A Series Expansion for the First Positive Zero of the Bessel Functions

By R. Piessens

Abstract. It is shown that the first positive zero \( j_{v,1} \) of the Bessel function \( J_v(x) \) is given by

\[
\frac{49(v + 1)}{1152} \frac{8363(v + 1)^4}{276480} \quad \text{for } -1 < v < 0.
\]

1. It is well known that, when \( v \) is real and \( v > -1 \), the Bessel function \( J_v(x) \) has an infinite number of zeros and that all zeros are real (Watson [9]). We denote the \( s \)th positive zero of \( J_v(x) \) by \( j_{v,s} \).

Several approximations, asymptotic expansions or bounds for the zeros of Bessel functions exist (see [1], [2], [4], [6], [7], [9]). Especially McMahon’s expansion for large zeros (see Abramowitz and Stegun [1]), Olver’s asymptotic expansion for large orders and Olver’s uniform asymptotic expansions (see Olver [6]) are interesting formulas, but, unfortunately, they are not applicable when \( s \) and \( v \) are small. The purpose of this note is to give a series expansion for \( j_{v,1} \) when \(-1 < v < 0\).

2. Cayley [3] noticed that Graeffe’s method for solving a polynomial equation can be applied for the efficient computation of

\[
\sum_{s=1}^{\infty} j_{v,s}^{-2r} = \sigma_v^{(r)}, \quad r = 1, 2, \ldots.
\]

An upper bound for \( j_{v,1} \) is given by Chambers [4]:

\[
j_{v,1} < (v + 1)^{1/2} \left[ (v + 2)^{1/2} + 1 \right].
\]

Further it is known that, when \( k > 1 \),

\[
\lim_{v \to -1} j_{v,k} = j_{1,k-1} > 0.
\]

Thus, the first term in the left side of (1) is dominant when \( v = -1 \), so that

\[
j_{v,1} = 2(v + 1)^{1/2} \phi_v(v) + o\left((v + 1)^{-1}\right), \quad v \to -1,
\]

where

\[
\phi_v(v) = \left[ \frac{1}{2^2(v + 1)^r \sigma_v^{(r)}} \right]^{1/2r}
\]

is analytic at \( v = -1 \).
By approximating $\phi_r(\nu)$ by a Taylor polynomial, we obtain

$$j_{r,1} = 2(\nu + 1)^{1/2} \sum_{k=0}^{r-1} C_k(\nu + 1)^k + o((\nu + 1)^{r-1}), \quad \nu \to -1,$$

where

$$C_k = \frac{1}{k!} \left. \frac{d^k \phi_r(\nu)}{d \nu^k} \right|_{\nu = -1}$$

is independent of $r$.

When $r \to \infty$, (6) becomes a series expansion for $j_{r,1}$, which, because of the presence of a branchpoint of $\phi_r(\nu)$ at $\nu = -2$, converges only in the interval $-1 < \nu < 0$.

Using REDUCE, which is a computer language for formula manipulation [5], we have computed $C_k$, $k = 0, 1, 2, 3, 4$, using (7), where $r = 8$ and

$$\phi_r(\nu) = \frac{(\nu + 2)^3(\nu + 3)^2(\nu + 4)^2(\nu + 5)(\nu + 6)(\nu + 7)(\nu + 8)}{429\nu^5 + 7640\nu^4 + 53752\nu^3 + 185430\nu^2 + 311387\nu + 202738},$$

The result is

$$j_{r,1} = 2(\nu + 1)^{1/2} \left[ 1 + \frac{(\nu + 1)^2}{4} - \frac{7(\nu + 1)^2}{96} + \frac{49(\nu + 1)^3}{1152} - \frac{8363(\nu + 1)^4}{276480} + \ldots \right].$$

In Table 1, we compare the exact values of $j_{r,1}$ with the approximate values given by (9), for $\nu = -3/4$, $-2/3$, $-1/2$, $-1/3$, $-1/4$ and also for $\nu = 0$ (although we were not able to prove the convergences of the expansion for $\nu = 0$).

<table>
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<tr>
<th>$\nu$</th>
<th>exact</th>
<th>approximation (9)</th>
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<tr>
<td>$-3/4$</td>
<td>1.058508</td>
<td>1.058489</td>
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<td>$-2/3$</td>
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</table>

3. An interesting application of (9) is the estimation of the smallest zero of Laguerre- and Gegenbauer-polynomials [8]. For example, the smallest zero $\xi_n$ of the Laguerre polynomials $L_n^{(\alpha)}(x)$ is approximated by (see Tricomi [8])

$$\xi_n = j_{a,1} \left[ 1 + \frac{2(\alpha^2 - 1) + j_{a,1}^2}{48k_n^2} \right],$$

where $k_n = n + (\alpha + 1)/2$. In Table 2, this approximation, where $j_{a,1}$ is replaced by (9), is compared with the exact value of $\xi_n$.  

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Table 2

*Values of the smallest zero of $I_n^{(a)}(x)$*

<table>
<thead>
<tr>
<th>$a$</th>
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</table>

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