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An Application of Matrices Over Finite Fields to Algebraic Number Theory

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Abstract. This paper ultilizes properties of matrices over finite fields to obtain information about the rank of the p-class group of certain algebraic number fields.

1. Introduction. Let K be a Galois extension of the field of rational numbers $\mathbf{Q}$ of degree p, where p is a prime number. Let A denote the p-class group of K, i.e., the Sylow p-subgroup of the ideal class group of K. (For $p = 2$, we shall be using the Sylow 2-subgroup of the strict (or narrow) ideal class group of K.) Let v denote the rank of A; i.e., $v = \dim_{\mathbf{F}_p}(A/pA)$, where $\mathbf{F}_p$ is the finite field with p elements. Let t denote the number of primes that ramify in $K/\mathbf{Q}$. It is a classical result that $v = t - 1$ if $p = 2$ (see [1, p. 247]), and in general $t - 1 \leqslant v \leqslant (p - 1)(t - 1)$. (See [5, Satz 30].) When $t = 1$, we get $v = 0$ for all p. For fixed $p \geqslant 3$ and $t \geqslant 2$, we shall show that v is usually equal to $t - 1$ and that in a probabilistic sense the expected value of v, denoted $E(v)$, satisfies $t - 1 < E(v) < t$. The techniques we use are similar to some of the techniques used by Rédei in [6] to specify the 4-rank of A in the quadratic case. In Section 2 we shall develop some results we need about matrices over finite fields, and in Section 3 we shall apply the results in Section 2 to obtain information about v.

2. Ranks of Matrices Over Finite Fields. Let M be an $m \times n$ matrix with entries in the finite field $\mathbf{F}_p$, where $m \leqslant n$ and p is a prime number. Next let N_r be the number of these $m \times n$ matrices M over $\mathbf{F}_p$ with rank $M = r$, where $0 \leqslant r \leqslant m$.

PROPOSITION 2.1.

$$N_r = \left[\prod_{j=1}^{r} (p^n - p^{j-1})\right] \sum_{\substack{i_1 + \cdots + i_r \leqslant m-r \\ \text{each } i_s \geqslant 0}} \left(\prod_{s=1}^{r} p^{si_s}\right).$$

(*For $r = 0$, we interpret this as $N_0 = 1$.*)

Proof. Let M be an $m \times n$ matrix over $\mathbf{F}_p$ with rank $M = r$. Let r_i be the rank of the first i rows of M, $1 \leqslant i \leqslant m$. Then $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_m = r$. Thus to each M with rank $M = r$, we have associated an ordered m-tuple $(r_1, r_2, \ldots, r_m)$. To determine N_r, it suffices to determine all possible m-tuples $(r_1, \ldots, r_m)$ and the number of M associated with each m-tuple. Let r_{k_s} be the first term in $(r_1, \ldots, r_m)$ with $r_{k_s} = s$ for

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$1 \leqslant s \leqslant r$. Let $i_0 = k_1 - 1$, $i_s = k_{s+1} - k_s - 1$ for $1 \leqslant s \leqslant r-1$, and $i_r = m - k_r$. Then $r_1 = \cdots = r_{i_0} = 0$ (if $k_1 > 1$), and for $s = 1,\ldots,r$, we have $r_{k_s} = r_{k_s+1} = \cdots = r_{k_s+i_s} = s$. We note that each $i_s \geqslant 0$, and

$$i_0 + i_1 + \cdots + i_r = m - r. \tag{2.1}$$

Also the $(r+1)$-tuple $(i_0, i_1,\ldots,i_r)$ determines the r-tuple $(k_1,\ldots,k_r)$, which determines the m-tuple $(r_1,\ldots,r_m)$. Now how many matrices M are associated with a given $(r_1,\ldots,r_m)$, or equivalently, with a given $(i_0, i_1,\ldots,i_r)$? For rows $1,\ldots,i_0$, there is only one possibility, namely rows with all entries equal to 0. For row k_1 there are $p^n - 1$ possibilities (only the row with all entries equal to 0 is excluded). Each of rows $k_1+1,\ldots,k_1+i_1$ must be contained in the space spanned by row k_1, and hence there are p possibilities for each such row. In general there are $p^n - p^{s-1}$ possibilities for row k_s (i.e., any row vector not in the $(s-1)$-dimensional space spanned by rows $1,\ldots,k_s - 1$) and p^s possibilities for each of rows $k_s+1,\ldots,k_s+i_s$ (i.e., any row vector contained in the space spanned by rows $1,\ldots,k_s$). Thus, for a given $(r+1)$-tuple $(i_0, i_1,\ldots,i_r)$, the number of possible matrices M is

$$1^{i_0}(p^n - 1)p^{i_1}\cdots(p^n - p^{s-1})(p^s)^{i_s}\cdots(p^n - p^{r-1})(p^r)^{i_r}.$$

Now allowing for all $(i_0, i_1,\ldots,i_r)$ satisfying Eq. (2.1) with each $i_s \geqslant 0$, we get

$$\begin{aligned} N_r &= \sum_{\substack{i_0+i_1+\cdots+i_r=m-r \\ \text{each } i_s\geqslant 0}} \left[\prod_{s=1}^{r}(p^n - p^{s-1})p^{si_s}\right] \\ &= \left[\prod_{j=1}^{r}(p^n - p^{j-1})\right] \sum_{\substack{i_1+\cdots+i_r\leqslant m-r \\ \text{each } i_s\geqslant 0}} \left(\prod_{s=1}^{r} p^{si_s}\right). \end{aligned}$$

Remark. Proposition 2.1 can be generalized to an arbitrary finite field with p^k elements by replacing p by p^k.

We shall now restrict our attention to $(t-1)\times t$ matrices M over $\mathbf{F}_p$, where $p \geqslant 3$ and $t \geqslant 2$ are fixed. Since there are $p^{t(t-1)}$ such matrices, the probability (which we denote by $R_{t,r}$) that a randomly chosen $(t-1)\times t$ matrix M over $\mathbf{F}_p$ has rank $M = r$ is given by

$$R_{t,r} = \frac{N_r}{p^{t(t-1)}} = \left[\prod_{j=1}^{r}\left(1 - \frac{1}{p^{t+1-j}}\right)\right]\cdot\frac{1}{p^{t(t-1-r)}}\cdot \sum_{\substack{i_1+\cdots+i_r\leqslant t-1-r \\ \text{each } i_s\geqslant 0}} \left(\prod_{s=1}^{r} p^{si_s}\right).$$

In subsequent calculations it will be convenient to let $e = t-1-r$ and $B_{t,e} = R_{t,r}$ (thus for example, $e = 0$ when $r = t-1$, and $B_{t,0} = R_{t,t-1}$). Then

$$B_{t,e} = \left[\prod_{j=1}^{t-1-e}\left(1 - \frac{1}{p^{t+1-j}}\right)\right]\cdot\frac{1}{p^{te}}\cdot \sum_{\substack{i_1+\cdots+i_{t-1-e}\leqslant e \\ \text{each } i_s\geqslant 0}} \left(\prod_{s=1}^{t-1-e} p^{si_s}\right). \tag{2.2}$$

For $t \geqslant 2$ and $0 \leqslant e \leqslant t-1$, we note that the probability $B_{t,e}$ satisfies $0 < B_{t,e} < 1$.

LEMMA 2.2.

$$\sum_{e=0}^{t-1} eB_{t,e} \leqslant \sum_{e=0}^{t-1} e(p-2)B_{t,e} < 1.$$

Proof. The first inequality is clear since we are assuming $p \geqslant 3$. We now let

$$W_{t,e} = \sum_{\substack{i_1 + \cdots + i_{t-1-e} \leqslant e \\ \text{each } i_s \geqslant 0}} \left(\prod_{s=1}^{t-1-e} p^{s i_s} \right).$$

(For $e = t - 1$, we interpret this as $W_{t,t-1} = 1$.) For $t > e \geqslant 1$, we have

$$\begin{aligned} W_{t,e} &\leqslant (1 + p + \cdots + p^{t-1-e}) W_{t-1,e-1} \\ &\leqslant (1 + p + \cdots + p^{t-1-e}) W_{t,e-1} = \frac{p^{t-e} - 1}{p - 1} \cdot W_{t,e-1}. \end{aligned}$$

Using Eq. (2.2), we then get

$$\begin{aligned} B_{t,e} &\leqslant B_{t,e-1} \left(1 - \frac{1}{p^{e+1}} \right)^{-1} \cdot \frac{1}{p^t} \cdot \frac{p^{t-e} - 1}{p - 1} = B_{t,e-1} \cdot \frac{1}{p - 1} \cdot \frac{p^{t-e} - 1}{p^t - p^{t-e-1}} \\ &= B_{t,e-1} \cdot \frac{1}{p - 1} \cdot \frac{1 - (p^{t-e})^{-1}}{p^e - p^{-1}} < \left(\frac{1}{p - 1} \right)^2 B_{t,e-1}. \end{aligned}$$

Then by induction we get

$$B_{t,e} < \left(\frac{1}{p - 1} \right)^{2e} B_{t,0} < \left(\frac{1}{p - 1} \right)^{2e} \quad \text{for } t > e \geqslant 1.$$

Finally

$$\begin{aligned} \sum_{e=0}^{t-1} e(p - 2) B_{t,e} &< \sum_{e=0}^{t-1} e(p - 2) \left(\frac{1}{p - 1} \right)^{2e} < \sum_{e=1}^{t-1} \frac{e}{(p - 1)^e} \cdot \frac{1}{(p - 1)^{e-1}} \\ &\leqslant \sum_{e=1}^{t-1} \frac{1}{2} \cdot \frac{1}{2^{e-1}} < \sum_{e=1}^{\infty} \frac{1}{2^e} = 1. \end{aligned}$$

Remark. If X is a random variable which assumes the value e $(0 \leqslant e \leqslant t - 1)$ with $\text{Prob}(X = e) = B_{t,e}$, then the expected value $E(X) = \sum_{e=0}^{t-1} e B_{t,e} < 1$ according to Lemma 2.2. It then follows that for an arbitrarily chosen $(t - 1) \times t$ matrix M over $\mathbf{F}_p$, the expected rank is greater than $t - 2$.

LEMMA 2.3. *Let $t \geqslant 2$ be arbitrary. For $p = 3$, $B_{t,0} > .840$; for $p = 5$, $B_{t,0} > .950$; for $p = 7$, $B_{t,0} > .976$; and for $p \geqslant 11$, $B_{t,0} > .99$.*

Proof. For all $p \geqslant 3$, $B_{t,0} = \prod_{j=1}^{t-1} (1 - 1/p^{t+1-j})$ from Eq. (2.2). By letting $k = t + 1 - j$, we get $B_{t,0} = \prod_{k=2}^{t} (1 - 1/p^k)$. Now for all $t \geqslant 2$,

$$B_{t,0} > \prod_{k=2}^{\infty} \left(1 - \frac{1}{p^k} \right) > 1 - \sum_{k=2}^{\infty} \frac{1}{p^k} = 1 - \left(\frac{1}{p^2} \right) \left(\frac{1}{1 - p^{-1}} \right) = 1 - \frac{1}{p^2 - p}.$$

When $p \geqslant 11$, it is clear that $B_{t,0} > .99$. For the cases $p = 3, 5, 7$, the product $\prod_{k=2}^{\infty} (1 - 1/p^k)$ was evaluated numerically to three decimal places to give the above results.

Table 2.1 gives values for $B_{t,e}$ when $t = 2, 3, 4$ and $p = 3, 5, 7, 11$.

TABLE 2.1. *Values of* $B_{t,e}$

	t \ e	0	1	2	3
$p = 3$	2	.8889	.1111		
	3	.8560	.1427	.0014	
	4	.8454	.1526	.0020	2×10^{-6}
$p = 5$	2	.9600	.0400		
	3	.9523	.0476	.0001	
	4	.9508	.0491	.0001	4×10^{-9}
$p = 7$	2	.9796	.0204		
	3	.9767	.0233	8×10^{-6}	
	4	.9763	.0237	1×10^{-5}	7×10^{-11}
$p = 11$	2	.9917	.0083		
	3	.9910	.0090	6×10^{-7}	
	4	.9909	.0091	6×10^{-7}	3×10^{-13}

LEMMA 2.4. *For all* $t \geqslant 2$ *and* $p \geqslant 3$, $B_{t,0} + B_{t,1} > .99$.

Proof. Since $B_{t,0} > .99$ if $p \geqslant 11$, it suffices to consider $p = 3, 5, 7$. We claim that $B_{t+1,1} > B_{t,1}$ for all $p \geqslant 3$ and $t \geqslant 2$. To show this, we use Eq. (2.2) to get

$$\begin{aligned} B_{t+1,1} &= B_{t,1}\left(1 - \frac{1}{p^{t+1}}\right) \cdot \frac{1}{p} \cdot \frac{p^{t-1} + \cdots + p + 1}{p^{t-2} + \cdots + p + 1} \\ &= B_{t,1} \frac{(p^{t+1} - 1)(p^{t-1} + \cdots + p + 1)}{p^{t+2}(p^{t-2} + \cdots + p + 1)} \\ &= B_{t,1} \frac{p^{2t} + p^{2t-1} + \cdots + p^{t+1} - p^{t-1} - p^{t-2} - \cdots - 1}{p^{2t} + p^{2t-1} + \cdots + p^{t+2}} \\ &> B_{t,1} \end{aligned}$$

since $p^{t+1} - p^{t-1} - p^{t-2} - \cdots - 1 > 0$. We now apply Lemma 2.3 and the results from Table 2.1. If $p = 7$, then for $t \geqslant 2$, $B_{t,0} + B_{t,1} > .976 + B_{2,1} > .99$. If $p = 5$, then for $t \geqslant 2$, $B_{t,0} + B_{t,1} > .950 + B_{2,1} = .99$. If $p = 3$, then for $t \geqslant 4$, $B_{t,0} + B_{t,1} > .840 + B_{4,1} > .99$. Also from Table 2.1 we see that $B_{2,0} + B_{2,1} > .99$ and $B_{3,0} + B_{3,1} > .99$. Hence the proof of Lemma 2.4 is complete.

3. Ranks of p-Class Groups. We first let K be a Galois extension of $\mathbf{Q}$ of degree 3, and we let A be the 3-class group of K. We assume that exactly t primes ramify in $K/\mathbf{Q}$, where $t \geqslant 2$, and we let f_K denote the conductor of K. (*Remark*: The prime divisors of the conductor are the ramified primes.) Employing the techniques described in Chapters IV and VI of [4], we see that $v = \operatorname{rank} A = 2(t-1) - r$, where r is the rank of a certain $t \times t$ matrix of Hilbert symbols, and we may think of this matrix as a $t \times t$ matrix over $\mathbf{F}_3$. Because of the product formula for Hilbert symbols, the last row of the matrix is completely determined by the preceding $(t-1)$ rows; hence we are considering a certain $(t-1) \times t$ matrix M over $\mathbf{F}_3$ associated with K. From [2] and [3], we see that M is equally likely to be any $(t-1) \times t$ matrix over $\mathbf{F}_3$ in the following sense. Let x be a large positive real

number, and let $S_x = \{K \mid \text{exactly } t \text{ primes ramify in } K/\mathbf{Q} \text{ and the conductor } f_K \leqslant x\}$. Assume S_x has the counting measure, and let W_x be the function which assigns to each $K \in S_x$ the associated matrix M. If H is an arbitrary $(t-1) \times t$ matrix over $\mathbf{F}_3$, let $V_x(H)$ be the probability that W_x takes the value H. Then $V_x(H) \to 1/3^{t(t-1)}$ as $x \to \infty$. The fact that this limit probability is the same for all H is the reason we say that each possible choice for M is equally likely.

Now let N_r be the number of $(t-1) \times t$ matrices over $\mathbf{F}_3$ that have rank $= r$, where $0 \leqslant r \leqslant t-1$. Let Y_x be the random variable which assigns to each $K \in S_x$ the rank of the matrix M associated with K. Then $\operatorname{Prob}(Y_x = r) \to N_r/3^{t(t-1)}$ as $x \to \infty$. Now recall that the 3-class group A of K has rank satisfying

$$v = \operatorname{rank} A = 2(t-1) - r = t - 1 + (t - 1 - r) = t - 1 + e,$$

where we have set $e = t - 1 - r$. Then the following proposition is a consequence of our results from Section 2.

PROPOSITION 3.1. *Let an integer $t \geqslant 2$ be fixed, and let x be a positive real number. Let S_x be the set of all cubic Galois extensions K of $\mathbf{Q}$ with exactly t ramified primes over $\mathbf{Q}$ and conductor $f_K \leqslant x$. Assume S_x has counting measure. If Z_x is the random variable which assigns to each $K \in S_x$ the rank of the 3-class group of K, then* $\operatorname{Prob}(Z_x = t - 1 + e) \to B_{t,e}$ *as $x \to \infty$, where $B_{t,e}$ is given by Eq.* (2.2) *with $p = 3$, and $0 \leqslant e \leqslant t - 1$. In particular*

$$\operatorname{Prob}(Z_x = t - 1) > .840 \quad \textit{and} \quad \operatorname{Prob}(Z_x = t - 1 \textit{ or } t) > .99$$

for all sufficiently large x.

Remark. For $t = 2, 3$, and 4, we can use Table 2.1 to get the limit probabilities for $v = \operatorname{rank} A = t - 1 + e$. For example, when $t = 2$, $\operatorname{Prob}(Z_x = 1)$ is approximately .8889 for large x.

Remark. When $\operatorname{rank} A = t - 1$, it is known that A is an elementary abelian 3-group (cf. [4]). Since $\operatorname{Prob}(Z_x = t - 1) > .840$, most cubic Galois extensions of $\mathbf{Q}$ with t ramified primes have elementary abelian 3-class groups with rank $= t - 1$.

From Lemma 2.2, the fact that $v = t - 1 + e$, and the fact that $B_{t,0} < 1$ for $t \geqslant 2$, we get the following result.

PROPOSITION 3.2. *With assumptions as in Proposition* 3.1, $t - 1 < E(Z_x) < t$ *for all sufficiently large x, where $E(Z_x)$ is the expected value of Z_x.*

For these cubic Galois extensions we can also obtain the following result.

PROPOSITION 3.3. *Let assumptions be as in Proposition* 3.1. *Let $L_{t,e,x}$ be the number of elements K in the set S_x whose 3-class group has* $\operatorname{rank} = t - 1 + e$, *where $0 \leqslant e \leqslant t - 1$. Then*

$$L_{t,e,x} \sim B_{t,e} \cdot \frac{1}{2} \cdot \frac{x(\log\log x)^{t-1}}{(t-1)!\log x}.$$

(*Here $F(x) \sim G(x)$ means $F(x)/G(x) \to 1$ as $x \to \infty$.*)

Proof. The factor

$$\frac{1}{2}\,\frac{x(\log\log x)^{t-1}}{(t-1)!\log x}$$

is an asymptotic estimate for the number of elements in S_x (see [3] for details), and the factor $B_{t,e}$ is introduced because we are counting only the elements K of S_x that have 3-class group with rank $= t - 1 + e$.

We are now ready to consider primes $p \geqslant 5$. We suppose that K is a Galois extension of $\mathbf{Q}$ of degree p; A is the p-class group of K; t is the number of primes that ramify in $K/\mathbf{Q}$ (and we are assuming $t \geqslant 2$); f_K is the conductor of K. Then employing the techniques from [4], we see that $v = \operatorname{rank} A$ satisfies $t - 1 + e \leqslant v \leqslant t - 1 + e(p - 2)$, where $e = t - 1 - r$ and r is the rank of a certain $(t - 1) \times t$ matrix over $\mathbf{F}_p$. Thus for $p \geqslant 5$ we have the inequalities $t - 1 + e \leqslant v \leqslant t - 1 + e(p - 2)$ instead of the equality $v = t - 1 + e$. However when $e = 0$, we do have the equality $v = t - 1$, and from our calculations in Section 2, we know that the cases $e = 0$ has the highest probability. Using our results from Section 2, we can obtain the following result.

PROPOSITION 3.4. *Let $p \geqslant 5$ be a prime number. Let an integer $t \geqslant 2$ be fixed, and let x be a positive real number. Let S_x be the set of all Galois extensions K of $\mathbf{Q}$ of degree p with exactly t ramified primes over $\mathbf{Q}$ and conductor $f_K \leqslant x$. Assume S_x has counting measure. If Z_x is the random variable which assigns to each $K \in S_x$ the rank of the p-class group A of K, then* $\operatorname{Prob}(Z_x = t - 1) \to B_{t,0}$ *as $x \to \infty$, where $B_{t,0}$ is given by Eq.* (2.2). *In particular, for all sufficiently large x,* $\operatorname{Prob}(Z_x = t - 1) > .950$ (*resp.*, .976; *resp.*, .99) *when $p = 5$ (resp., $p = 7$; resp., $p \geqslant 11$). Furthermore $t - 1 < E(Z_x) < t$ for all sufficiently large x, where $E(Z_x)$ is the expected value of Z_x. Finally if $L_{t,x}$ is the number of elements K in S_x whose p-class group has rank $= t - 1$, then*

$$L_{t,x} \sim B_{t,0} \cdot \frac{1}{p-1} \cdot \frac{x(\log\log x)^{t-1}}{(t-1)!\log x}.$$

Remark. When $\operatorname{rank} A = t - 1$, it is known that A is an elementary abelian p-group (cf. [4]). Thus most Galois extensions of $\mathbf{Q}$ of degree p with t ramified primes have elementary abelian p-class groups with rank $= t - 1$.

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1. Z. BOREVICH & I. SHAFAREVICH, *Number Theory*, Academic Press, New York, 1966.

2. F. GERTH, "Consequences of l-th power reciprocity." (To appear.)

3. F. GERTH, "Counting certain number fields with prescribed l-class numbers," *J. Reine Angew. Math.*, v. 337, 1982, pp. 195–207.

4. G. GRAS, "Sur les l-classes d'idéaux dans les extensions cycliques relatives de degré premier l," *Ann. Inst. Fourier (Grenoble)*, v. 23, no. 3, 1973, pp. 1–48, and v. 23, no. 4, 1973, pp. 1–44.

5. E. INABA, "Über die Struktur der l-Klassengruppe zyklischer Zahlkörper von Primzahlgrad l", *J. Fac. Sci. Imp. Univ. Tokyo, Sect.* I, v. 4, 1940, pp. 61–115.

6. L. RÉDEI, "Über einige Mittelwertfragen im quadratischen Zahlkörper," *J. Reine Angew. Math.*, v. 174, 1936, pp. 15–55.