Estimates for the Chebyshev Function $\psi(x) - \theta(x)$

By N. Costa Pereira*

Abstract. A simple approximation for the difference $\psi(x) - \theta(x)$ is established by elementary methods. This approximation is used to obtain several estimates for $\psi(x) - \theta(x)$ which are sharper than those previously given in the literature.

Let $\psi$ and $\theta$ be defined, as usual, by

$$\psi(x) = \sum_{p^a \leq x} \log p; \quad \theta(x) = \sum_{p \leq x} \log p$$

with $x > 0$, $p$ prime and $a$ a positive integer. In this paper we show that it is possible to approximate the difference $\psi - \theta$ in terms of $\psi$ in quite a simple form. As consequences we deduce some estimates for $\psi - \theta$ which improve those given in [3], [4], and [5].

**Theorem 1.** For every $x > 0$ we have

1. $\psi(x) - \theta(x) \leq \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}),$
2. $\psi(x) - \theta(x) \geq \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/7}).$

**Proof.** From the well-known identity

$$\psi(x) = \sum_{k \geq 1} \theta(x^{1/k}),$$

we deduce

$$\psi(x^{1/2}) = \sum_{k \geq 1} \theta(x^{1/2k}).$$

Substituting (4) in (3) we get

$$\psi(x) = \theta(x) + \psi(x^{1/2}) + \sum_{k \geq 1} \theta(x^{1/2k+1})$$

or

$$\psi(x) - \theta(x) = \psi(x^{1/2}) + \sum_{k \geq 1} \theta(x^{1/2k-3})$$

$$+ \sum_{k \geq 1} \theta(x^{1/6k-1}) + \sum_{k \geq 1} \theta(x^{1/6k+1}).$$

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From (3) we can still deduce
\[
\psi(x^{1/3}) = \sum_{k \geq 1} \theta(x^{1/3k}) = \sum_{k \geq 1} \theta(x^{1/6k - 1}) + \sum_{k \geq 1} \theta(x^{1/6k})
\]
and (5) is transformed into
\[
(6) \quad \psi(x) - \theta(x) = \psi(x^{1/2}) + \psi(x^{1/3}) + \sum_{k \geq 1} \theta(x^{1/6k - 1})
\]
\[- \sum_{k \geq 1} \theta(x^{1/6k}) + \sum_{k \geq 1} \theta(x^{1/6k + 1}).
\]

Since \(\theta\) is an increasing function, from (6) we get the inequalities
\[
(7) \quad \psi(x) - \theta(x) \leq \psi(x^{1/2}) + \psi(x^{1/3}) + \sum_{k \geq 1} \theta(x^{1/5k})
\]
and
\[
(8) \quad \psi(x) - \theta(x) \geq \psi(x^{1/2}) + \psi(x^{1/3}) + \sum_{k \geq 1} \theta(x^{1/7k}).
\]

Using (3) again, we obtain
\[
\psi(x^{1/5}) = \sum_{k \geq 1} \theta(x^{1/5k}); \quad \psi(x^{1/7}) = \sum_{k \geq 1} \theta(x^{1/7k}).
\]

Substituting these identities respectively in (7) and (8), we get (1) and (2), which proves the theorem.

Other estimates for \(\psi - \theta\) could be obtained with the methods used in the preceding proof. As an example, observing that
\[
\sum_{k \geq 1} \theta(x^{1/6k - 1}) + \sum_{k \geq 1} \theta(x^{1/6k + 1})
\]
\[\geq \sum_{k \geq 1} \theta(x^{1/10k - 5}) + \sum_{k \geq 1} \theta(x^{1/10k}) = \psi(x^{1/5}),
\]
we get from (6),
\[
\psi(x) - \theta(x) \geq \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}) - \psi(x^{1/6}),
\]
which is sharper than (2) for large values of \(x\). Combining this inequality with (1), we have
\[
\psi(x) - \theta(x) = \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}) - \xi \psi(x^{1/6})
\]
with \(0 \leq \xi \leq 1\).

The inequalities given by Theorem 1 are, however, sharp enough for our present purposes. In order to apply Theorem 1, we have first constructed a table of \(\psi(x) - \theta(x)\) in the range \(0 < x \leq 3 \cdot 10^6\) with an accuracy of five decimals (Table I). Since \(\psi(x) - \theta(x)\) is a step function that only jumps where \(x = p^\alpha\) with \(p\) prime and \(\alpha\) an integer greater than one, we only need to tabulate \(\psi(x) - \theta(x)\) for those values of \(x\).

We will also use two simple lemmas.
**Lemma 1.** If an interval $I$ is the union of a finite collection of intervals $I_k = (m_k, n_k)$, and there are positive constants $a_k, b_k, c_k, L^+$ and a constant $\varepsilon > 0$, such that $a_k > 1 + \varepsilon$ and

\[
\psi(x^{1/2}) < a_k x^{1/2}; \quad \psi(x^{1/3}) < b_k x^{1/3}, \\
\psi(x^{1/5}) < c_k x^{1/5} \quad \text{for } x \in I_k, \\
m_k \geq (4c_k/5(a_k - 1 - \varepsilon))^{10/3}, \\
L^+ \geq \sup_k \{(a_k - 1 - \varepsilon)n_k^{1/6} + b_k + c_k n_k^{-2/15}\},
\]

we have

\[
\psi(x) - \theta(x) < (1 + \varepsilon)x^{1/2} + L^+ x^{1/3} \quad \text{for } x \in I.
\]

**Proof.** If $x \in I_k$, from Theorem 1 and (9) it follows that

\[
\psi(x) - \theta(x) < a_k x^{1/2} + b_k x^{1/3} + c_k x^{1/5} \quad \text{if } x \in I_k,
\]

or,

\[
\psi(x) - \theta(x) < (1 + \varepsilon)x^{1/2} \\
((a_k - 1 - \varepsilon)x^{1/6} + b_k + c_k x^{-2/15})x^{1/3} \quad \text{if } x \in I_k.
\]

The coefficient of $x^{1/3}$ in this last expression increases monotonically in $I_k$ provided that

\[
x \geq (4c_k/5(a_k - 1 - \varepsilon))^{10/3} \quad \text{for } x \in I_k
\]

and this condition is implied by (10). Thus, we get

\[
\psi(x) - \theta(x) < (1 + \varepsilon)x^{1/2} + ((a_k - 1 - \varepsilon)n_k^{1/6} + b_k + c_k n_k^{-2/15})x^{1/3}
\]

if $x \in I_k$, and the lemma follows from (11).

**Lemma 2.** If an interval $J$ is the union of a finite collection of intervals $J_k = (m_k, n_k)$, and there are positive constants $a'_k, b'_k, d_k, L^-$ and a constant $\varepsilon > 0$ such that $a'_k < 1 - \varepsilon$ and

\[
\psi(x^{1/2}) > a'_k x^{1/2}; \quad \psi(x^{1/3}) > b'_k x^{1/3}, \\
\psi(x^{1/7}) > d_k x^{1/7} \quad \text{for } x \in J_k, \\
L^- \leq \inf_k \{(a'_k - 1 + \varepsilon)n_k^{1/6} + b'_k + d_k n_k^{-4/21}\},
\]

we have

\[
\psi(x) - \theta(x) > (1 - \varepsilon)x^{1/2} + L^- x^{1/3} \quad \text{for } x \in J.
\]

**Proof.** If $x \in J_k$, from Theorem 1 and (12), we have

\[
\psi(x) - \theta(x) > a'_k x^{1/2} + b'_k x^{1/3} + d_k x^{1/7}
\]

or

\[
\psi(x) - \theta(x) > (1 - \varepsilon)x^{1/2} \\
+ ((a'_k - 1 + \varepsilon)x^{1/6} + b'_k + d_k x^{-4/21})x^{1/3} \quad \text{if } x \in J_k.
\]
The coefficient of $x^{1/3}$ in this last expression clearly decreases when $x$ increases and we get, if $x \in J_k$,

$$
\psi(x) - \theta(x) > (1 - \epsilon)x^{1/2} + \left( (a'_{k} - 1 + \epsilon)n_k^{1/6} + b'_{k} + d_k n_k^{4/21} \right) x^{1/3}.
$$

Now the lemma follows from (13).

It is now possible to show the following

**Theorem 2.** We have

\begin{align*}
(14) & \quad \psi(x) - \theta(x) < \sqrt{x} + \frac{4}{3} \sqrt[3]{x} \quad \text{if } 0 < x \leq 10^8, \\
(15) & \quad \psi(x) - \theta(x) > \sqrt{x} + \frac{2}{3} \sqrt[3]{x} \quad \text{if } 2187 \leq x \leq 10^8.
\end{align*}

**Proof.** Inequalities (14) and (15) can be verified directly from our Table I for $x \leq 3 \cdot 10^6$. To prove (14) in this range we start with $x > 0$ and we get

$$
\sqrt{x} + \frac{4}{3} \sqrt[3]{x} > 0 = \lim_{t \rightarrow 4^+} \psi(t) - \theta(t) = \psi(x) - \theta(x) \quad \text{if } 0 < x < 4.
$$

Next, we take $x \geq 4$ and Table I gives

$$
\sqrt{x} + \frac{4}{3} \sqrt[3]{x} > 4.116 > \lim_{t \rightarrow 25^+} \psi(t) - \theta(t) \geq \psi(x) - \theta(x) \quad \text{if } 4 \leq x < 25.
$$

Continuing in the same way and noticing that $\psi(3 \cdot 10^6) - \theta(3 \cdot 10^6)$ is given by the last entry in the table, we establish (14) for $x \leq 3 \cdot 10^6$.

To verify (15) in the range $2187 \leq x \leq 3 \cdot 10^6$ we start at the end of Table I. Taking $x \leq 3 \cdot 10^6$, we have

$$
\sqrt{x} + \frac{2}{3} \sqrt[3]{x} < 1828.201 < \psi(x_1) - \theta(x_1),
$$

with $x_1 = 2765569$, and this gives (15) for $x_1 \leq x \leq 3 \cdot 10^6$. Next, taking $x < x_1$ we have

$$
\sqrt{x} + \frac{2}{3} \sqrt[3]{x} < 1756.578 < \psi(x_2) - \theta(x_2),
$$

with $x_2 = 2571353$, and this gives (15) for $x_2 \leq x < x_1$. This process can be continued down to $x = 2187$ and (15) is established for $x \leq 3 \cdot 10^6$.

If $3 \cdot 10^6 < x \leq 10^8$ we have

$$
1732 < x^{1/2} \leq 10^4; \quad 144 < x^{1/3} \leq 465; \quad 19 < x^{1/5} < 40; \quad 8 < x^{1/7} < 14.
$$

From the Appel and Rosser [1] table of $\theta(x)$, Lehmer's [2] table of primes, and Table I, it follows that we can apply Lemma 2 to the interval $J = J_1 = (3 \cdot 10^6, 10^8]$ with

$$
\epsilon = 0; \quad L^* = 2/3; \quad a'_1 = 0.98708; \quad b'_1 = 0.94842; \quad d_1 = 0.71200,
$$

and this proves (15).

To prove (14) in the interval $J = (3 \cdot 10^6, 10^8]$, we consider the intervals $I_1 = (3 \cdot 10^6, 83.5 \cdot 10^6]$ and $I_2 = (83.5 \cdot 10^6, 10^8]$. Using the same tables it is possible to apply Lemma 2 with

$$
\epsilon = 0; \quad L^* = 4/3; \quad a_1 = 1.00990; \quad a_2 = 1.00463; \quad b_1 = b_2 = 1.03591; \quad c_1 = c_2 = 1.02938,
$$

and this establishes (14).
From Theorem 2, we can deduce the following estimates for $\psi(x)$ in the range $0 < x \leq 10^8$:

**Theorem 3.** We have

\[(16) \quad \psi(x) < x + 0.656 \sqrt{x} + \frac{4}{3} \frac{\sqrt{x}}{x} \quad \text{if} \quad 0 < x \leq 10^8,
\]
\[(17) \quad \psi(x) > x - 0.833 \sqrt{x} + \frac{2}{3} \frac{\sqrt{x}}{x}
\]
\[\quad \text{if} \quad 1427 \leq x \leq 3298, \; 3299 \leq x \leq 19371 \text{ or } 19373 \leq x \leq 10^8.\]

**Proof.** From the Appel and Rosser [1] tables of $\theta(x)$ and $(x - \theta(x))/\sqrt{x}$, we have

\[\frac{x - \theta(x)}{\sqrt{x}} > 0.344 \quad \text{if} \quad 0 < x \leq 10^8,
\]
and

\[\frac{x - \theta(x)}{\sqrt{x}} < 1.833 \quad \text{if} \quad 19801 \leq x \leq 10^8.
\]

These inequalities, together with (14) and (15), prove (16) for $0 < x \leq 10^8$ and (17) for $19801 \leq x \leq 10^8$. Directly from [1], [2] and Table I, we can verify that (17) still holds when $1427 \leq x \leq 3298, \; 3299 \leq x \leq 19371 \text{ and } 19373 \leq x < 19801$.

We have applied Theorem 3 together with the tables [1], [2] and Table I to get some close estimates for $\psi(x)$ in the range $0 < x \leq 10^8$. In Table II, we have listed triplets $(n, \lambda^-, \lambda^+)$ such that

\[\lambda^- x < \psi(x) < \lambda^+ x \quad \text{if} \quad n \leq x \leq 10^8.
\]

To evaluate the lower bounds $\lambda^-$, we observe that

\[\psi(x)/x > \psi(n_k)/n_{k+1} \quad \text{for} \quad n_k \leq x < n_{k+1},
\]
where $n_k$ and $n_{k+1}$ are two consecutive prime powers. Then, if $n_k$ is the largest prime power (not exceeding $10^8$) such that $\psi(n_{k-1})/n_k < \lambda^-$, it follows that $\psi(x) > \lambda^- x$ for $n_k \leq x \leq 10^8$; it also follows that this inequality fails immediately below $n_k$.

Now, in the range $10^8 \leq x < 10^{16}$, we can get better estimates for $\psi(x) - \theta(x)$ than those given by (14) and (15).

**Theorem 4.** We have

\[(18) \quad \psi(x) - \theta(x) < \sqrt{x} + \frac{6}{5} \frac{3\sqrt{x}}{x} \quad \text{if} \quad 10^8 \leq x \leq 10^{16},
\]
\[(19) \quad \psi(x) - \theta(x) > \sqrt{x} + \frac{6}{7} \frac{3\sqrt{x}}{x} \quad \text{if} \quad 10^8 \leq x \leq 10^{16}.
\]

**Proof.** To prove (18) we divide $I = [10^8, 10^{16}]$ into intervals $I_k = [m_k, n_k]$ for $k = 1, \ldots, 8$ and $I_9 = [m_9, 10^{16}]$. Using the tables [1], [2] together with (16) and Tables I and II, it is now easy to verify that we can apply Lemma 1 with $\epsilon = 0$, $L^+ = 6/5$, and the following constants $m_k, n_k, a_k, b_k, c_k$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Similarly, to prove (19), we divide $J = [10^8, 10^{16}]$ into intervals $J_k = [m_k, n_k)$ for $k = 1, \ldots, 7$ and $J_8 = [m_8, 10^{16}]$. Again using (1), (2) together with (17) and Tables I and II, we can apply Lemma 2 with $\varepsilon = 0$, $L_\approx = 6/7$ and the following constants $m_k, n_k, a'_k, b'_k, d_k$.

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<th>$n_k$</th>
<th>$a'_k$</th>
<th>$b'_k$</th>
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With the aid of a computer, it can be easily verified that inequalities (18) and (19) still hold, respectively, for $8236167 < x < 10^8$ and $2036329 < x < 10^8$ but fail immediately below these bounds. We conclude

$$\psi(x) - \theta(x) < \sqrt{x} + \frac{6}{5}\sqrt[3]{x} \quad \text{if} \quad 8236167 \leq x \leq 10^{16},$$

and

$$\psi(x) - \theta(x) > \sqrt{x} + \frac{6}{7}\sqrt[3]{x} \quad \text{if} \quad 2036329 \leq x \leq 10^{16}.$$
(22) \(|\psi(x) - x| < 0.00119721x \) if \(10^8 \leq x < e^{18.43}\),
(23) \(|\psi(x) - x| < 0.0011930x \) if \(e^{18.43} \leq x < e^{18.44}\),
(24) \(|\psi(x) - x| < 0.0011885x \) if \(e^{18.44} \leq x < e^{18.45}\),
(25) \(|\psi(x) - x| < 0.0011839x \) if \(e^{18.45} \leq x < e^{18.5}\),
(26) \(|\psi(x) - x| < 0.0011615x \) if \(e^{18.5} \leq x < e^{18.7}\),
(27) \(|\psi(x) - x| < 0.0010765x \) if \(e^{18.7} \leq x < e^{19}\),
(28) \(|\psi(x) - x| < 0.00096161x \) if \(x \geq e^{19}\).

We can now prove the following

**Theorem 5.** We have

(29) \(|\psi(x) - \theta(x)| < 1.001 \sqrt{x} + 1.1 \sqrt[3]{x} \) if \(x \geq 10^{16}\),
(30) \(|\psi(x) - \theta(x)| < 1.001 \sqrt{x} + 0.9 \sqrt[3]{x} \) if \(x \geq e^{38}\),
(31) \(|\psi(x) - \theta(x)| > 0.999 \sqrt{x} + 0.9 \sqrt[3]{x} \) if \(x \geq 10^{16}\),
(32) \(|\psi(x) - \theta(x)| > 0.999 \sqrt{x} + 0.9 \sqrt[3]{x} \) if \(x \geq e^{38}\).

**Proof.** To prove (29) we divide \(I = [10^{16}, e^{38}]\) into intervals \(I_1 = [10^{16}, e^{36.88}]\), \(I_2 = [e^{36.88}, e^{37}]\), \(I_3 = [e^{37}, e^{38}]\). Using [1], [2] and Tables I and II together with estimates (22) to (27), we see that it is possible to apply Lemma 1 with \(\epsilon = 0.001\), \(L^+ = 1.1\), and
divide \(I = [10^{16}, e^{38}]\) into intervals \(I_1 = [10^{16}, e^{36.88}]\), \(I_2 = [e^{36.88}, e^{37}]\), \(I_3 = [e^{37}, e^{38}]\). Using [1], [2] and Tables I and II together with estimates (22) to (27), we see that it is possible to apply Lemma 1 with \(\epsilon = 0.001\), \(L^+ = 1.1\), and

\[a_1 = 1.00119721, \quad a_2 = 1.0011885, \quad a_3 = 1.0011615,\]
\[b_1 = b_2 = 1.00052, \quad b_3 = 1.00121,\]
\[c_1 = 1.00450, \quad c_2 = c_3 = 1.01364.\]

Now, if \(x \geq e^{38}\), it follows from (1), (22) to (28) and Table II that

\[\psi(x) - \theta(x) < 1.001 x^{1/2} + (0.00003839 x^{1/6} + 1.00115 + 1.00990 x^{-2/15}) x^{1/3}\]
\[< 1.001 x^{1/2} + x^{1/3},\]

and this proves (29) and (30).

To prove (31), we divide \(J = [10^{16}, e^{38}]\) into intervals \(J_1 = [10^{16}, e^{37}]\), \(J_2 = [e^{37}, e^{38}]\). Using inequalities (22) to (27) together with Table II, it is easily seen that we can apply Lemma 2 with \(\epsilon = 0.001\), \(L^+ = 0.9\), and

\[a_1' = 0.99880279, \quad a_2' = 0.9988385, \quad b_1' = b_2' = 0.99870, \quad d_1 = d_2 = 0.94842.\]

Finally, if \(x \geq e^{38}\), it follows from (2), (22) to (28) and Table II that

\[\psi(x) - \theta(x) > 0.999 x^{1/2} + (0.00003839 x^{1/6} + 0.99880) x^{1/3}\]
\[> 0.999 x^{1/2} + x^{1/3},\]

and this proves (31) and (32).


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Table I
ESTIMATES FOR THE CHEBYSHEV FUNCTION $\psi(x) - \theta(x)$

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Table I (continued)
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