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On Common Zeros of Legendre's Associated Functions

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Abstract. In this paper it is proved that any two given Legendre associated functions $P_n^m(\mu)$ and $P_n^s(\mu)$, where $n \geqslant 1$ is an integer and where one of the integers m or s may be 0 (and $m \neq \pm s$), have either no zero in common or exactly one common zero, namely $\mu = 0$. An auxiliary result states that the $n - m$ zeros of P_n^m known to lie in the open interval $(-1, 1)$ lie in fact in the open interval $(-c, c)$, where $\pm c$ are the two zeros of $n(n+1) - m^2/(1-\mu^2)$ which is one of the coefficients in the Legendre associated equation satisfied by P_n^m. Some monotonicity behavior of P_n^m is simultaneously described.

The proof of the main result is based on properties of Prüfer polar coordinates.

1. Introduction. It was in dealing with a mathematical study of binary cell cleavage in (animal) embryology [1], that the question arose as to whether two or more different Legendre associated functions (or "polynomials") could have some common zeros. There emerged simultaneously the auxiliary question of the position of the zeros of any given Legendre associated function.

The functions under consideration are the $P_n^m(\mu)$, where m and n are integers, usually with $0 \leqslant m \leqslant n$, and $\mu = \cos\theta$, $-1 \leqslant \mu \leqslant 1$, $0 \leqslant \theta \leqslant \pi$. The value $m = 0$ yields Legendre's polynomials $P_n(\mu)$. The function $P_n^m(\mu)$ is a solution over the open interval $(-1, 1)$ of Legendre's associated equation

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dy}{d\mu}\right] + \left[n(n+1) - m^2/(1-\mu^2)\right]y = 0. \tag{1}$$

It is known that P_n^m has $n - m$ distinct simple zeros in the open interval $(-1, 1)$. If we call $q(\mu)$ the coefficient of y in Eq. (1), then a first result, which is a lemma but with some interest of its own, states that when $m \geqslant 1$ these zeros are positioned between the two zeros $\pm c$ of $q(\mu)$. We take the opportunity to note (as a corollary) some immediate consequences about the behavior of P_n^m over the intervals from c to 1 and -1 to $-c$.

The *main result* deals with common zeros: any two given P_n^m and P_n^s, where one of m or s may be 0 and $m \neq \pm s$, have either no zero in common or exactly one common zero, namely $\mu = 0$.

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The basic tools in establishing the main result are Prüfer polar coordinates; details on this subject may be found in Sagan [3, pp. 152–158].

2. The Position of the Zeros of P_n^m. If we write $p(\mu) = 1 - \mu^2$ and $\pm c = \pm[1 - m^2/n(n+1)]^{1/2}$ for the two zeros of $q(\mu)$, then over the open interval $(c, 1)$ we have $p(\mu) > 0$ and $q(\mu) < 0$. We recall that $P_n^m(\pm 1) = 0$ when $m \geqslant 1$, while $P_n(1) = 1$, $P_n(-1) = (-1)^n$, and that the zeros of P_n^m occur in symmetrical pairs about the origin; in fact

$$P_n^m(\mu) = (-1)^{n-m} P_n^m(-\mu). \tag{2}$$

See for instance Hobson [2, pp. 89–95].

LEMMA 1. *Any Legendre associated function $P_n^m(\mu)$, where $1 \leqslant m < n$ are integers, which is known to have $n - m$ zeros in the open interval $(-1, 1)$, has all such zeros in the open interval $(-c, c)$, where $\pm c$ are the two zeros of $q(\mu) = n(n+1) - m^2/(1-\mu^2)$.*

Proof. We write for brevity $y(\mu) = P_n^m(\mu)$. Suppose y had a zero μ_1 in $[c, 1)$. Since $y(1) = 0$, then by Rolle's theorem, y' would have a zero μ_2 in $(\mu_1, 1)$. If we then multiply Eq. (1) by y and integrate (by parts), we obtain

$$pyy'|_{\mu_1}^{\mu_2} = \int_{\mu_1}^{\mu_2} p(y')^2\, d\mu - \int_{\mu_1}^{\mu_2} qy^2\, d\mu,$$

which is a contradiction since the left side is zero while the right side is positive.

The "symmetry" expressed by Eq. (2) completes the lemma. Q.E.D.

The lemma may have some interest as a statement about an upper (and a lower) bound for the zeros of P_n^m, although one may easily check with numerical examples that it is rather coarse. We mention some consequences of the above lemma.

COROLLARY 1. *Consider the situation given in Lemma* 1. *Then*:

(i) $P_n^m(\pm c) \neq 0$;

(ii) $P_n^m(\mu)$ *is strictly monotone over* $[c, 1]$ *and* $[-1, c]$;

(iii) $P_n^m(\mu)$ *has a nonvanishing derivative over* $(c, 1)$ *and* $(-1, -c)$.

Proof. Statement (i) comes from the proof of Lemma 1. To deal with statements (ii) and (iii), first consider the case $y(c) = P_n^m(c) > 0$. Suppose that y were increasing for some values in $[c, 1)$. Then, by the continuity and differentiability of y and by Rolle's theorem, there would be an interval $[a, b]$ within $[c, 1)$ and a value $\mu_2 \in (a, b)$ such that $y(a) = y(b) > 0$, $y'(a) > 0$ and $y'(\mu_2) = 0$. Then an integration as in the proof of Lemma 1, with limits a and μ_2, would lead to a contradiction. Thus y is nonincreasing over $[c, 1]$.

Suppose now that there were a μ_2 in $(c, 1)$ with $y'(\mu_2) = 0$. Here $y(\mu_2) > 0$ by Lemma 1. Then there is a value b in $(\mu_2, 1)$ for which $y(b) > 0$ and $y'(b) < 0$. Again an integration, with limits μ_2 and b, leads to a contradiction, so that (ii) and (iii) are established for the case $y(c) > 0$.

The case $y(c) < 0$ is settled similarly and the results for $[-1, -c]$ and $(-1, -c)$ follow from the "symmetry" expressed by Eq. (2). Q.E.D.

3. On the Common Zeros of P_n^m and P_n^s. We may now state our main result.

PROPOSITION 1. *Any two nonzero Legendre associated functions $P_n^m(\mu)$ and $P_n^s(\mu)$, where n is an integer $\geqslant 1$ and m, s are integers with $m \neq \pm s$, have over the open interval $(-1, 1)$ either no common zero or exactly one common zero, namely $\mu = 0$. And this last case occurs if and only if $n - |m|$ and $n - |s|$ are both odd and positive.*

Proof. A negative m or s may be replaced by its absolute value since in general P_n^m and P_n^{-m} are simply related by a constant [2, p. 99]. Any P_n^m with $m > n$ is identically 0, while it only has zeros at $\mu = \pm 1$ if $m = n \neq 0$ [2, p. 94]. (We remark, for completeness, that if $m = n = 0$, the function is the constant 1.)

Hence there is no loss of generality if we assume that our integers satisfy $0 \leqslant s < m \leqslant n - 1$.

P_n^m satisfies Eq. (1) with $p_1(\mu) = 1 - \mu^2$ and $q_1(\mu) = n(n+1) - m^2/(1-\mu^2)$, while P_n^s satisfies Eq. (1) with $p_2(\mu) = 1 - \mu^2 = p_1(\mu)$ and $q_2(\mu) = n(n+1) - s^2/(1-\mu^2)$. We have $q_1(\mu) < q_2(\mu)$ for $-1 < \mu < 1$.

Let c and $-c$ stand for the two zeros of $q_1(\mu)$. According to Lemma 1, the zeros of P_n^m in $(-1, 1)$ lie in fact in $(-c, c)$. Thus our search of the common zeros is restricted to $(-c, c)$. Since P_n^m and P_n^s each have a finite number of zeros in $(-1, 1)$, there is a closed interval $[a, b]$ within $(-c, c)$ which contains all the zeros of P_n^m and P_n^s which lie in $(-c, c)$. If in $[a, b]$, $-\mu_0$ and μ_0 were a symmetrical pair of nonzero common zeros of P_n^m and P_n^s, then we would have $\phi_1(\pm\mu_0) = \phi_2(\pm\mu_0) = \pi/2 \pmod{\pi}$, where $\phi_1(\mu)$ and $\phi_2(\mu)$ are Prüfer polar coordinates of P_n^m and P_n^s, respectively [3, p. 153]. On the other hand, $\phi_1(-\mu_0) = \phi_2(-\mu_0)$ implies that $\phi_2(\mu) < \phi_1(\mu)$ for all $\mu > -\mu_0$ in our interval $[a, b]$; see [3, p. 158]. This contradiction shows that over $[a, b]$, there can be at most one common zero, namely $\mu = 0$. The last statement is immediate. Q.E.D.

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