Odd Triperfect Numbers

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Abstract. We prove that an odd triperfect number has at least ten distinct prime factors.

1. A positive number $N$ is called a triperfect number if $\sigma(N) = 3N$, where $\sigma(N)$ is the sum of the positive divisors of $N$. Six even triperfect numbers are known:

$2^{14} \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151,$
$2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127,$
$2^9 \cdot 3 \cdot 11 \cdot 31,$
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73,$
$2^5 \cdot 3 \cdot 7,$
$2^3 \cdot 3 \cdot 5.$

However, the existence of an odd triperfect (OT) number is an open question. McDaniel [1] and Cohen [2] proved that an OT number has at least nine distinct prime factors, and Beck and Najar [3] showed that it exceeds $10^{50}$.

In this paper using the technique of [4], we prove

Theorem. If $N$ is OT, $N$ has at least ten distinct prime factors.

2. Throughout this paper we let

$$N = \prod_{i=1}^{r} p_i^{a_i},$$

where $p_i$'s are odd primes, $p_1 < \cdots < p_r$ and $a_i$'s are positive integers.

The following lemmas are easy to prove:

Lemma 1. If $N$ is OT,

(1) $a_i$'s are even for $1 \leq i \leq r$.

Lemma 2. If $N$ is OT and $q$ is a prime factor of $\sigma(p_i^{a_i})$ for some $i$, then $q = 3$ or $q = p_j$ for some $j$, $1 \leq j \leq r$.

Lemma 3. If $N$ is OT and $r = 9$, $p_8 < 80$.
As in [4] we define

\[ S(N) = \sigma(N)/N, \]
\[ a(p) = \min\{a | a \text{ is even and } p^{a+1} > 10^{11}\}, \]
\[ b_i = \min\{a_i, a(p_i)\}, \]
\[ M = \prod_{i=1}^{r} p_i^{b_i}. \]

**Lemma 4.** If \( N \) is OT, then

\[ \log 3 - r \cdot 10^{-11} < \log S(M) \leq \log 3. \]

*Proof.* Since \( M | N \), \( S(M) \leq S(N) = 3 \) and so \( \log S(M) \leq \log 3 \). In [4] we proved that if \( a > a(p) \), then

\[ 0 < \log S(p^a) - \log S(p^{a(p)}) < 10^{-11}. \]

Hence

\[ 0 < \log S(N) - \log S(M) < r \cdot 10^{-11}, \]

and we have

\[ \log 3 - r \cdot 10^{-11} = \log S(N) - r \cdot 10^{-11} < \log S(M). \]

Q.E.D.

**Corollary.** If \( N \) is OT, \( L = M/p_{p^r} \) and if \( p_r > 3500 \), then

\[ \log 3 - r \cdot 10^{-11} - \log S(3499^2) < \log S(L) < \log 3. \]

*Proof of Theorem.* We used a computer (PDP 11 at the University of Toledo) to find

\[ M = \prod_{i=1}^{9} p_i^{a_i}, \]

satisfying (1), \( a_i \leq a(p_i) \) for \( 1 \leq i \leq 9 \), (2) with \( r = 9 \), and \( p_9 < 3500 \). There were 71 such \( M \)'s; however, all of them had a factor \( p_{p^r} \) such that \( a_i < a(p_i) \), \( \sigma(p_{p^r}) \) had a prime factor \( q > 3 \), and \( q \neq p_j \) for \( 1 \leq j \leq 9 \).

Next we tried to find

\[ L = \prod_{i=1}^{8} p_i^{a_i}, \]

satisfying (1), \( a_i \leq a(p_i) \) for \( 1 \leq i \leq 8 \), and (3) with \( r = 9 \). There were 12689 such \( L \)'s; however, 12473 of them had a factor \( p_{p^r} \) such that \( a_i < a(p_i) \), \( \sigma(p_{p^r}) \) had a prime factor \( q > 3 \), and \( q \neq p_j \) for \( 1 \leq j \leq 8 \), and

\[ \log S(L) + \log S(q^2) > \log 3. \]

The remaining 216 of them had the following properties: there exist two consecutive primes \( u \) and \( v \) such that \( 3500 < u < v \),

\[ \log S(L) + \log S(u^2) > \log 3, \text{ and } \]
\[ \log S(L) + \log v/(v-1) < \log 3 - 9 \cdot 10^{-11}. \]
These three cases show that if $N$ is an odd integer with $r = 9$, then $N$ cannot be OT. Q.E.D.

The computer time was over five hours.