{\pi}{\mu}} \left( \frac{2n - 1}{\mu} \right)!! \left[ -\gamma - \log 4\mu + 2\delta(n, 1) \right]
\end{equation*}

\begin{equation*}
= \sqrt{\frac{\pi}{\mu}} \frac{1}{\mu^n} \left[ - \left( \frac{1}{n} \right) \gamma + \log 4\mu \right] + 2(-1)^n \sum_{k=1}^{n} (-1)^k S_n^{(k)} \kappa 2^{-k}.
\end{equation*}

Thus, by comparison,

\begin{equation}
\sum_{k=1}^{n} (-1)^k S_n^{(k)} k 2^{-k} = (-1)^n \frac{(2n - 1)!!}{2^n} \delta(n, 1).
\end{equation}
Similarly, for $m = 2$, one finds

$$
(48) \sum_{k=2}^{n} (-1)^k S_n^{(k)} k(k-1) 2^{-k} = (-1)^n \frac{(2n-1)!!}{2^n} \left[ \delta^2(n, 1) - \delta(n, 2) \right].
$$

In general, as in the case of $R_m^*(\mu, n)$, a comparison of the result obtained from (12) with expression (44) shows that

$$
(49) \sum_{k=\rho}^{n} (-1)^k S_n^{(k)} \frac{k!}{(k-\rho)!} 2^{-k} = (-1)^n \frac{(2n-1)!!}{2^n} \Omega_n^{(\rho)}(\delta(n, 1), \ldots, \delta(n, \rho)),
$$

where $\Omega_n^{(\rho)}$ is defined by (31), and $\Omega_n^{(\rho)} \equiv 0$ for $0 < n < \rho$.

Formula (44) can be evaluated easily by formula manipulation. Expressions for $R_m^*(\mu, n)$, $m = 0(1)5$, $n = 0(1)5$, are given in Table 3. $R_m^*(\mu, n)$ can also be found in [6, No. 4.3523].

**Table 3**

Let

$$
J_{m,n} = \sqrt{\frac{\mu}{\pi}} \mu_n e^{-\mu t} t^{n-1/2} \log^m t \, dt = \sqrt{\frac{\mu}{\pi}} \mu^* R_m^*(\mu, n) \quad (\text{Re} \mu > 0)
$$

and

$$
C = \gamma + \log 4 \mu.
$$

Then

$$
J_{00} = 1
$$
$$
J_{0n} = \frac{(2n-1)!!}{2^n} \quad (n > 0)
$$
$$
J_{10} = -C
$$
$$
J_{1n} = \frac{(2n-1)!!}{2^n} \left( -C + 2 \sum_{j=1}^{n} \frac{1}{2j-1} \right) \quad (n > 0)
$$

$$
J_{20} = C^2 + \frac{\pi^2}{2}
$$
$$
J_{21} = \frac{1}{2} C^2 - 2C + \frac{\pi^2}{4}
$$
$$
J_{22} = \frac{3}{4} C^2 - 4C + \frac{3}{8} \pi^2 + 2
$$
$$
J_{23} = \frac{15}{8} C^2 - \frac{23}{2} C + \frac{15}{16} \pi^2 + 9
$$
$$
J_{24} = \frac{105}{16} C^2 - 44C + \frac{105}{32} \pi^2 + 43
$$
$$
J_{25} = \frac{945}{32} C^2 - \frac{1689}{8} C + \frac{945}{64} \pi^2 + 475/2
$$

$$
J_{30} = -C^3 - \frac{3}{2} \pi^2 C - 14 \xi(3)
$$
$$
J_{31} = -\frac{1}{2} C^3 + 3C^2 - \frac{3}{4} \pi^2 C - 7 \xi(3) + \frac{3}{2} \pi^2
$$
$$
J_{32} = -\frac{3}{4} C^3 + 6C^2 - 3 \left( \frac{3}{8} \pi^2 + 2 \right) C - \frac{21}{2} \xi(3) + 3 \pi^2
$$
$$
J_{33} = -15 \frac{C^3}{8} + \frac{69}{4} C^2 - 9 \left( \frac{5}{16} \pi^2 + 3 \right) C - \frac{105}{4} \xi(3) + \frac{69}{8} \pi^2 + 6
$$
\[ J_{44} = \frac{105}{16} C^4 - 88C^3 + 3 \left( \frac{105}{16} \pi^2 + 86 \right) C^2 + 3 \left( \frac{245}{2} \xi(3) - 44\pi^2 - 64 \right) C \]
\[ - 1232\xi(3) + \frac{735}{64} \pi^4 + 129\pi^2 + 24 \]
\[ J_{45} = \frac{945}{32} C^4 - \frac{1689}{4} C^3 + 15 \left( \frac{189}{32} \pi^2 + 95 \right) C^2 + 3 \left( \frac{2205}{4} \xi(3) - \frac{1689}{8} \pi^2 - 460 \right) C \]
\[ - \frac{11823}{2} \xi(3) + \frac{6615}{128} \pi^4 + \frac{1425}{2} \pi^2 + 300 \]

\[ J_{50} = - C^5 - 5\pi^2 C^3 - 140\xi(3) C^2 - \frac{35}{4} \pi^4 C - 70\xi(3) \pi^2 - 744\xi(5) \]

\[ J_{51} = - \frac{1}{2} C^5 + 5C^4 - \frac{5}{2} \pi^2 C^3 + 5(3\pi^2 - 14\xi(3)) C^2 \]
\[ + 35 \left( \xi(3) - \frac{\pi^4}{8} \right) C - 35\xi(3) \pi^2 - 372\xi(5) + \frac{35}{4} \pi^4 \]

\[ J_{52} = - \frac{3}{4} C^5 + 10C^4 - 5 \left( \frac{3}{4} \pi^2 + 4 \right) C^3 + 15(2\pi^2 - 7\xi(3)) C^2 \]
\[ + 5 \left( 112\xi(3) - \frac{21}{16} \pi^4 - 6\pi^2 \right) C - \frac{105}{2} \xi(3) \pi^2 - 280\xi(3) - 558\xi(5) + \frac{35}{2} \pi^4 \]

\[ J_{53} = - \frac{15}{8} C^5 + \frac{115}{4} C^4 - 15 \left( \frac{5}{8} \pi^2 + 6 \right) C^3 + 15 \left( \frac{23}{4} \pi^2 - \frac{35}{2} \xi(3) + 4 \right) C^2 \]
\[ + 5 \left( 322\xi(3) - \frac{105}{32} \pi^4 - 27\pi^2 \right) C - \frac{525}{4} \xi(3) \pi^2 \]
\[ - 1260\xi(3) - 1395\xi(5) + \frac{805}{16} \pi^4 + 30\pi^2 \]

\[ J_{54} = - \frac{105}{16} C^5 + 110C^4 - 5 \left( \frac{105}{16} \pi^2 + 86 \right) C^3 + 15 \left( 22\pi^2 - \frac{245}{4} \xi(3) + 32 \right) C^2 \]
\[ + 5 \left( 1232\xi(3) - \frac{735}{64} \pi^4 - 129\pi^2 - 24 \right) C - \frac{3675}{8} \xi(3) \pi^2 \]
\[ - 6020\xi(3) - \frac{9765}{2} \xi(5) + \frac{385}{2} \pi^4 + 240\pi^2 \]

\[ J_{55} = - \frac{945}{32} C^5 + \frac{8445}{16} C^4 - 25 \left( \frac{189}{32} \pi^2 + 95 \right) C^3 + 15 \left( \frac{1689}{16} \pi^2 - \frac{2205}{8} \xi(3) + 230 \right) C^2 \]
\[ + 15 \left( \frac{3941}{2} \xi(3) - \frac{2205}{128} \pi^4 - \frac{475}{2} \pi^2 - 100 \right) C \]
\[ - \frac{33075}{16} \xi(3) \pi^2 - 33250\xi(5) - \frac{87885}{4} \xi(5) + \frac{59115}{64} \pi^4 + 1725\pi^2 + 120 \]
5. Two Related Integrals. Substituting \( t = T^2 \) in \( \tilde{R}_m(\mu, n) \), we find that
\[
\int_0^\infty e^{-\mu T^2} T^{2n+1} \log^n T \, dT = \frac{1}{2^{m+1}} \tilde{R}_m(\mu, n),
\]
and, by the same substitution in \( R_m^*(\mu, n) \),
\[
\int_0^\infty e^{-\mu T^2} T^{2n} \log^n T \, dT = \frac{1}{2^{m+1}} R_m^*(\mu, n).
\]
Three particular combinations of these integrals for \( m = 1 \) are given in [6, No. 4.3551, 4.3553, 4.3554].

6. A Related Contour Integral. When deriving the asymptotic form of an integral of modified Bessel functions appearing in a problem of heat conduction, Ritchie and Sakakura [12] were led to the contour integral
\[
I_n^m(\mu) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{\mu z} z^{-n-1} \log^m z \, dz,
\]
where \( \Re \mu > 0 \), \( m, n \) integers. The contour of integration is one which begins at \( -\infty - i\alpha \), proceeds around the origin in a counterclockwise direction and ends at \( -\infty + i\alpha \), with arbitrary \( \alpha > 0 \). For \( m > 0 \), they give the expression
\[
I_n^m(\mu) = \mu^{-n} \sum_{j=0}^k A_j^n m \log^j \mu,
\]
where the coefficients \( A_j^n m \) are given implicitly in [12] as derivatives of gamma functions, and can therefore be expressed by the polygamma function (8). For \( n > 0 \), the contour of (52) can be contracted to the cut along the negative real axis, and we can write, using (1),
\[
I_n^m(\mu) = \frac{(-1)^{n-1}}{2\pi i} \int_0^\infty e^{-\mu t} t^{n-1} \left[ (\log t - i\pi)^m - (\log t + i\pi)^m \right] \, dt
\]
\[
= (-1)^n \sum_{j=0}^k \left[ (-1)^j \left( \frac{m}{2j+1} \right) e^{-\mu t} t^{k-1} (\pi^2 + \log^2 t)^{-1} \right] dt.
\]

For \( m = -k \), \( k > 0 \), the authors of [12] derive an asymptotic series for \( I_n^{-k}(\mu) \), valid for \( \mu \to \infty \). The coefficients of this series are again given by derivatives of the gamma function. Under the further restriction \( n \geq 0 \), they present an alternative asymptotic series.

Wood [13] shows that \( I_n^{-k}(t) \) \( (k > 0, \text{integer}) \) can be expressed by combinations of the Ramanujan integral \( \phi(\mu) = \phi(0)(\mu) \) [7] and its derivatives
\[
\phi^{(k)}(\mu) = \left( \frac{d}{d\mu} \right)^k \phi(\mu) = (-1)^k \int_0^\infty e^{-\mu t} t^{1-k} (\pi^2 + \log^2 t)^{-1} \, dt.
\]
In particular one may show, analogously to (54), that \( I_0^{-1}(\mu) = -\phi(\mu) \). Wood also gives the first few coefficients of an asymptotic expansion for \( \phi^{(k)}(\mu) \) \( (k \geq 1) \) as \( \mu \to \infty \). (Note that the labels on two curves in Figure 3 of [13] should read \( n = 2 \), \( k = -2 \); \( n = 2 \), \( k = -3 \), instead of \( n = 2 \), \( k = 2 \); \( n = 3 \), \( k = 3 \), respectively. These
curves agree with formulae (2f) and (2g) in [13]. The author was not able to identify the curve labelled $n = 1, k = 2$ in Figure 3.)

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2. W. Börsch-Supan, “On the evaluation of the function $\phi(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\ln u + \lambda u} du$ for real values of $\lambda$,” J. Res. Nat. Bur. Standards, v. 65B, 1961, pp. 245–250.