Analysis of Some Finite Elements for the Stokes Problem

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Abstract. We study some finite elements which are used in the approximation of the Stokes problem, so as to obtain error estimates of optimal order.


I. Introduction. Let \( \Omega \) be a bounded polyhedral domain in \( \mathbb{R}^d \), \( d = 2 \) or \( 3 \). We consider the standard variational formulation of the stationary Stokes equations: for \( f \) given in \( H^{-1}(\Omega)^d \), find \( (u, p) \) in \( H^1_0(\Omega)^d \times L^2_0(\Omega) \) such that

\[
\begin{align*}
\forall v & \in H^1_0(\Omega)^d, \quad v(\text{grad} u, \text{grad} v) - (p, \text{div} v) = (f, v), \\
\forall q & \in L^2_0(\Omega), \quad (q, \text{div} u) = 0,
\end{align*}
\]

where we denote by \((\cdot, \cdot)\) the inner product of \( L^2(\Omega) \) (or \( L^2(\Omega)^d \) or \( L^2(\Omega)^d \)). Hereafter \( L^2_0(\Omega) \) is the space \( \{q \in L^2(\Omega); \int_\Omega q \, dx = 0\} \). Now let \( h \) be a real positive parameter tending to zero. We introduce two finite-dimensional subspaces \( X_h \) and \( M_h \) of \( H^1_0(\Omega)^d \) and \( L^2_0(\Omega) \) respectively, satisfying the usual condition: for any \( q_h \) in \( M_h \), \( q_h \neq 0 \), there exists \( v_h \) in \( X_h \) such that \( (q_h, \text{div} v_h) \neq 0 \). We consider the discretized problem: find \( (u_h, p_h) \) in \( X_h \times M_h \) such that

\[
\begin{align*}
\forall v_h & \in X_h, \quad v_h(\text{grad} u_h, \text{grad} v_h) - (p_h, \text{div} v_h) = (f, v_h), \\
\forall q_h & \in M_h, \quad (q_h, \text{div} u_h) = 0.
\end{align*}
\]

We recall that problem (1.1) (respectively problem (1.2)) has a unique solution \( (u, p) \) in \( H^1_0(\Omega)^d \times L^2_0(\Omega) \) (respectively \( (u_h, p_h) \) in \( X_h \times M_h \)). Moreover, when \( (u, p) \) belongs to the space \( H^{m+1}(\Omega)^d \times H^m(\Omega) \), it is well-known (see [7]) that the error estimate

\[
\|u - u_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq C h^m(\|u\|_{m+1,\Omega} + \|p\|_{m,\Omega})
\]

holds whenever the following additional hypotheses are satisfied:

(H1) for any \( q \) in \( H^m(\Omega) \cap L^2_0(\Omega) \), one has

\[
\inf_{q_h \in M_h} \|q - q_h\|_{0,\Omega} \leq C h^m \|q\|_{m,\Omega};
\]

(H2) there exists a linear operator \( \Pi_h \) from \( H^{m+1}(\Omega)^d \cap H^1_0(\Omega)^d \) into \( X_h \) such that

\[
\forall v \in H^{m+1}(\Omega)^d \cap H^1_0(\Omega)^d, \quad \left\{ \begin{array}{l}
\forall q_h \in M_h, \quad (q_h, \text{div} (v - \Pi_h v)) = 0, \\
\|v - \Pi_h v\|_{1,\Omega} \leq C h^m \|v\|_{m+1,\Omega};
\end{array} \right.
\]

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(H3) for each $q_h$ in $M_h$, there exists a function $v_h$ in $X_h$ such that

$$ \langle \text{div } v_h, q_h \rangle \geq \beta \| q_h \|_{0, \Omega} \| v_h \|_{1, \Omega}, $$

where $\beta > 0$ is a constant independent of $h$.

Our aim is to give some examples of finite-element spaces such that hypotheses (H1), (H2) and (H3) are satisfied. To this end, we introduce a family $(T_h)_h$ of triangulations of $\Omega$, where $T_h$ is made of $d$-simplices with diameters bounded by $h$.

For any integer $k$, $P_k(K)$ denotes the space of polynomials of degree $\leq k$ on $K$. We set

$$ M^h = \{ q_h \in L^2(\Omega) ; \forall K \in T_h, q_h/K \in P_{m-1}(K) \}. $$

Then hypothesis (H1) is satisfied (see [2] for instance). Finally, we set

$$ X_h = \{ v_h \in C^0(\Omega)^d \cap H^1_0(\Omega)^d ; \forall K \in T_h, v_h/K \in P_K \}; $$

hereafter we study some examples of spaces $P_K$ introduced by Fortin [6] such that hypotheses (H2) and (H3) are satisfied.

More precisely, we give in Section II an example of a simplicial element of order $m = 1$ and, in Section III, an example of a three-dimensional tetrahedral element of order $m = 2$.

From now on we denote by $\| \cdot \|_{m, \Omega}$ and $| \cdot |_{m, \Omega}$ the usual norm and seminorm on the Sobolev space $H^m(\Omega)$.

II. A Simplicial Element of Order 1 ($d = 2$ or 3). Let us consider a $d$-simplex $K$ with vertices $a_1, \ldots$ and $a_{d+1}$. For $1 \leq i \leq d + 1$, we denote by $\lambda_i$ the barycentric coordinate associated with $a_i$, by $F_i$ the face which does not contain $a_i$, and by $n_i$ the unit outward normal to $F_i$, and we set

$$ p_i = n_i \prod_{j=1, j \neq i}^{d+1} \lambda_j. $$

Then, we consider

$$ P_K = P_1(K)^d \oplus \text{Span}(p_i, 1 \leq i \leq d + 1). $$

(Note that dim $P_K = (d + 1)^2$.) As far as the degrees of freedom are concerned, we can choose the values at the vertices $a_i$, $1 \leq i \leq d + 1$, and the flux through the faces $F_i$, $1 \leq i \leq d + 1$.

**Lemma II.1.** For any $v$ in $C^0(K)^d$, there exists a unique $\Pi_K v$ in $P_K$ such that

$$ \begin{cases}
\Pi_K v(a_i) = v(a_i), \\
\int_{F_i} (v - \Pi_K v) \cdot n_i \, d\sigma = 0,
\end{cases} \quad 1 \leq i \leq d + 1. $$

Moreover, $\Pi_K v_{|F_i}$ depends only on $v_{|F_i}$, $1 \leq i \leq d + 1$.

**Proof.** Let us denote by $\tilde{\Pi}_K v$ the classical Lagrange interpolate of $v$ in $P_1(K)^d$, i.e.,

$$ \tilde{\Pi}_K v = \sum_{i=1}^{d+1} v(a_i) \lambda_i. $$
Then, as the \( p_i \)'s are equal to 0 at any vertex, one has

\[
\Pi_K v = \tilde{\Pi}_K v + \sum_{i=1}^{d+1} \alpha_i p_i,
\]

(II.3) with \( \alpha_i = \left( \int_{F_i} (v - \tilde{\Pi}_K v) \cdot n_i \, d\sigma \right) / \int_{F_i} \prod_{j=1, j \neq i}^{d+1} \lambda_j \, d\sigma. \)

Moreover, on \( F_i \),

\[
\Pi_K v_{/F_i} = \sum_{j=1, j \neq i}^{d+1} v(a_j) \lambda_j + \alpha_i p_i,
\]

so that \( \Pi_K v_{/F_i} \) depends only on \( v(a_j), j \neq i \), and on \( \int_{F_i} v \cdot n_i \, d\sigma \).

Now, for each \( h \), we consider a triangulation \( \mathcal{T}_h \) of \( \Omega \) made of \( d \)-simplices with diameters bounded by \( h \) and we assume that the family \( (\mathcal{T}_h)_h \) is regular, i.e., (see [2]) there exists a constant \( \sigma \) such that

(II.4) \[ \forall h, \forall K \in \mathcal{T}_h, \quad h_K \leq \sigma \rho_K, \]

where \( h_K \) is the diameter of \( K \), and \( \rho_K \) the diameter of the sphere inscribed in \( K \).

With each \( K \) in \( \mathcal{T}_h \), we associate the space \( P_K \) defined by (II.1); then Lemma II.1 allows us to define an operator \( \Pi_h \) from \( C^0(\Omega)^d \cap H_0^1(\Omega)^d \) into \( X_h \) by

(II.5) \[ \forall K \in \mathcal{T}_h, \quad \Pi_h v_{/K} = \Pi_K v. \]

**Lemma II.2.** The operator \( \Pi_h \) satisfies (H2) for \( m = 1 \).

**Proof.** Clearly, one has

\[
\int_K \text{div}(v - \Pi_K v) \, dx = \sum_{i=1}^{d+1} \int_{F_i} (v - \Pi_K v) \cdot n_i \, d\sigma = 0,
\]

so that \( \forall q_h \in M_h^{(1)}, (q_h, \text{div}(v - \Pi_h v)) = 0 \).

Moreover, we know that (see [2], for instance), for \( k = 0 \) and \( 1 \),

\[
|v - \tilde{\Pi}_K v|_{k, K} \leq C h^{2-k}|v|_{2, K}.
\]

Let us compute \( \Pi_K v - \tilde{\Pi}_K v = \sum_{i=1}^{d+1} \alpha_i p_i \). We consider an affine invertible mapping \( F_K: \hat{x} \mapsto x = B_K \hat{x} + b_K \) which maps the \( d \)-simplex \( \hat{K} = \{ \hat{x} \in \mathbb{R}^d; \forall i, 1 \leq i \leq d, \hat{x}_i \geq 0 \text{ and } \sum_{i=1}^d \hat{x}_i \leq 1 \} \) onto \( K \), and use the notations \( x = F_K(\hat{x}), v = \hat{v} \circ F_K^{-1} \). Clearly, one has

\[
|p_i|_{k, K} = \int_K \left\| D^k \left( \prod_{j=1, j \neq i}^{d+1} \lambda_j \right) \right\|^2 \, dx
\]

\[
\leq C \int_K \left\| D^k \left( \prod_{j=1, j \neq i}^{d+1} \lambda_j \right) \right\|^2 \|B_K^{-1}\|_{2k} \|\det B_K\| d\hat{x} \leq C |\det B_K| \|B_K\|^{-2k}
\]

so that, by the regularity of the family \( (\mathcal{T}_h)_h \),

(II.6) \[ |p_i|_{k, K} \leq C h_K^{d/2-k}. \]

But, since

\[
\int_{F_i} \prod_{j=1, j \neq i}^{d+1} \lambda_j \, d\sigma = |\det B_{K/F_i}| \int_{F_i} \prod_{j=1, j \neq i}^{d+1} \hat{\lambda}_j \, d\hat{\sigma},
\]
we obtain by (II.3)

$$|\alpha| \leq C |\det B_{K/F}|^{-1} \int_{F_i} |v - \Pi_K v| d\sigma \leq C \int_{F_i} |\hat{v} - \hat{\Pi}_K \hat{v}| d\sigma;$$

therefore, as $P_1(\hat{K})^d$ is invariant under $\hat{\Pi}_K$,

$$|\alpha| \leq C |\hat{v}|_{2, K} \leq C |\det B_K|^{-1/2} \|B_K\|^2 \|v\|_{1, K} \leq Ch^{-d/2}_K \|v\|_{2, K}.$$ 

The previous inequalities yield, for $k = 0$ and 1,

$$|v - \Pi_K v|_{k, K} \leq Ch^{2-k}_K \|v\|_{2, K},$$

so that

$$\|v - \Pi_K v\|_{1, \Omega} \leq Ch|v|_{2, \Omega}.$$ 

We recall the proof of the following inequality only for the reader's convenience.

**Lemma II.3.** For any $v$ in $H^1(K)$, we have

(II.7) $$\|v\|_{0, F_i} \leq C |\text{mes } F_i|^{1/2} \hbar^{-d/2} \left\{ \|v\|_{0, K} + h_K \|v\|_{1, K} \right\}.$$ 

*Proof.* As the trace mapping is continuous from $H^1(K)$ into $L^2(F_i)$,

$$\|v\|_{0, F_i}^2 = |\det B_{K/F}| \int_{F_i} \hat{v}^2 d\hat{\sigma} \leq C |\det B_{K/F}| \left\{ \|\hat{v}\|_{0, K}^2 + |\hat{v}|_{1, K}^2 \right\}$$

$$\leq C |\text{mes } F_i| \hbar^{-d} \left\{ \|v\|_{0, K}^2 + h_K \|v\|_{1, K}^2 \right\}.$$ 

Let us now study the hypothesis (H3). We know (see [7, Chapter I, Lemma 3.2]) that, for each $q_h$ in $M_1^{(1)}$, there exists $v$ in $H^1_0(\Omega)^d$ such that

(II.8) $$\text{div } v = q_h \quad \text{and} \quad \|v\|_{1, \Omega} \leq C \|q_h\|_{0, \Omega}.$$ 

Hence, the hypothesis (H3) is an immediate consequence of the following

**Lemma II.4.** For any $v$ in $H^1_0(\Omega)^d$, there exists $v_h$ in $X_h$ such that

(II.9) $$\forall q_h \in M_1^{(1)}, \quad \left\{ (q_h, \text{div}(v - v_h)) = 0 \right. \quad \left. \text{and} \|v_h\|_{1, \Omega} \leq C \|v\|_{1, \Omega}. \right\}$$ 

*Proof.* Let us denote by $w_h$ the interpolate of $v$ in the space

$$\left\{ u_h \in C^0(\overline{\Omega}) \cap H^1_0(\Omega); \forall K \in \mathcal{T}_h, u_{h/K} \in P_1(K) \right\}^d,$$

defined by local regularization as in [4] (see [1] for an explicit generalization to the case $d = 3$). By the regularity of the family $(\mathcal{T}_h)_h$, we know that the following local interpolation error holds

(II.10) $$\|v - w_h\|_{0, K} + h_K \|w_h\|_{1, K} \leq Ch \|v\|_{1, \Delta_K},$$

where $\Delta_K$ is the union of all $K'$ in $\mathcal{T}_h$ such that $K \cap K' \neq \emptyset$; moreover, each element of $\mathcal{T}_h$ is contained in at most $M$ subsets $\Delta_K$, where $M$ is an integer independent of $h$.

Then, we consider the element $v_h$ in $V_h$ defined by

$$v_h(a_i) = w_h(a_i),$$

$$\int_{F_i} (v - v_h) \cdot n_i d\sigma = 0, \quad 1 \leq i \leq d + 1,$$
or, in other words, equal on $K$ to

$$\left\{ \begin{array}{l}
\nu_h/K = w_h + \sum_{i=1}^{d+1} \alpha_i p_i \\
\text{with } \alpha_i = \left( \int_{F_i} (v - w_h) \cdot n_i \, d\sigma \right) / \int_{F_i} \prod_{j=1, j \neq i}^{d+1} \lambda_j \, d\sigma.
\end{array} \right.$$  

Clearly, one has $\forall q_h \in M_h^{(1)}$, $(q_h, \text{div}(v - v_h)) = 0$. Moreover, by (II.6),

$$\|v_h\|_{1,K} \leq \|w_h\|_{1,K} + \sum_{i=1}^{d+1} |\alpha_i| \|p_i\|_{1,K} \leq \|w_h\|_{1,K} + C h_k^{d/2} \sum_{i=1}^{d+1} |\alpha_i|.$$  

But, we also have

$$|\alpha_i| \leq C |\det B_{K/F_i}|^{-1} \int_{F_i} (v - w_h) \cdot n_i \, d\sigma \leq C |\text{mes } F_i|^{-1/2} \|v - w_h\|_{0,F_i}.$$  

Lemma II.3 implies

$$(\text{II.11}) \quad |\alpha_i| \leq C h_k^{d/2} \left\{ \|v - w_h\|_{0,K} + h_k \|v - w_h\|_{1,K} \right\}.$$  

Finally, we obtain

$$\|v_h\|_{1,K} \leq \|w_h\|_{1,K} + h_k^{-1} \left\{ \|v - w_h\|_{0,K} + h_k \|v - w_h\|_{1,K} \right\},$$

which, together with (II.10), yields $\|v_h\|_{1,\Omega} \leq C \|v\|_{1,\Omega}$.

As assumptions (H1) to (H3) are satisfied with $m = 1$, this element can be used to solve the Stokes problem with an $O(h)$-error estimate.

Remark II.1. In the two-dimensional case, we can also consider a triangulation $\mathcal{T}_h$ of $\Omega$ made of triangles and convex quadrilaterals. Then, if $K$ is a triangle, the space $P_K$ is defined by (II.1). If $K$ is a convex quadrilateral with vertices $a_1, \ldots$ and $a_4$, there exists an invertible mapping $F_K$ in $\tilde{Q}_2^2$ which maps the unit square $\tilde{K} = [0, 1]^2$ onto $K$ ($\tilde{Q}_2$ is the space of polynomials spanned by $x_1, x_2, x_3 = 1 - x_1$ and $x_4 = 1 - x_2$); for $1 \leq i \leq 4$, we denote by $F_i$ the edge with vertices $a_{i-1}$ and $a_i$ (of course, $a_0 = a_4$) and by $n_i$ the unit outward normal to $F_i$, and we set

$$p_i = n_i \left( \tilde{q}_i \circ F_K^{-1} \right), \quad \tilde{q}_i = \prod_{j=1, j \neq i}^4 \tilde{x}_j.$$  

Then, we consider

$$(\text{II.12}) \quad P_K = Q_1(K)^2 \oplus \text{Span} \{p_i, 1 \leq i \leq 4\},$$

where $Q_1(K) = \{ \tilde{p} \circ F_K^{-1}, \tilde{p} \in \tilde{Q}_1 \}$ (Note that dim $P_K = 12$). The degrees of freedom can be chosen as previously. If the family $(\mathcal{T}_h)_h$ is regular (see [3] for instance), the previous results are still valid.

III. A Tetrahedral Element of Order 2 ($d = 3$). Let us consider a tetrahedron $K$ with vertices $a_1, \ldots$ and $a_4$. We use the same notations as in Section II, in particular, we set

$$p_i = n_i \left( \prod_{j=1, j \neq i}^4 \lambda_j \right), \quad 1 \leq i \leq 4;$$  

we also introduce the points $a_{ij} = \frac{1}{2} (a_i + a_j), 1 \leq i < j \leq 4$. Then, we consider

$$(\text{III.1}) \quad P_K = P_2(K)^3 \oplus \text{Span} \{p_i, 1 \leq i \leq 4\} \oplus \text{Span} \{\lambda_1 \lambda_2 \lambda_3 \lambda_4\}. $$
(Note that \( \dim P_K = 37 \).) Let us remark that this space generalizes in the three-dimensional case the space studied in [5] for \( d = 2 \). As far as the degrees of freedom are concerned, we choose the values at the vertices \( a_i, 1 \leq i \leq 4 \), and at the midpoints \( a_{ij}, 1 \leq i < j \leq 4 \), the flux through the faces \( F_i, 1 \leq i \leq 4 \), and the moments \( \int_K x_l \text{div}(\cdot) \, dx, 1 \leq l \leq 3 \).

**Lemma III.1.** For any \( v \) in \( \mathcal{V}_0(K)^3 \cap H^1(K)^3 \), there exists a unique \( \Pi_K v \) in \( P_K \) such that

\[
\begin{align*}
\Pi_K v(a_i) &= v(a_i), & 1 \leq i \leq 4, \\
\Pi_K v(a_{ij}) &= v(a_{ij}), & 1 \leq i < j \leq 4, \\
\int_{F_i} (v - \Pi_K v) \cdot n_i \, d\sigma &= 0, & 1 \leq i \leq 4, \\
\int_K x_l \text{div}(v - \Pi_K v) \, dx &= 0, & 1 \leq l \leq 3.
\end{align*}
\]

Moreover, \( \Pi_K v/F_i \) depends only on \( v/F_i, 1 \leq i \leq 4 \).

**Proof.** Let us denote by \( \tilde{\Pi}_K v \) the classical Lagrange interpolate of \( v \) in \( P_2(K)^3 \), i.e.,

\[
\tilde{\Pi}_K v = \sum_{i=1}^{4} v(a_i) \lambda_i (2\lambda_i - 1) + \sum_{1 \leq i < j \leq 4} v(a_{ij}) 4 \lambda_i \lambda_j.
\]

Then, as the \( p_i \)'s and \( \lambda_1 \lambda_2 \lambda_3 \lambda_4 \) are equal to 0 on any edge, \( \Pi_K v \) can be written

\[
\Pi_K v = \tilde{\Pi}_K v + \sum_{i=1}^{4} \alpha_i p_i + \beta \lambda_1 \lambda_2 \lambda_3 \lambda_4.
\]

Since \( \lambda_1 \lambda_2 \lambda_3 \lambda_4 \) is equal to 0 on \( \partial K \), we have

\[
\alpha_i = \left( \int_{F_i} (v - \tilde{\Pi}_K v) \cdot n_i \, d\sigma \right) / \int_{F_i} \prod_{j=1, j \neq i}^{4} \lambda_j \, d\sigma, & 1 \leq i \leq 4.
\]

Then, setting

\[
\Pi_K v = \tilde{\Pi}_K v + \sum_{i=1}^{4} \alpha_i p_i,
\]

and using the Green's formula, we obtain

\[
\beta_i = -\left( \int_K x_l \text{div}(v - \tilde{\Pi}_K v) \, dx \right) / \int_K \lambda_1 \lambda_2 \lambda_3 \lambda_4 \, dx, & 1 \leq l \leq 3.
\]

Moreover, on \( F_i \), one has

\[
\Pi_K v/F_i = \tilde{\Pi}_K v/F_i + \alpha_i p_i,
\]

so that \( \Pi_K v/F_i \) depends only on \( v/F_i \).

Now, for each \( h \), we consider a triangulation \( \mathcal{T}_h \) of \( \Omega \) made of tetrahedra with diameters bounded by \( h \) and we assume that the family \( (\mathcal{T}_h)_h \) is regular.

With each \( K \) in \( \mathcal{T}_h \), we associate the space \( P_K \) defined by (III.1); then Lemma III.1 allows us to define an operator \( \Pi_h \) from \( \mathcal{V}_0(\Omega)^3 \cap H^1_0(\Omega)^3 \) into \( X_h \) by (II.5).

**Lemma III. 2.** The operator \( \Pi_h \) satisfies (H2) for \( m = 2 \).
Proof. Clearly, one has

\[ \int_K \text{div}(v - \Pi_K v) \, dx = \int_K x_i \text{div}(v - \Pi_K v) \, dx = 0, \quad 1 \leq l \leq 3, \]

so that \( \forall q_h \in M_h^{(2)}, (q_h, \text{div}(v - \Pi_K v)) = 0 \).

Moreover, we know that (see [2]), for \( k = 0 \) and \( 1 \),

\[ |v - \hat{\Pi}_K v|_{k,K} \leq Ch^{3-k} |v|_{3,K}. \]

Let us compute \( \Pi_K v - \hat{\Pi}_K v = \sum_{i=1}^4 \alpha_i p_i \). As in Section II,

\[ |\alpha_i| \leq C \int_{F_i} |\hat{\psi} - \hat{\Pi}_K \hat{\psi}| \, d\hat{\sigma}; \]

therefore, as \( P_2(\hat{K}) \) is invariant under \( \hat{\Pi}_K \),

\[ |\alpha_i| \leq C |\psi|_{3,\hat{K}} \leq Ch^{3/2} |v|_{3,K}. \]

The previous inequalities, together with (II.6), yield

\[ |v - \Pi_K v|_{k,K} \leq Ch^{3-k} |v|_{3,K}. \]

Finally, we compute \( \Pi_K v - \Pi_K v = \beta \lambda_1 \lambda_2 \lambda_3 \lambda_4 \). Clearly, one has

(III.7) \[ |\lambda_1 \lambda_2 \lambda_3 \lambda_4| \leq C \left( \int_K \|D^k(\lambda_1 \lambda_2 \lambda_3 \lambda_4)\|^2 \|B_K\|^{2k} |\det B_K| \, dx \right)^{1/2} \]

\[ \leq C h^{2/k}. \]

and, by (III.6),

\[ |\beta| \leq C |\det B_K|^{-1} \left| \int_K x_i \text{div}(v - \Pi_K v) \, dx \right|. \]

We use Green’s formula

\[ |\beta| \leq C |\det B_K|^{-1} \left| \int_K (v - \Pi_K v) \, dx \right| + \int_{\partial K} x_i (v - \Pi_K v) \cdot n \, d\sigma \]

\[ \leq C \left( |\det B_K|^{-1/2} |v - \Pi_K v|_{0,K} + |\det B_K|^{-1} \int x_i (v - \Pi_K v) \cdot n \, d\sigma \right). \]

But we remark that, since \( x = B_K \hat{x} + b_K \),

\[ \int_{\partial K} x_i (v - \Pi_K v) \cdot n \, d\sigma = \int_{\partial K} (B_K \hat{x})_i (v - \Pi_K v) \cdot n |\det B_K/\partial K| \, d\hat{\sigma} \]

\[ + b_K \int_{\partial K} (v - \Pi_K v) \cdot n |\det B_K/\partial K| \, d\hat{\sigma}. \]

Therefore,

\[ \left| \int_{\partial K} x_i (v - \Pi_K v) \cdot n \, d\sigma \right| \leq \|B_K\| \int_{\partial K} |v - \Pi_K v| \, d\sigma \]

\[ + |b_K| \left| \int_{\partial K} (v - \Pi_K v) \cdot n \, d\sigma \right|. \]
Since the last term is equal to 0, we obtain

\[ |\beta| \leq C \left( |\text{det } B_K|^{-1/2} \|v - \Pi_K v\|_{0,K} + |\text{det } B_K|^{-1/2} \|B_K\| \sum_{i=1}^4 |\text{mes } F_i|^{1/2} \|v - \Pi_K v\|_{0,F_i} \right) \]

so that, by Lemma II.3,

\[ |\beta| \leq \left\{ h^{-3/2}_K h^3_K + h^{-3}_K h^2_K \right\} |v|_{3,K} \leq Ch^{3/2}_K |v|_{3,K}. \]

The previous inequalities yield, for \( k = 0 \) and 1,

\[ |v - \Pi_K v|_{k,K} \leq Ch^{3-k}_K |v|_{3,K}. \]

By (II.8), the hypothesis (H3) is an immediate consequence of

**Lemma III.3.** For any \( v \) in \( H^1_0(\Omega)^3 \), there exists \( v_h \) in \( X_h \) such that

\[ (v_h, \text{div}(v - v_h)) = 0 \]

and \( \|v_h\|_{1,\Omega} \leq C\|v\|_{1,\Omega}. \)

**Proof.** Let us denote by \( w_h \) the interpolate of \( v \) in the space

\[ \left\{ u_h \in \mathcal{C}^0(\overline{\Omega}) \cap H^1_0(\Omega); \forall K \in \mathcal{T}_h, u_{h,K} \in P_2(K) \right\}, \]

defined by local regularization as in [1], so that (II.10) is still satisfied.

Then, we consider the element \( v_h \) in \( V_h \) equal on \( K \) to

\[ v_h = w_h + \sum_{i=1}^4 \alpha_i p_i + \beta \lambda_1 \lambda_2 \lambda_3 \lambda_4 \]

with

\[ \alpha_i = \left( \int_{F_i} (v - w_h) \cdot n_i \, d\sigma \right) / \int_{F_i} \prod_{j \neq i} \lambda_j \, d\sigma, \]

\[ \beta_i = -\int_K x_i \text{div} \left( v - w_h - \sum_{i=1}^4 \alpha_i p_i \right) \, dx / \int_K \lambda_1 \lambda_2 \lambda_3 \lambda_4 \, dx. \]

Clearly, one has \( \forall q_h \in M_h^{(2)}, (q_h, \text{div}(v - v_h)) = 0 \). Moreover, by (II.6) and (III.7),

\[ \|v_h\|_{1,K} \leq \|w_h\|_{1,K} + Ch^{1/2}_K \left( \sum_{i=1}^4 |\alpha_i| + |\beta| \right). \]

The \( \alpha_i \)'s still satisfy (II.11). We also have

\[ |\beta| \leq C|\text{det } B_K|^{-1} \left\{ \left| \int_K \left( v - w_h - \sum_{i=1}^4 \alpha_i p_i \right) \, dx \right| \right. \]

\[ + \left. \left| \int_{\partial K} x_i \left( v - w_h - \sum_{i=1}^4 \alpha_i p_i \right) \cdot n \, d\sigma \right| \right\}. \]
By the same way as in the proof of Lemma III.2,

\[ |\beta_i| \leq C \left[ |\det B_K|^{-1/2} \left\| v - w_h - \sum_{i=1}^{4} \alpha_i p_i \right\|_{0,K} + |\det B_K|^{-1} \| B_K \| \right. 
\times \left. \sum_{i=1}^{4} |\text{mes } F_i| h^{-3/2} \left( \left\| v - w_h - \sum_{i=1}^{4} \alpha_i p_i \right\|_{0,K} + h_K \| v - w_h - \sum_{i=1}^{4} \alpha_i p_i \|_{1,K} \right) \right] 
\leq C \left( h_K^{-2/3} \| v - w_h \|_{0,K} + h_K^{-1/2} \| v - w_h \|_{1,K} + \sum_{i=1}^{4} |\alpha_i| \right). \]

Finally, we obtain

\[ \| v_h \|_{1,K} \leq \| w_h \|_{1,K} + C h_K^{-1} \left( \| v - w_h \|_{0,K} + h_K \| v - w_h \|_{1,K} \right), \]

which, together with (II.10), yields \( \| v_h \|_{1,\Omega} \leq C \| v \|_{1,\Omega}. \)

Consequently, this element can be used to solve the Stokes problem in the three-dimensional case with an \( O(h^2) \)-error estimate.

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