A Nonaveraging Set of Integers With a Large Sum of Reciprocals

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Abstract. A set of integers is constructed with no three elements in arithmetic progression and with a rather large sum of reciprocals.

It is a famous open question due to Erdös [2] whether every infinite sequence of positive integers \( a_i (i = 1, 2, \ldots) \) such that

\[ \sum_{i=1}^{\infty} \frac{1}{a_i} = \infty \]

contains arbitrarily long arithmetic progressions. It is not even known whether there exist sequences \( a_i \) containing no three terms in arithmetic progression (called for the sake of brevity nonaveraging sets) such that the sum \( \sum_{i=1}^{\infty} 1/a_i \) is arbitrarily large.

Gerver (see [4]) constructed sequences containing no \( k \)-term arithmetic progression with the sum of reciprocals greater than \((1 - \epsilon)k \cdot \log k\), where any \( \epsilon > 0 \) is appropriate for all but a finite number of integers \( k \geq 3 \).

A well known nonaveraging set apparently first studied by G. Szekeres (see [3]), consists of the numbers \( 1 + 3^{a_1} + 3^{a_2} + \cdots + 3^{a_k} \), where \( k \geq 0 \) and \( 0 < a_1 < a_2 < \cdots < a_k \). Denoting it by \( S \) we have

\[ 3.00793 < \sum_{a \in S} \frac{1}{a} < 3.00794 \]

(cf. [5] where the value of the sum is given as 3.007).

The aim of this note is to construct a nonaveraging set of integers with the sum of reciprocals appreciably greater than \( \sum_{a \in S} 1/a \). The construction uses the idea of Behrend [1]. Let, for \( p, q > 0, r \geq 0 \), \( B(p, q, r) \) be the set of all integers of the form

\[ \sum_{i=1}^{q} k_i (2p - 1)^{i-1} \]

where \( 0 \leq k_i < p \) for \( i = 1, 2, \ldots, q \) and \( \sum_{i=1}^{q} T_p(k_i) = r \) with

\[ T_p(k) = \frac{(k - [(p + 1)/2]) \cdot (k - [(p - 1)/2])}{2} \]

Lemma 1. The set \( B(p, q, r) \) is nonaveraging. Moreover, for \( s \in B(p, q, r) \) we have

\[ 0 \leq s < \frac{1}{2}(2p - 1)^q \]

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The proof is similar to that of Behrend [1] whose $k^2$ has been replaced here by $T_p(k_i)$.

Let $Z$, $T$ be two finite sets of nonnegative integers, and let $z$, $t$ be the greatest elements of $Z$, $T$ respectively. Define $(Z, T)$ by the formula

$$(Z, T) = Z \cup T + m + z + 1 \cup T + 3m + 2t + z + 3 \cup T + 3m + 4t + z + 4,$$

where $m = \max(z, t)$ and $T + x = \{a + x : a \in T\}$.

**Lemma 2.** If $Z$, $T$ are nonaveraging, so is $(Z, T)$.

The proof is by straightforward verification.

Now we give the construction of our set. Put

$$Z_0 = \{n \in S : n \leq 21523361\},$$
$$Z_1 = (Z_0, B(4, 9, 5)), \quad Z_2 = (Z_1, B(4, 10, 5))$$

and let for $n \geq 3$

$$Z_n = (Z_{n-1}, B(6, n + 6, r_n)), \quad \text{where } r_n = \begin{cases} \left\lfloor \frac{4(n + 6)}{3} \right\rfloor & \text{for } n \neq 5 \\ 15 & \text{for } n = 5. \end{cases}$$

By definition $Z_0 \subset Z_1 \subset Z_2 \subset \cdots$ and Lemmas 1 and 2 imply that the set $Z = \bigcup_{n=0}^{\infty} Z_n$ is nonaveraging.

Computation performed on the computer ODRA 1305 of the Wrocław University shows that $\sum_{a \in Z} 1/a > 3.00849$. Thus we have established

**Theorem.** There exists a nonaveraging set of integers with the sum of reciprocals greater than 3.00849.

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