# A Note on the Moment Generating Function
# for the Reciprocal Gamma Distribution

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**Abstract.** In this note we consider the function $\varphi(t) = \int_0^\infty e^{-tx}/\Gamma(x)\,dx$ and use the Euler-Maclaurin expansion with the step-length $h = 1/4$ to obtain some useful (from a numerical point of view) formulae. Numerical values of $\varphi(t)$ correct to 11D are given for $t = 0.0(0.1)5.0$.

**Introduction.** In [3] we analyzed the function

$$\varphi(t) = \int_0^\infty \frac{e^{-tx}}{\Gamma(x)}\,dx$$

and used the Euler-Maclaurin expansion to obtain some interesting (from a numerical point of view) formulae ((3.6) and (3.7)). These cases corresponded to the step-lengths $h = 1$ and $h = \frac{1}{2}$. Using a little more sophisticated analysis, also the case $h = \frac{1}{4}$ may be investigated.

**1. The Euler-Maclaurin Summation Formula With Step-Length $h = \frac{1}{4}$.** We first must sum the expression

$$(1.1) \qquad\qquad F(t) = \sum_{k=0}^\infty \frac{e^{-kt/4}}{\Gamma(k/4)}.$$

It is evident that

$$(1.2) \qquad\qquad F(t) = \varphi_1(t) + \varphi_2(t) + \varphi_3(t) + \varphi_4(t),$$

where

$$\varphi_1(t) = \sum_{j=1}^\infty \frac{e^{-tj}}{\Gamma(j)} = e^{-t+e^{-t}},$$

$$\varphi_2(t) = \sum_{j=0}^\infty \frac{e^{-t(j+1/2)}}{\Gamma(j+1/2)} = \frac{e^{-t/2}}{\sqrt{\pi}} + \left(2N(2^{1/2}e^{-t/2}) - 1\right)e^{-t+e^{-t}},$$

$$\varphi_3(t) = \sum_{j=0}^\infty \frac{e^{-(j+1/4)t}}{\Gamma(j+1/4)},$$

$$\varphi_4(t) = \sum_{j=0}^\infty \frac{e^{-(j+3/4)t}}{\Gamma(j+3/4)}.$$

In the expression for $\varphi_2(t)$, $N(\cdot)$ denotes the standardized normal distribution function, defined in [3, Eq. (3.5)].

It remains to give useful alternative analytical expressions for the functions $\varphi_3(t)$ and $\varphi_4(t)$. We write $\varphi_3(t)$ in the form

$$(1.3) \qquad \varphi_3(t) = \frac{e^{-t/4}}{\Gamma(1/4)} + \frac{e^{-t/4}}{\Gamma(1/4)} \sum_{j=0}^{\infty} \frac{4^{j+1}e^{-(j+1)t}}{(4j+1)(4j-3)\cdots 1}.$$

Putting $I = \int_0^u e^{-x^4/4}\, dx$ and using integration by parts, we get

$$(1.4) \qquad I = \int_0^u e^{-x^4/4}\, dx = \sum_{j=0}^{\infty} \frac{u^{4j+1}e^{-u^4/4}}{(4j+1)(4j-3)\cdots 1}.$$

Thus we have

$$(1.5) \qquad e^{u^4/4}\int_0^u e^{-x^4/4}\, dx = \sum_{j=0}^{\infty} \frac{u^{4j+1}}{(4j+1)(4j-3)\cdots 1}.$$

Comparing (1.3) and (1.5), we get after some calculations

$$(1.6) \qquad \varphi_3(t) = \frac{1}{\Gamma(1/4)}\left\{ e^{-t/4} + e^{-t+e^{-t}}(\sqrt{2})^3 \int_0^{\sqrt{2}\,e^{-t/4}} e^{-x^4/4}\, dx \right\}.$$

In Section 2 we will discuss different techniques to evaluate numerically the integral occurring in (1.6).

Now write

$$(1.7) \qquad \varphi_4(t) = \frac{e^{-3t/4}}{\Gamma(3/4)} + \frac{e^{-3t/4}}{\Gamma(3/4)} \sum_{j=0}^{\infty} \frac{4^{j+1}e^{-(j+1)t}}{(4j+3)(4j-1)\cdots (3)}.$$

Using integration by parts, we may prove that

$$(1.8) \qquad e^{u^4/4}\int_0^u e^{-x^4/4}x^2\, dx = \sum_{j=0}^{\infty} \frac{u^{4j+3}}{(4j+3)(4j-1)\cdots (3)}.$$

Comparing (1.7) and (1.8), we get

$$(1.9) \qquad \varphi_4(t) = \frac{1}{\Gamma(3/4)}\left\{ e^{-3t/4} + \sqrt{2}\,e^{-t+e^{-t}} \int_0^{\sqrt{2}\,e^{-t/4}} x^2\, e^{-x^4/4}\, dx \right\}.$$

We will return to the integral in (1.9) in Section 2.

Applying the Euler-Maclaurin summation formula to the function $\varphi(t)$ with a step-length $h = \frac{1}{4}$, we get, after some manipulations,

$$(1.10) \qquad \varphi(t) = \frac{1}{4}\sum_{i=1}^{4} \varphi_i(t) + \sum_{j=0}^{\infty} \frac{(-t)^j}{j!} \sum_{k=[(j+1)/2]+1}^{\infty} \frac{B_{2k}}{2k}\left(\frac{1}{4}\right)^{2k} a_{2k-1-j}.$$

(The coefficients $a_n$ occurring in (1.10) are defined in [3, Eq. (3.23)]). Using the methods developed in Section 2 for computing the functions $\varphi_3(t)$ and $\varphi_4(t)$, we tabulated $\varphi(t)$ to 15D in the interval [0, 5.0]. See Table I, where we give only 12 decimals.

**2. Calculation of Some Integrals.** To calculate the functions $\varphi_3(t)$ and $\varphi_4(t)$ as given by (1.6) and (1.9) we need some fast and accurate methods to compute the integrals

$$(2.1) \qquad I = \int_0^t e^{-x^4/4}\, dx \quad \text{and} \quad J = \int_0^t x^2 e^{-x^4/4}\, dx.$$

We start with $I$. Our first technique to evaluate this integral stems from a paper by Kerridge and Cook [2]. To generalize their arguments we must study the polynomials

$$(2.2) \qquad p_n(x) = (-1)^n e^{x^4/4} D_x^n e^{-x^4/4}.$$

The polynomials $\{p_n(x)\}$ satisfy the recurrence relation

$$(2.3) \qquad p_{n+1}(x) = x^3 p_n(x) - 3nx^2 p_{n-1}(x) + 3xn(n-1)p_{n-2}(x)$$
$$- n(n-1)(n-2)p_{n-3}(x),$$

with starting values $p_0 = 1$, $p_1 = x^3$, $p_2 = x^6 - 3x^2$, $p_3 = x^9 - 9x^5 + 6x$. We now make the following Taylor series expansion

$$(2.4) \qquad I = \int_0^t e^{-x^4/4} \, dx = \int_0^t \sum_{n=0}^{\infty} \left(x - \frac{t}{2}\right)^n \frac{\left(D_x^n e^{-x^4/4}\right)_{x=t/2}}{n!} \, dx.$$

Carrying out the integration, we get

$$(2.5) \qquad I = te^{-t^4/64} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}t\right)^{2n}}{(2n+1)!} p_{2n}\left(\tfrac{1}{2}t\right).$$

Define the polynomials $\theta_n(x)$ as

$$(2.6) \qquad \theta_n(x) = \frac{x^n}{n!} p_n(x).$$

Then

$$(2.7) \qquad I = \int_0^t e^{-x^4/4} \, dx = te^{-t^4/64} \sum_{n=0}^{\infty} \frac{\theta_{2n}\left(\frac{1}{2}t\right)}{2n+1},$$

and the polynomials $\theta_n(x)$ satisfy the simpler recurrence relation

$$(2.8) \qquad \theta_{n+1}(x) = \frac{x^4}{n+1}\left(\theta_n(x) - 3\theta_{n-1}(x) + 3\theta_{n-2}(x) - \theta_{n-3}(x)\right),$$

with starting values $\theta_0 = 1$, $\theta_1 = x^4$, $\theta_2 = \frac{1}{2}(x^8 - 3x^4)$, $\theta_3 = \frac{1}{6}(x^{12} - 9x^8 + 6x^4)$. To find the other technique, we study the expansion

$$(2.9) \qquad e^{-a^2(x-1/2)^4/2} = e^{-a^2/32} e^{a[ax(1-x)]/4 - [ax(1-x)]^2/2}.$$

Now remember that the Hermite polynomials may be defined by the following generating function (see, e.g., Kendall and Stuart [1, p. 155])

$$(2.10) \qquad e^{tz - t^2/2} = \sum_{n=0}^{\infty} \frac{t^n H_n(z)}{n!}.$$

Therefore

$$(2.11) \qquad e^{-a^2(x-1/2)^4/2} = e^{-a^2/32} \sum_{n=0}^{\infty} \frac{(x(1-x))^n}{n!} a^n H_n\left(\frac{a}{4}\right).$$

Integrating this identity between the limits 0 and 1, we get after a few rearrangements

$$(2.12) \qquad \frac{1}{\sqrt{a}} \int_{-\sqrt{a}/2}^{+\sqrt{a}/2} e^{-u^4/2} \, du = e^{-a^2/32} \sum_{n=0}^{\infty} \frac{a^n H_n(a/4)}{n!(2n+1)\binom{2n}{n}}.$$

Making the proper manipulations, we finally get

$$(2.13) \qquad I = \int_0^t e^{-z^4/4}\,dz = te^{-t^4/4} \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n+1)\binom{2n}{n}} \psi_n\!\left(\frac{t^2}{\sqrt{2}}\right),$$

where $\psi_n(x)$ is defined by

$$(2.14) \qquad \psi_n(x) = \frac{x^n}{n!} H_n(x).$$

$\psi_n(x)$ satisfies the recurrence relation

$$(2.15) \qquad \psi_{n+1}(x) = \frac{x^2}{n+1}\left(\psi_n(x) - \psi_{n-1}(x)\right).$$

If we instead consider the expression $a(x - \tfrac{1}{2})^2 e^{-a^2(x-1/2)^4/2}$ and proceed along the lines indicated by (2.9)–(2.12), we get

$$(2.16) \qquad J = \int_0^t z^2 e^{-z^4/4}\,dz = t^3 e^{-t^4/4} \sum_{n=0}^{\infty} \frac{\psi_n\!\left(t^2/\sqrt{2}\right)2^{2n}}{(2n+1)\binom{2n}{n}}$$

$$- 4t^3 e^{-t^4/4} \sum_{n=0}^{\infty} \frac{\psi_n\!\left(t^2/\sqrt{2}\right)2^{2n}}{(2n+3)\binom{2n+2}{n+1}}.$$

Some simplifications finally yield

$$(2.17) \qquad J = \int_0^t z^2 e^{-z^4/4}\,dz = t^3 e^{-t^4/4} \sum_{n=0}^{\infty} \frac{\psi_n\!\left(t^2/\sqrt{2}\right)2^{2n}}{(2n+3)(2n+1)\binom{2n}{n}}.$$

To obtain a formula for $J$ similar to (2.7) we must introduce the polynomials

$$(2.18) \qquad P_n(x) = (-1)^n e^{x^4/4} D_x^n x^2 e^{-x^4/4}.$$

In terms of the polynomials $p_n(x)$ we may write

$$(2.19) \qquad P_n(x) = x^2 p_n(x) - 2nx p_{n-1}(x) + n(n-1) p_{n-2}(x).$$

If we define the functions $\langle \xi_n(x)\rangle$ as

$$(2.20) \qquad \xi_n(x) = x^n P_n(x)/n!,$$

we observe that

$$(2.21) \qquad \xi_n(x) = x^2\big(\theta_n(x) - 2\theta_{n-1}(x) + \theta_{n-2}(x)\big) = x^2 \Delta^2 \theta_n(x),$$

and we get

$$(2.22) \qquad J = \int_0^t x^2 e^{-x^4/4}\,dx = te^{-t^4/64} \sum_{n=0}^{\infty} \frac{\xi_{2n}\!\left(\tfrac{1}{2}t\right)}{2n+1}.$$

It is evident that for small values of $t$ we may use the simple formula

$$(2.23) \qquad \int_0^t x^a e^{-x^4/4}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+a+1}}{n!4^n(4n+a+1)}; \qquad a \in \{0,2\}.$$

When calculating the integrals $I$ and $J$ we found formulae (2.7) and (2.22) to be of the greatest value. The convergence in the series (2.13) and (2.17) turned out to be rather slow. For small values of $t$ also the formula (2.23) was useful. The resulting numerical values of $\varphi(t)$ correct to (at least) 11D appear in Table I. The reason why

we give 12D in Table I is that a use of Watson's Lemma [3, Eq. (3.23)] indicates that for $t = 5.0$ we have a precision of 14D. A comparison with Table IV of [3] confirms that Table IV correctly yields 10D.

## TABLE I

*Values of $\varphi(t)$ using the polynomials $p_n(x)$.*

| $t$ | $\varphi(t)$ | $t$ | $\varphi(t)$ | $t$ | $\varphi(t)$ |
|-----|------------|-----|------------|-----|------------|
| 0.0 | 2.807770242028 | 1.7 | 0.300933511958 | 3.4 | 0.091421032961 |
| 0.1 | 2.326237047400 | 1.8 | 0.275394801591 | 3.5 | 0.086581618381 |
| 0.2 | 1.946771821817 | 1.9 | 0.252780695525 | 3.6 | 0.082102419937 |
| 0.3 | 1.644358498906 | 2.0 | 0.232680797724 | 3.7 | 0.077949589282 |
| 0.4 | 1.400823696157 | 2.1 | 0.214751780293 | 3.8 | 0.074093068791 |
| 0.5 | 1.202793433329 | 2.2 | 0.198705012619 | 3.9 | 0.070506103326 |
| 0.6 | 1.040305961681 | 2.3 | 0.184296711962 | 4.0 | 0.067164822585 |
| 0.7 | 0.905856615825 | 2.4 | 0.171320056208 | 4.1 | 0.064047882795 |
| 0.8 | 0.793731332327 | 2.5 | 0.159598832949 | 4.2 | 0.061136158425 |
| 0.9 | 0.699535729986 | 2.6 | 0.148982298581 | 4.3 | 0.058412476217 |
| 1.0 | 0.619858414145 | 2.7 | 0.139340995782 | 4.4 | 0.055861385110 |
| 1.1 | 0.552027547158 | 2.8 | 0.130563334193 | 4.5 | 0.053468956707 |
| 1.2 | 0.493932984351 | 2.9 | 0.122552782029 | 4.6 | 0.051222611791 |
| 1.3 | 0.443895013087 | 3.0 | 0.115225549144 | 4.7 | 0.049110969132 |
| 1.4 | 0.400566564690 | 3.1 | 0.108508667380 | 4.8 | 0.047123713408 |
| 1.5 | 0.362859707863 | 3.2 | 0.102338393501 | 4.9 | 0.045251479573 |
| 1.6 | 0.329889922708 | 3.3 | 0.096658875253 | 5.0 | 0.043485751382 |

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