Some Inequalities for Elementary Mean Values

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Abstract. Upper and lower bounds for the difference between the arithmetic and harmonic means of \( n \) positive numbers are obtained in terms of \( n \) and the largest and smallest of the numbers. Also, results of S. H. Tung [2], are used to obtain upper and lower bounds for the elementary mean values \( M_p \) of Hardy, Littlewood, and Pólya.

1. In 1975, S. H. Tung proved the following theorem [2]:

Let \( 0 < b = x_1 \leq x_2 \leq \cdots \leq x_n = B \). Let \( A \) and \( G \) be the arithmetic and geometric means, respectively, of \( x_1, \ldots, x_n \). Then

\[
\frac{n}{n-1} \left( B^{1/2} - b^{1/2} \right)^2 \leq A - G \leq g(b, B),
\]

where \( g(b, B) = cb + (1 - c)B - b'B^{1-c} \), and

\[
c = \frac{\log((b/B - b) \log B/b)}{\log B/b}.
\]

We will derive somewhat similar bounds for the difference between the arithmetic and the harmonic means of \( n \) positive numbers.

2. In [1, Chapter 2] Hardy, Littlewood, and Pólya discussed the elementary mean values, which are defined as follows:

Let \( x_1, x_2, \ldots, x_n \) be positive numbers, and let \( p \) be a real number. Then \( M_p(x_1, \ldots, x_n) \) is defined as \( [n^{-1} \sum_{k=1}^{n} x_k^p]^{1/p} \), if \( p \neq 0 \); \( M_0(x_1, \ldots, x_n) \) is defined as \( (\prod_{k=1}^{n} x_k)^{1/n} \). We denote \( M_1 \), the arithmetic mean, by \( A \); \( M_0 \), the geometric mean, by \( G \); and \( M_{-1} \), the harmonic mean, by \( H \). Since \( M_p(kx_1, \ldots, kx_n) = kM_p(x_1, \ldots, x_n) \) for all \( p \) and for all \( k > 0 \), we may, without loss of generality, assume \( x_1 = 1 \).

Theorem 1. Let \( 1 = x_1 \leq x_2 \leq \cdots \leq x_n = B \). Then

\[
\frac{(B - 1)^2}{n(n + 1)} \leq A(1, \ldots, B) - H(1, \ldots, B) \leq (B^{1/2} - 1)^2.
\]

Proof. For each \( k, 2 \leq k \leq n, \) let

\[
A_k = A(x_1, x_2, \ldots, x_{k-1}, x_k) \quad \text{and} \quad H_k = H(x_1, x_2, \ldots, x_{k-1}, x_k).
\]

Fix \( x_1, x_2, \ldots, x_{n-2}, x_n \), and let \( x_{n-1} = x \) vary in \([1, B]\). Let

\[
D(x) = A_n - H_n = \frac{(n-1)A_{n-1} + x}{n} - H_n - \frac{nxH_{n-1}}{(n-1)x + H_{n-1}}.
\]

Computation of \( D'(x) \) shows that \( x = H_{n-1} \) is its only positive zero, and standard methods of analysis show that a minimum for \( D(x) \) is attained at \( x = H_{n-1} \).
Therefore,

\[ A_n - H_n \geq D(H_n^{-1}) = n^{-1}(n - 1)(A_{n-1} - H_{n-1}). \]

This process may be repeated, giving

\[ A_n - H_n \geq \frac{n - 1}{n} (A_{n-1} - H_{n-1}) \geq \frac{n - 2}{n} (A_{n-2} - H_{n-2}) \]

\[ \geq \cdots \geq \frac{2}{n} (A_2 - H_2) = \frac{(B - 1)^2}{n(B + 1)}. \]

The maximum of \( D(x) \) must occur at an endpoint, 1 or \( B \), as each of the variables \( x_2, x_3, \ldots, x_n \) in turn varies from 1 to \( B \). So

\[ A_n - H_n \leq \frac{nB - (B - 1)k}{n} - \frac{nB}{(B - 1)k + n} = F(k), \]

for some \( k, 0 \leq k \leq n \). The maximum of \( F(x) \) on \([0, n]\) will, then, be an upper bound for \( A_n - H_n \). Again, computation of \( F'(x) \) and standard methods of analysis show that a maximum is attained for \( x = n(B^{1/2} - 1)/(B - 1) \). Hence

\[ A_n - H_n \leq F(n(B^{1/2} - 1)/(B - 1)) = (B^{1/2} - 1)^2. \]

This completes the proof of Theorem 1.

Upper and lower bounds for \( G_n - H_n \) may be obtained using Theorem 1 and Tung's Theorem, since

\[ G_n - H_n = (A_n - H_n) - (A_n - G_n). \]

3. Tung's Theorem may be used to obtain upper and lower bounds for the elementary mean values \( M_p \), by using the relation

\[ M_p(x_1, \ldots, x_n) = \langle A(x_1^p, \ldots, x_n^p) \rangle^{1/p}. \]

(See [1].)

**Theorem 2.** Let \( 1 = x_1 \leq x_2 \leq \cdots \leq x_n = B \), and let \( p > 0 \). Then

\[ \left[ n^{-1}(B^{p/2} - 1)^2 + G^p \right]^{1/p} \leq M_p(1, \ldots, B) \leq \left[ g(1, B^p) + G^p \right]^{1/p}, \]

where \( G = G(1, \ldots, B) \), and \( g \) is the function defined in Tung's Theorem.

**Theorem 3.** Let \( 1 = x_1 \leq x_2 \leq \cdots, x_n = B \), and let \( p < 0 \). Then

\[ \left[ g(B^p, 1) + G^p \right]^{1/p} \leq M_p(1, \ldots, B) \leq \left[ n^{-1}(1 - B^{p/2})^2 + G^p \right]^{1/p}, \]

where \( G = G(1, \ldots, B) \) and \( g \) is the function of Tung's Theorem.

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