Analysis of Some Finite Elements for the Stokes Problem

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Abstract. We study some finite elements which are used in the approximation of the Stokes problem, so as to obtain error estimates of optimal order.


I. Introduction. Let Ω be a bounded polyhedral domain in \( \mathbb{R}^d \), \( d = 2 \) or \( 3 \). We consider the standard variational formulation of the stationary Stokes equations: for \( f \) given in \( H^{-1}(\Omega)^d \), find \( (u, p) \) in \( H^1_0(\Omega)^d \times L^2_0(\Omega) \) such that

\[
\begin{align*}
\forall v & \in H^1_0(\Omega)^d, \quad v(\nabla u, \nabla v) - (p, \text{div} \, v) = (f, v), \\
\forall q & \in L^2_0(\Omega), \quad (q, \text{div} \, u) = 0,
\end{align*}
\]

where we denote by \((\cdot, \cdot)\) the inner product of \( L^2(\Omega) \) (or \( L^2(\Omega)^d \) or \( L^2(\Omega)^d \)). Hereafter \( L^2_0(\Omega) \) is the space \( \{ q \in L^2(\Omega); \int_{\Omega} q \, dx = 0 \} \). Now let \( h \) be a real positive parameter tending to zero. We introduce two finite-dimensional subspaces \( X_h \) and \( M_h \) of \( H^1_0(\Omega)^d \) and \( L^2_0(\Omega) \) respectively, satisfying the usual condition: for any \( q_h \in M_h \), \( q_h \neq 0 \), there exists \( v_h \in X_h \) such that \((q_h, \text{div} \, v_h) \neq 0\). We consider the discretized problem: find \( (u_h, p_h) \) in \( X_h \times M_h \) such that

\[
\begin{align*}
\forall v_h & \in X_h, \quad v(\nabla u_h, \nabla v_h) - (p_h, \text{div} \, v_h) = (f, v_h), \\
\forall q_h & \in L^2_0(\Omega), \quad (q_h, \text{div} \, u_h) = 0.
\end{align*}
\]

We recall that problem (1.1) (respectively problem (1.2)) has a unique solution \( (u, p) \) in \( H^1(\Omega)^d \times L^2(\Omega) \) (respectively \( (u_h, p_h) \) in \( X_h \times M_h \)). Moreover, when \( (u, p) \) belongs to the space \( H^{m+1}(\Omega)^d \times H^m(\Omega) \), it is well-known (see [7]) that the error estimate

\[
\|u - u_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq C h^m \left( \|u\|_{m+1,\Omega} + \|p\|_{m,\Omega} \right)
\]

holds whenever the following additional hypotheses are satisfied:

(H1) for any \( q \) in \( H^m(\Omega) \cap L^2_0(\Omega) \), one has

\[
\inf_{q_h \in M_h} \|q - q_h\|_{0,\Omega} \leq C h^m \|q\|_{m,\Omega};
\]

(H2) there exists a linear operator \( \Pi_h \) from \( H^{m+1}(\Omega)^d \cap H^1_0(\Omega)^d \) into \( X_h \) such that

\[
\forall v \in H^{m+1}(\Omega)^d \cap H^1_0(\Omega)^d, \quad \begin{cases} \forall q_h \in M_h, \quad (q_h, \text{div} \, (v - \Pi_h v)) = 0, \\
\|v - \Pi_h v\|_{1,\Omega} \leq C h^m \|v\|_{m+1,\Omega}; \end{cases}
\]

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(H3) for each \( q_h \) in \( M_h \), there exists a function \( v_h \) in \( X_h \) such that
\[
(\text{div} \, v_h, q_h) \geq \beta \|q_h\|_{0,\Omega} \|v_h\|_{1,\Omega},
\]
where \( \beta > 0 \) is a constant independent of \( h \).

Our aim is to give some examples of finite-element spaces such that hypotheses (H1), (H2) and (H3) are satisfied. To this end, we introduce a family \((\mathcal{T}_h)_h\) of triangulations of \( \overline{\Omega} \), where \( \mathcal{T}_h \) is made of \( d \)-simplices with diameters bounded by \( h \).

For any integer \( k \), \( P_k(K) \) denotes the space of polynomials of degree \( \leq k \) on \( K \). We set
\[
M_h^{(m)} = \left\{ q_h \in L^2_0(\Omega); \forall K \in \mathcal{T}_h, q_h/K \in P_{m-1}(K) \right\}.
\]
Then hypothesis (H1) is satisfied (see [2] for instance). Finally, we set
\[
X_h = \left\{ v_h \in \mathcal{C}^0(\overline{\Omega})^d \cap H^1_0(\Omega)^d; \forall K \in \mathcal{T}_h, v_h/K \in P_K \right\};
\]
hereafter we study some examples of spaces \( P_K \) introduced by Fortin [6] such that hypotheses (H2) and (H3) are satisfied.

More precisely, we give in Section II an example of a simplicial element of order \( m = 1 \) and, in Section III, an example of a three-dimensional tetrahedral element of order \( m = 2 \).

From now on we denote by \( \| \cdot \|_{m,\Omega} \) and \( | \cdot |_{m,\Omega} \) the usual norm and seminorm on the Sobolev space \( H^m(\Omega) \).

II. A Simplicial Element of Order 1 (\( d = 2 \) or 3). Let us consider a \( d \)-simplex \( K \) with vertices \( a_1, \ldots, a_d+1 \). For \( 1 \leq i \leq d+1 \), we denote by \( \lambda_i \), the barycentric coordinate associated with \( a_i \), by \( F_i \) the face which does not contain \( a_i \), and by \( n_i \), the unit outward normal to \( F_i \), and we set
\[
p_i = n_i \prod_{j=1, j \neq i}^{d+1} \lambda_j.
\]
Then, we consider
\[
(\text{II.1}) \quad P_K = P_1(K)^d \oplus \text{Span}\{p_i, 1 \leq i \leq d+1\}.
\]
(\( \text{Note that dim } P_K = (d+1)^2 \).) As far as the degrees of freedom are concerned, we can choose the values at the vertices \( a_i, 1 \leq i \leq d+1 \), and the flux through the faces \( F_i, 1 \leq i \leq d+1 \).

**Lemma II.1.** For any \( v \) in \( \mathcal{C}^0(K)^d \), there exists a unique \( \Pi_K v \) in \( P_K \) such that
\[
(\text{II.2}) \quad \begin{cases} 
\Pi_K v(a_i) = v(a_i), \\
\int_{F_i} (v - \Pi_K v) \cdot n_i \, d\sigma = 0, \quad 1 \leq i \leq d+1.
\end{cases}
\]
Moreover, \( \Pi_K v/F_i \) depends only on \( v/F_i, 1 \leq i \leq d+1 \).

**Proof.** Let us denote by \( \tilde{\Pi}_K v \) the classical Lagrange interpolate of \( v \) in \( P_1(K)^d \), i.e.,
\[
\tilde{\Pi}_K v = \sum_{i=1}^{d+1} v(a_i) \lambda_i.
\]
Then, as the $p_i$'s are equal to 0 at any vertex, one has

$$\Pi_K \nu = \Pi_K \nu + \sum_{i=1}^{d+1} \alpha_i p_i, \tag{II.3}$$

with $\alpha_j = \left( \int_{F_i} (\nu - \Pi_K \nu) \cdot n_i d\sigma \right) / \int_{F_i} \prod_{j=1, j \neq i}^{d+1} \lambda_j d\sigma$.

Moreover, on $F_i$,

$$\Pi_K \nu_{/F_i} = \sum_{j=1, j \neq i}^{d+1} \nu(a_j) \lambda_j + \alpha_i p_i,$$

so that $\Pi_K \nu_{/F_i}$ depends only on $\nu(a_j), j \neq i$, and on $\int_{F_i} \nu \cdot n_i d\sigma$.

Now, for each $h$, we consider a triangulation $\mathcal{T}_h$ of $\Omega$ made of $d$-simplices with diameters bounded by $h$ and we assume that the family $(\mathcal{T}_h)_h$ is regular, i.e., (see [2]) there exists a constant $\sigma$ such that

$$\forall h, \forall K \in \mathcal{T}_h, \quad h_K \leq \sigma \rho_K, \tag{II.4}$$

where $h_K$ is the diameter of $K$, and $\rho_K$ the diameter of the sphere inscribed in $K$.

With each $K$ in $\mathcal{T}_h$, we associate the space $P_K$ defined by (II.1); then Lemma II.1 allows us to define an operator $\Pi_h$ from $\mathcal{C}^0(\Omega)^d \cap H_0^1(\Omega)^d$ into $X_h$ by

$$\forall K \in \mathcal{T}_h, \quad \Pi_h \nu_{/K} = \Pi_K \nu. \tag{II.5}$$

**Lemma II.2.** The operator $\Pi_h$ satisfies (H2) for $m = 1$.

**Proof.** Clearly, one has

$$\int_K \text{div}(\nu - \Pi_K \nu) dx = \sum_{i=1}^{d+1} \int_{F_i} (\nu - \Pi_K \nu) \cdot n_i d\sigma = 0,$$

so that $\forall q_h \in M_h^{(1)}, (q_h, \text{div}(\nu - \Pi_h \nu)) = 0$.

Moreover, we know that (see [2], for instance), for $k = 0$ and 1,

$$|\nu - \Pi_K \nu|_{1,k,K} \leq C h^{2-k} |\nu|_{2,k}.$$

Let us compute $\Pi_K \nu - \Pi_K \nu = \sum_{i=1}^{d+1} \alpha_i p_i$. We consider an affine invertible mapping $F_K: \hat{x} \mapsto x = B_K \hat{x} + b_K$ which maps the $d$-simplex $\hat{K} = \{ \hat{x} \in \mathbb{R}^d; \forall i, 1 \leq i \leq d, \hat{x}_i \geq 0 \text{ and } \sum_{i=1}^d \hat{x}_i \leq 1 \}$ onto $K$, and use the notations $x = F_K(\hat{x}), \nu = \hat{\nu} \circ F_K\^{-1}$.

Clearly, one has

$$|p_i|_{k,k,K}^2 = \int_K \left\| D^k \left( \prod_{j=1, j \neq i}^{d+1} \lambda_j \right) \right\|^2 ||\nu||_{1,k} = C \int_K \left\| D^k \left( \prod_{j=1, j \neq i}^{d+1} \lambda_j \right) \right\|^2 \|B_K\|^{2k} \leq C |\det B_K| \|B_K\|^{-2k},$$

so that, by the regularity of the family $(\mathcal{T}_h)_h$,

$$|p_i|_{k,k,K} \leq C h_K^{d/2-k}. \tag{II.6}$$

But, since

$$\int_{F_i} \prod_{j=1, j \neq i}^{d+1} \lambda_j d\sigma = |\det B_{K/F_i}| \int_{\hat{F}_i} \prod_{j=1, j \neq i}^{d+1} \lambda_j d\hat{\sigma},$$
we obtain by (II.3)
\[ |\alpha_1| \leq C|\text{det } B_{K/F_k}|^{-1} \int_{F_k} |v - \Pi_K v| d\sigma \leq C \int_{F_k} |v - \Pi_K v| d\delta; \]
therefore, as \( P_1(\hat{K})^d \) is invariant under \( \hat{\Pi}_K \),
\[ |\alpha_1| \leq C|v|_{2,K} \leq C|\text{det } B_K|^{-1/2} \| B_K \|^{\frac{d}{2}} \| v \|_{2,K} \leq C h_{K}^{d/2} \| v \|_{2,K}. \]
The previous inequalities yield, for \( k = 0 \) and \( 1 \),
\[ |v - \Pi_K v|_{k,K} \leq C h_{K}^{d-k} \| v \|_{2,K}, \]
so that
\[ \| v - \Pi_K v \|_{1,\Omega} \leq C h|v|_{2,\Omega}. \]

We recall the proof of the following inequality only for the reader’s convenience.

**Lemma II.3.** For any \( v \) in \( H^1(K) \), we have
\[ (II.7) \quad \| v \|_{0,F_i} \leq C|\text{mes } F_i|^{1/2} h_{K}^{d/2} \{ \| v \|_{0,K} + h_K \| v \|_{1,K} \}. \]

**Proof.** As the trace mapping is continuous from \( H^1(\hat{K}) \) into \( L^2(\hat{F}_i) \),
\[ \| v \|_{0,F_i}^2 = |\text{det } B_{K/F_i}| \int_{F_i} \hat{v}^2 d\delta \leq C|\text{det } B_{K/F_i}| \{ \| \hat{v} \|_{0,K} + |\hat{v}|_{1,K}^2 \} \]
\[ \leq C|\text{mes } F_i|^{1/2} h_{K}^{d/2} \{ \| v \|_{0,K}^2 + h_{K}^{2} \| v \|_{1,K}^2 \}. \]

Let us now study the hypothesis (H3). We know (see [7, Chapter I, Lemma 3.2]) that, for each \( q_h \) in \( M_h^{(1)} \), there exists \( v \) in \( H_0^1(\Omega)^d \) such that
\[ (II.8) \quad \text{div } v = q_h \quad \text{and} \quad \| v \|_{1,\Omega} \leq C\| q_h \|_{0,\Omega}. \]
Hence, the hypothesis (H3) is an immediate consequence of the following

**Lemma II.4.** For any \( v \) in \( H_0^1(\Omega)^d \), there exists \( v_h \) in \( X_h \) such that
\[ (II.9) \quad \forall q_h \in M_h^{(1)}, \quad \begin{cases} (q_h, \text{div}(v - v_h)) = 0 \\ \text{and } \| v_h \|_{1,\Omega} \leq C\| v \|_{1,\Omega}. \end{cases} \]

**Proof.** Let us denote by \( w_h \) the interpolate of \( v \) in the space
\[ \{ u_h \in C^0(\overline{\Omega}) \cap H_0^1(\Omega); \forall K \in s_h, u_{h/K} \in P_1(K) \}^d, \]
defined by local regularization as in [4] (see [1] for an explicit generalization to the case \( d = 3 \)). By the regularity of the family \((s_h)_h\), we know that the following local interpolation error holds
\[ (II.10) \quad \| v - w_h \|_{0,K} + h_K \| v_h \|_{1,K} \leq C h_K \| v \|_{1,\Delta_K}, \]
where \( \Delta_K \) is the union of all \( K' \) in \( s_h \) such that \( K \cap K' \neq \emptyset \); moreover, each element of \( s_h \) is contained in at most \( M \) subsets \( \Delta_K \), where \( M \) is an integer independent of \( h \).

Then, we consider the element \( v_h \) in \( V_h \) defined by
\[ \begin{cases} v_h(a_i) = w_h(a_i), \\ \int_{F_i} (v - v_h) \cdot n_i d\sigma = 0, \quad 1 \leq i \leq d + 1, \end{cases} \]
or, in other words, equal on $K$ to

$$v_{h/K} = w_h + \sum_{i=1}^{d+1} \alpha_i p_i,$$

with

$$\alpha_i = \left( \int_{F_i} (v - w_h) \cdot n_i \, d\sigma \right) / \int_{F_i} \prod_{j=1, j \neq i}^{d+1} \lambda_j \, d\sigma.$$

Clearly, one has $\forall q_h \in M_h^{(1)}$, $(q_h, \text{div}(v - w_h)) = 0$. Moreover, by (II.6),

$$\|v_h\|_{1,K} \leq \|w_h\|_{1,K} + \sum_{i=1}^{d+1} |\alpha_i| \|p_i\|_{1,K} \leq \|w_h\|_{1,K} + Ch_K^{d/2-1} \sum_{i=1}^{d+1} |\alpha_i|.$$

But, we also have

$$|\alpha_i| \leq C|\det B_{K/F_i}|^{-1} \int_{F_i} (v - w_h) \cdot n_i \, d\sigma \leq C|\text{mes } F_i|^{-1/2} \|v - w_h\|_{0,F_i}.$$

Lemma II.3 implies

$$|\alpha_i| \leq C h_K^{d/2} \left\{ \|v - w_h\|_{0,K} + h_K \|v - w_h\|_{1,K} \right\}.$$

Finally, we obtain

$$\|v_h\|_{1,K} \leq \|w_h\|_{1,K} + h_K^{-1} \left\{ \|v - w_h\|_{0,K} + h_K \|v - w_h\|_{1,K} \right\},$$

which, together with (II.10), yields $\|v_h\|_{1,\Omega} \leq C \|v\|_{1,\Omega}$.

As assumptions (H1) to (H3) are satisfied with $m = 1$, this element can be used to solve the Stokes problem with an $O(h)$-error estimate.

Remark II.1. In the two-dimensional case, we can also consider a triangulation $\mathcal{T}_h$ of $\Omega$ made of triangles and convex quadrilaterals. Then, if $K$ is a triangle, the space $P_K$ is defined by (II.1). If $K$ is a convex quadrilateral with vertices $a_1, \ldots$ and $a_4$, there exists an invertible mapping $F_K$ in $\hat{Q}_2$ which maps the unit square $\hat{K} = [0, 1]^2$ onto $K$ ($\hat{Q}_2$ is the space of polynomials spanned by $x_1, x_2, x_3 = 1 - x_1$ and $x_4 = 1 - x_2$); for $1 \leq i \leq 4$, we denote by $F_i$ the edge with vertices $a_{i-1}$ and $a_i$ (of course, $a_0 = a_4$) and by $n_i$ the unit outward normal to $F_i$, and we set

$$P_i = n_i(\hat{q}_i \circ F_K^{-1}), \quad \hat{q}_i = \prod_{j=1, j \neq i}^{4} \hat{x}_j.$$

Then, we consider

$$P_K = Q_1(K)^2 \oplus \text{Span}\{p_i, 1 \leq i \leq 4\},$$

where $Q_1(K) = \{ p \circ F_K^{-1}, p \in \hat{Q}_1 \}$. (Note that $\dim P_K = 12$.) The degrees of freedom can be chosen as previously. If the family $(\mathcal{T}_h)_h$ is regular (see [3] for instance), the previous results are still valid.

III. A Tetrahedral Element of Order 2 ($d = 3$). Let us consider a tetrahedron $K$ with vertices $a_1, \ldots$ and $a_4$. We use the same notations as in Section II, in particular, we set

$$P_i = n_i(\hat{q}_i \circ F_K^{-1}), \quad \hat{q}_i = \prod_{j=1, j \neq i}^{4} \hat{x}_j.$$
(Note that \( \dim P_K = 37 \).) Let us remark that this space generalizes in the three-dimensional case the space studied in [5] for \( d = 2 \). As far as the degrees of freedom are concerned, we choose the values at the vertices \( a_i, 1 \leq i \leq 4 \), and at the midpoints \( a_{ij}, 1 \leq i < j \leq 4 \), the flux through the faces \( F_i, 1 \leq i \leq 4 \), and the moments \( \int_K x_l \text{div}(\cdot) \, dx, 1 \leq l \leq 3 \).

**Lemma III.1.** For any \( v \) in \( \mathcal{G}^0(K)^3 \cap H^1(K)^3 \), there exists a unique \( \Pi_K v \) in \( P_K \) such that

\[
\begin{align*}
\Pi_K v(a_i) &= v(a_i), & 1 \leq i \leq 4, \\
\Pi_K v(a_{ij}) &= v(a_{ij}), & 1 \leq i < j \leq 4, \\
\int_{F_i} (v - \Pi_K v) \cdot \mathbf{n}_i \, d\sigma &= 0, & 1 \leq i \leq 4, \\
\int_K x_l \text{div}(v - \Pi_K v) \, dx &= 0, & 1 \leq l \leq 3.
\end{align*}
\]

Moreover, \( \Pi_K v/F_i \) depends only on \( v/F_i, i \leq i \leq 4 \).

**Proof.** Let us denote by \( \bar{\Pi}_K v \) the classical Lagrange interpolate of \( v \) in \( P_2(K)^3 \), i.e.,

\[
\bar{\Pi}_K v = \sum_{i=1}^4 v(a_i) \lambda_i (2\lambda_i - 1) + \sum_{1 \leq i < j \leq 4} v(a_{ij}) 4\lambda_i \lambda_j.
\]

Then, as the \( \mathbf{p}_i \)'s and \( \lambda_1 \lambda_2 \lambda_3 \lambda_4 \) are equal to 0 on any edge, \( \Pi_K v \) can be written

\[
\Pi_K v = \bar{\Pi}_K v + \sum_{i=1}^4 \alpha_i \mathbf{p}_i + \beta \lambda_1 \lambda_2 \lambda_3 \lambda_4.
\]

Since \( \lambda_1 \lambda_2 \lambda_3 \lambda_4 \) is equal to 0 on \( \partial K \), we have

\[
\alpha_i = \left( \int_{F_i} (v - \bar{\Pi}_K v) \cdot \mathbf{n}_i \, d\sigma \right) / \int_{F_i} \prod_{j=1, j \neq i}^4 \lambda_j \, d\sigma, \quad 1 \leq i \leq 4.
\]

Then, setting

\[
\bar{\Pi}_K v = \Pi_K v + \sum_{i=1}^4 \alpha_i \mathbf{p}_i,
\]

and using the Green's formula, we obtain

\[
\beta_l = -\left( \int_K x_l \text{div}(v - \bar{\Pi}_K v) \, dx \right) / \int_K \lambda_1 \lambda_2 \lambda_3 \lambda_4 \, dx, \quad 1 \leq l \leq 3.
\]

Moreover, on \( F_i \), one has

\[
\Pi_K v/F_i = \bar{\Pi}_K v/F_i + \alpha_i \mathbf{p}_i,
\]

so that \( \Pi_K v/F_i \) depends only on \( v/F_i \).

Now, for each \( h \), we consider a triangulation \( \mathcal{T}_h \) of \( \bar{\Omega} \) made of tetrahedra with diameters bounded by \( h \) and we assume that the family \( \{ \mathcal{T}_h \} \) is regular.

With each \( K \) in \( \mathcal{T}_h \), we associate the space \( P_K \) defined by (III.1); then Lemma III.1 allows us to define an operator \( \Pi_h \) from \( \mathcal{G}^0(\bar{\Omega})^3 \cap H^1_0(\Omega)^3 \) into \( X_h \) by (II.5).

**Lemma III.2.** The operator \( \Pi_h \) satisfies (H2) for \( m = 2 \).
Proof. Clearly, one has
\[
\int_K \text{div}(v - \Pi_K v) \, dx = \int_K x_i \text{div}(v - \Pi_K v) \, dx = 0, \quad 1 \leq l \leq 3,
\]
so that \( \forall q_h \in M_h^{(2)}, (q_h, \text{div}(v - \Pi_K v)) = 0 \).

Moreover, we know that (see [2]), for \( k = 0 \) and \( 1 \),
\[
|v - \hat{\Pi}_K v|_{k,K} \leq Ch_k^{-k}|v|_{3,K}.
\]

Let us compute \( \Pi_K v - \hat{\Pi}_K v = \sum_{i=1}^4 \alpha_i p_i \). As in Section II,
\[
|\alpha_i| \leq C \int_{\hat{F}_i} |\hat{v} - \hat{\Pi}_K \hat{v}| d\hat{\sigma};
\]
therefore, as \( P_2(\hat{K})^3 \) is invariant under \( \hat{\Pi}_K \),
\[
|\alpha_i| \leq C|\hat{v}|_{3,\hat{K}} \leq Ch_k^{3/2}|v|_{3,K}.
\]

The previous inequalities, together with (II.6), yield
\[
|v - \Pi_K v|_{k,K} \leq Ch_k^{2-k} |v|_{3,K}.
\]

Finally, we compute \( \Pi_K v - \hat{\Pi}_K v = \beta \lambda_1 \lambda_2 \lambda_3 \lambda_4 \). Clearly, one has
\[
(III.7) \quad |\lambda_1 \lambda_2 \lambda_3 \lambda_4|_{k,K} \leq C \left( \int_K \|D^k(\lambda_1 \lambda_2 \lambda_3 \lambda_4)\|^2 \|B_K\|^{2k} |\det B_K| \, d\hat{x} \right)^{1/2}
\]
\[
\leq Ch_k^{2/2-k},
\]
and, by (III.6),
\[
|\beta| \leq C|\det B_K|^{-1} \left\| \int_K x_i \text{div}(v - \Pi_K v) \, dx \right\|.
\]

We use Green's formula
\[
|\beta| \leq C|\det B_K|^{-1} \left\{ \left| \int_K (v - \Pi_K v) \, dx \right| + \left| \int_{\partial K} x_i (v - \Pi_K v) \cdot n \, d\sigma \right| \right\}
\]
\[
\leq C \left( |\det B_K|^{1/2} \|v - \Pi_K v\|_{0,K} + |\det B_K|^{-1} \left| \int_{\partial K} x_i (v - \Pi_K v) \cdot n \, d\sigma \right| \right).
\]

But we remark that, since \( x = B_K \hat{x} + b_K \),
\[
\int_{\partial K} x_i (v - \Pi_K v) \cdot n \, d\sigma = \int_{\partial K} (B_K \hat{x})_i (v - \Pi_K v) \cdot \hat{n} |\det B_K/\partial K| \, d\hat{\sigma}
\]
\[
+ b_{K i} \int_{\partial K} (v - \Pi_K v) \cdot \hat{n} |\det B_K/\partial K| \, d\hat{\sigma}.
\]

Therefore,
\[
\left| \int_{\partial K} x_i (v - \Pi_K v) \cdot n \, d\sigma \right| \leq \|B_K\| \int_{\partial K} |v - \Pi_K v| \, d\sigma
\]
\[
+ |b_K| \left| \int_{\partial K} (v - \Pi_K v) \cdot n \, d\sigma \right|.
\]
Since the last term is equal to 0, we obtain
\[
|\beta| \leq C \left( |\det B_K|^{-1/2} \|v - \Pi_K v\|_{0,K} + |\det B_K|^{-1/2} \|B_K\| \sum_{i=1}^{4} \|\text{mes } F_i\|^{1/2} \|v - \Pi_K v\|_{0,F_i} \right)
\]
so that, by Lemma II.3,
\[
|\beta| \leq \left\{ h_K^{-3/2} h^3 + h_K^{-3} h^3 h_K^{1/2} \right\} |v|_{3,k} \leq Ch_3^{1/2} |v|_{3,k}.
\]
The previous inequalities yield, for \( k = 0 \) and \( 1 \),
\[
|v - \Pi_K v|_{k,k} \leq Ch_3^{3-k} |v|_{3,k}.
\]
By (II.8), the hypothesis (H3) is an immediate consequence of

**Lemma III.3.** For any \( v \) in \( H^1_0(\Omega)^3 \), there exists \( v_h \) in \( X_h \) such that

\[
(\forall q_h \in M_h^{(2)}, \quad \begin{cases} (q_h, \text{div}(v - v_h)) = 0 \\ \text{and } \|v_h\|_{1,\Omega} \leq C \|v\|_{1,\Omega} \end{cases})
\]

**Proof.** Let us denote by \( w_h \) the interpolate of \( v \) in the space
\[
\left\{ u_h \in C^0(\overline{\Omega}) \cap H^1_0(\Omega); \forall K \in \mathcal{T}_h, u_{h,K} \in P_2(K) \right\}^3,
\]
defined by local regularization as in [1], so that (II.10) is still satisfied.
Then, we consider the element \( v_h \) in \( V_h \) equal on \( K \) to
\[
v_h = w_h + \sum_{i=1}^{4} \alpha_i p_i + \beta \lambda_1 \lambda_2 \lambda_3 \lambda_4
\]
with
\[
\alpha_i = \left( \int_{F_i} (v - w_h) \cdot n \, d\sigma \right) / \int_{F_i} \prod_{j=1, j \neq i}^{4} \lambda_j \, d\sigma,
\]
\[
\beta_i = -\int_{K} x_i \text{div} \left( v - w_h - \sum_{i=1}^{4} \alpha_i p_i \right) \, dx / \int_{K} \lambda_1 \lambda_2 \lambda_3 \lambda_4 \, dx.
\]
Clearly, one has \( \forall q_h \in M_h^{(2)}, (q_h, \text{div}(v - v_h)) = 0 \). Moreover, by (II.6) and (III.7),
\[
\|v_h\|_{1,k} \leq \|w_h\|_{1,k} + Ch_3^{1/2} \left( \sum_{i=1}^{4} |\alpha_i| + |\beta| \right).
\]
The \( \alpha_i \)'s still satisfy (II.11). We also have
\[
|\beta| \leq C |\det B_K|^{-1} \left( \left| \int_{K} \left( v - w_h - \sum_{i=1}^{4} \alpha_i p_i \right) \, dx \right| + \left| \int_{\partial K} x_i \left( v - w_h - \sum_{i=1}^{4} \alpha_i p_i \right) \cdot n \, d\sigma \right| \right).
\]
By the same way as in the proof of Lemma III.2,

\[ |\beta| \leq C \left[ |\det B_K|^{-1/2} \left\| v - w_h - \sum_{i=1}^{4} \alpha_i p_i \right\|_{0,K} + |\det B_K|^{-1} \left\| B_K \right\| \right. \]

\[ \times \sum_{i=1}^{4} |\text{mes } E_i| h^{-3/2} \left( \left\| v - w_h - \sum_{i=1}^{4} \alpha_i p_i \right\|_{0,K} + h_K \left\| v - w_h - \sum_{i=1}^{4} \alpha_i p_i \right\|_{1,K} \right) \]

\[ \leq C \left( h^{-3/2} \left\| v - w_h \right\|_{0,K} + h^{-1/2} \left\| v - w_h \right\|_{1,K} + \sum_{i=1}^{4} |\alpha_i| \right). \]

Finally, we obtain

\[ \left\| v_h \right\|_{1,K} \leq \left\| w_h \right\|_{1,K} + Ch^{-1} \left\{ \left\| v - w_h \right\|_{0,K} + h_K \left\| v - w_h \right\|_{1,K} \right\}, \]

which, together with (II.10), yields \( \left\| v_h \right\|_{1,\Omega} \leq C \left\| v \right\|_{1,\Omega}. \)

Consequently, this element can be used to solve the Stokes problem in the three-dimensional case with an \( O(h^2) \)-error estimate.

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