On the Largest Zeroes of Orthogonal Polynomials for Certain Weights

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Abstract. The asymptotic growth of the largest zero of the orthogonal polynomials for the weights $W(x) = |x|^b \exp(-k \log |x|^c)$ is investigated.

1. Introduction. Freud [3], [4] investigated the largest zeroes of orthogonal polynomials for weights on $(-\infty, \infty)$. Nevai and Dehesa [5] studied the sums of powers of zeroes of orthogonal polynomials. Here we investigate the asymptotic growth of the largest zeroes for the weights

$$W(x) = |x|^b \exp(-k \log |x|^c), \quad x \in (-\infty, \infty)$$

where $c > 1; k > 0; b \in (-\infty, \infty)$

and

$$W(x) = \begin{cases} x^b \exp(-k \log x^c), & x \in (0, \infty), \\ 0, & x \in (-\infty, 0]. \end{cases}$$

where $c > 1; k > 0; b \in (-\infty, \infty)$.

When $c = 2$ and $b = 0$ in (1.2), $W(x)$ yields the Stieltjes-Wigert polynomials (Chihara [1, 2]), and Chihara [2] has remarked that very little is known about their zeroes.

2. Notation. Given a nonnegative measurable function $W(x)$ on $(-\infty, \infty)$ for which all moments

$$\mu_n(W) = \int_{-\infty}^{\infty} x^n W(x) \, dx, \quad n = 0, 1, 2, \ldots,$$

exist, its orthogonal polynomials are

$$p_n(W; x) = \gamma_n(W) \prod_{j=1}^{n} (x - x_{jn}(W)), \quad n = 0, 1, 2, \ldots,$$

satisfying

$$\int_{-\infty}^{\infty} p_n(W; x) p_m(W; x) W(x) \, dx = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

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We let \( X_n(W) = \max\{x_n(W) : j = 1, 2, \ldots, n\} \), \( n = 1, 2, \ldots \). Further, for each positive \( t \), \( \xi_t \) denotes the smallest possible number (if it exists) such that
\[
(\xi_t)^t W(\xi_t) = \max\{x^t W(x) : x \in (0, \infty)\},
\]
and assuming that
\[
\int_0^\pi \log W(\xi \cos \theta) \, d\theta < \infty, \quad \xi \in (0, \infty),
\]
we define
\[
G_\xi(W) = \exp\left\{\pi^{-1} \int_0^\pi \log W(\xi \cos \theta) \, d\theta\right\}, \quad \xi \in (0, \infty).
\]

3. The Largest Zero.

**Lemma 3.1.** Let \( W(x) \) be given by (1.1). Then
\[
\lim_{n \to \infty} \frac{\mu_{2^n}(W)}{d(2n + b + 1)^{2(\cdot c) - 1/2(c - 1)^{-1}}} \exp\left\{f(2n + b + 1)^{-1/c} - 1\right\} = 1,
\]
where \( d = 2\{2\pi(c_1 - 1)^{-1}(ck)^{1/(1 - c)}\}^{1/2} \) and \( f = (c - 1)(ck)^{1/(1 - c)} \).

**Proof.**
\[
(\log_2 2n)^{1/(c - 1)} - 1 \approx \frac{f}{2}(2n + b + 1)^{-1/c} \log(W) = 1.
\]

Following is our main result.

**Theorem 3.2.** Let \( W(x) \) be given by (1.1). Then
\[
\lim_{n \to \infty} \frac{\mu_{2^n}(W)}{2n^{1/(c - 1)}} = 1.
\]

**Proof.** (i) By Lemma 3 in Freud [3, p. 95],
\[
\log X_n(W) \approx \left(\log \mu_{2^n-2}(W) - \log \mu_{2^n-4}(W)\right)/2
\]
\[
(2n + b - 1)^{c/(c - 1)} - (2n + b - 3)^{c/(c - 1)} + O(n^{-1}) \text{ (by Lemma 3.1)}
\]
\[
(2n - kc)^{1/(c - 1)} + O\left(n^{(2-c)/(c-1)}\right)
\]
Next, for any \( \xi > 0 \) and \( A > 1 \), Theorem 2 in Freud [4, p. 52] shows that
\[
X_n(W) \approx A_\xi + \frac{4}{3\pi} \left(\frac{2}{\xi}\right)^{n-1} \int_{A_\xi}^{\infty} x^{2n-1} W(x) \, dx.
\]
Freud states this under the additional assumption that \( W(x) \) is positive in \((-\infty, \infty)\), but his proof is valid if (2.2) holds. It is easily seen that for some positive constant \( K_0 \), independent of \( \xi \),

\[
G_1(W) \geq K_0^{-1}W(\xi), \quad \xi \in (0, \infty).
\]

Then taking \( \xi = \xi_{2(n+s)} \) and \( A = 2^n/s \) where \( s \in (0, \infty) \), we obtain, from (2.1), (3.3) and (3.4),

\[
X_n(W) \leq A_2 \xi_{2(n+s)} + \frac{4K_0}{3\pi} \left( \frac{2}{\xi_{2(n+s)}} \right)^{2n-1} \xi_{2(n+s)} \int_{A_2 \xi_{2(n+s)}}^\infty x^{-1/2} \, dx \\
= \xi_{2(n+s)} \left[ 2^{n/s} + K_0(3\pi s)^{-1} \right].
\]

Next, for large \( t \in (0, \infty) \), \( \xi_t \) is a root of \( d[x'W(x)]/dx = 0 \) so \( \log \xi_t = [(t + b)/kc]^{1/(c-1)} \). Taking \( s = n^\delta \) in (3.5) where

\[
0 < \delta < 1 \quad \text{and} \quad 1 - \delta(c - 1)^{-1},
\]

we obtain

\[
\log X_n(W) \leq \left[ \left( 2n + 2n^\delta + b \right)/kc \right]^{1/(c-1)} + n^{1-\delta}\log 2 + o(1).
\]

The result follows from (3.2), (3.6) and (3.7).

(ii) follows from (i) and Theorem 1 in Freud [3, p. 91]. □

Since

\[
\{X_n(W)\}^m \leq \sum_{j=1}^n |x_j(W)|^m \leq n\{X_n(W)\}^m, \quad m > 0, n = 1, 2, \ldots,
\]

we deduce that, for \( m > 0 \),

\[
\lim_{n \to \infty} \left( \frac{kc}{2n} \right)^{1/(c-1)} \log \left\{ \sum_{j=1}^n |x_j(W)|^m \right\} = m,
\]

which provides a contrast to the results of Nevai and Dehesa [5, Theorem 1].

**Corollary 3.3.** Let \( W(x) \) be given by (1.2). Then the conclusions (i), (ii) of Theorem 3.2 remain true.

**Proof.** Let

\[
W^*(x) = |x| W(x^2) = |x|^{2b+1} \exp(-k_1 |\log |x||^c), \quad x \in (-\infty, \infty),
\]

where \( k_1 = k2^c \). Then, by Theorem 3.2,

\[
\lim_{n \to \infty} \left( \frac{k_1c}{4n} \right)^{1/(c-1)} \log X_{2n}(W^*) = 1,
\]

\[
\lim_{n \to \infty} \left( \frac{k_1c}{4n} \right)^{1/(c-1)} \log \{\gamma_{2n-j-1}(W^*)/\gamma_{2n-j}(W^*)\} = 1, \quad j = 0, 1.
\]

Further, the substitution \( x = u^2 \) yields \( p_n(W; u^2) = p_{2n}(W^*; u) \) and hence

\[
X_{n}(W) = \left\{X_{2n}(W^*) \right\}^2, \quad \gamma_n(W) = \gamma_{2n}(W^*),
\]

and the conclusions follow from (3.8) and (3.9). □
For real $b$, and fixed positive $k$, let

$$W_b(x) = \begin{cases} k \pi^{-1/2} x^b \exp(-k^2(\log x)^2), & x \in (0, \infty), \\ 0, & x \in (-\infty, 0] \end{cases}$$

Wigert [7] explicitly found $p_n(W_0; x)$, $n = 1, 2, \ldots$, while Chihara [2] constructed discrete solutions of the moment problem corresponding to $W_0$, which provided some information regarding the distribution of $\{x_n(W_0)\}_{n=1}^\infty$. Using the relation

$$W_b(x) = \alpha^{b^2} W_0(x/\alpha^{2b}), \quad x \in (-\infty, \infty),$$

where $\alpha = \exp(1/4k^2)$, it follows that

$$P_n(W_b; x) = \alpha^{-b/2} p_n(W_0; x/\alpha^{2b}), \quad n = 1, 2, \ldots,$$

and hence the results of Wigert [7] and Chihara [2] for $W_0(x)$ generalize to $W_b(x)$, any $b \in (-\infty, \infty)$.

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