Odd Triperfect Numbers Are Divisible
By Eleven Distinct Prime Factors

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Abstract. We prove that an odd triperfect number has at least eleven distinct prime factors.

1. Introduction. A positive number $N$ is called a triperfect number if $\sigma(N) = 3N$ where $\sigma(N)$ is the sum of the positive divisors of $N$. Six even triperfect numbers are known:

$$2^{14} \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151,$$
$$2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127,$$
$$2^9 \cdot 3 \cdot 11 \cdot 31,$$
$$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73,$$
$$2^5 \cdot 3 \cdot 7,$$
$$2^3 \cdot 3 \cdot 5.$$

However, the existence of an odd triperfect (OT) number is an open question. McDaniel [4] and Cohen [2] proved that an OT number has at least nine distinct prime factors; the author proved that it has at least ten prime factors [3], and Beck and Najar [1] showed that it exceeds $10^{50}$.

In this paper we prove

**Theorem.** If $N$ is OT, $N$ has at least eleven distinct prime factors.

2. Proof of Theorem. Throughout this paper we let

$$N = \prod_{i=1}^{10} p_i^{a_i},$$

where $p_i$'s are odd primes, $p_1 < \cdots < p_{10}$ and $a_i$'s are positive integers. We call $p_i^{a_i}$ a component of $N$ and write $p_i^{a_i} \mid | N$.

The following lemmas are easy to prove:

**Lemma 1.** If $N$ is OT, $a_i$'s are even for $1 \leq i \leq 10$.

**Lemma 2.** If $N$ is OT and $q$ is a prime factor of $\sigma(p_i^{a_i})$ for some $i$, then $q = 3$ or $q = p_j$ for some $j$, $1 \leq j \leq 10$.

The following lemmas are stated in [5].
Lemma 3. Suppose \( q \) is a prime, \( q \geq 2 \) and \( a \geq 1 \). Then \( \sigma(q^a) \) has a prime factor \( p \) such that \( a + 1 \) is the order of \( q \) modulo \( p \) except for \( q = 2 \) and \( a = 5 \) and for \( q = a \) a Mersenne prime and \( a = 1 \). In particular \( a + 1 | p - 1 \).

Lemma 4. Suppose \( p \) is a Fermat prime (3, 5, 17, etc.), \( q \) is an odd prime and \( a \) is even. If \( p^b | \sigma(q^a) \), then \( q \equiv 1 \pmod{p} \), \( p^b | a + 1 \), and \( \sigma(q^a) \) has \( b \) distinct prime factors congruent to \( 1 \) modulo \( p \).

Lemma 5. If \( N \) is OT, \( 17 \nmid N \).

Proof. Suppose \( N \) is OT. Since the three smallest primes \( \equiv 1 \pmod{17} \) are 103, 137, and 239 and

\[
\begin{align*}
3 & 5 & 7 & 11 & 13 & 17 & 19 & 23 & 29 & 103 & 137 & 239 < 3, \\
2 & 4 & 6 & 10 & 12 & 16 & 18 & 22 & 28 & 102 & 136 & 238 < 3,
\end{align*}
\]

\( N \) has at most two primes \( \equiv 1 \pmod{17} \). Suppose \( p^a \) and \( q^b \) are components of \( N \) and \( p = q = 1 \pmod{17} \). If \( 17^c \nmid N \) and \( c \geq 4 \), then \( 17^2 | \sigma(p^a) \) or \( 17^2 | \sigma(q^b) \), and, by Lemma 4, \( N \) would have two more primes \( \equiv 1 \pmod{17} \), a contradiction. Hence \( 17^4 \nmid N \). Suppose \( 17^2 | N \). Then \( N \) has a component \( 307^d \) because \( \sigma(17^2) = 307 \). Then \( 17 \nmid \sigma(307^d) \) because \( 16661 \cdot 36857 | \sigma(307^{16}) \), \( \sigma(307^{16}) | \sigma(307^d) \) and

\[
\begin{align*}
3 & 5 & 7 & 11 & 13 & 17 & 19 & 307 & 16661 & 36857 < 3, \\
\end{align*}
\]

Hence \( N \) has another component \( p^b \) such that \( 17^2 | \sigma(p^b) \). Then we get a contradiction again. Hence \( \nmid N \). Q.E.D.

The proof of the following lemma is easy.

Lemma 6. If \( N \) is OT, \( p_9 \leq 283 \).

Lemma 7. If \( N \) is OT and \( 5^a | N \), then \( a = 2 \), \( 5^2 | \sigma(P_1^{a_0}) \) and \( p_{10} \geq 311 \).

Proof. Suppose \( N \) is OT, \( p^b \) is a component of \( N \) and \( 5 | \sigma(p^b) \). By Lemma 4, \( p \equiv 1 \pmod{5} \), \( 5 | b + 1 \) and \( \sigma(p^4) | \sigma(p^b) \). If \( 61 \leq p \leq 281 \), then \( \sigma(p^4) \) has a prime factor \( q \) such that

\[
\begin{align*}
3 & 5 & 7 & 11 & 13 & 19 & 23 & 29 & p & q & 1 < 3, \quad \text{or}
\end{align*}
\]

\( \sigma(p^4) \) has prime factors \( q \) and \( r \) such that

\[
\begin{align*}
\end{align*}
\]

Hence \( p = 11, 31 \) or 41 or \( p \geq 311 \).

Suppose \( p = 11, 31 \) or 41. If \( 5^2 | \sigma(p^b) \), \( 5^2 | b + 1 \) by Lemma 4. Then \( \sigma(p^{2a}) | \sigma(p^b) \) and \( \sigma(p^{24}) \) has two distinct prime factors \( \geq 283 \), contradicting Lemma 6. Hence \( 5^2 \nmid \sigma(p^b) \). Since \( 3221 | \sigma(11^4) \), \( 17351 | \sigma(31^4) \) and \( 579281 | \sigma(41^4) \), \( 5^2 \nmid \sigma(p_{10}^{a_1}) \) and \( p_{10} = 3221, 17351 \) or 579281 and 5 | \( \sigma(p_{10}^{a_0}) \). However, \( \sigma(p_{10}^{a_0}) \) has a prime factor \( \geq 283 \), contradicting Lemma 6. Hence \( p \geq 311 \) and \( 5^a | \sigma(p^b) \).

If \( a \geq 4 \), then by Lemma 4, \( N \) would have four more primes \( \equiv 1 \pmod{5} \), which is a contradiction because

\[
\begin{align*}
3 & 5 & 7 & 11 & 13 & 19 & 31 & 41 & 61 & 311 < 3, \\
2 & 4 & 6 & 10 & 12 & 18 & 30 & 40 & 60 & 310 < 3.
\end{align*}
\]

Hence \( a = 2 \). Q.E.D.
LEMMA 8. If \( N \) is OT, \( p_9 \leq 71 \).

Proof. By Lemma 6, \( 31 = \sigma(5^2) \mid N \). Since

\[
\begin{array}{ccccccccccc}
3 & \sigma(5^2) & 7 & 11 & 13 & 19 & 23 & 31 & 73 & 311 \\
\frac{3}{2} & 5^2 & 6 & 10 & 12 & 18 & 22 & 30 & 72 & 310
\end{array}
\]

\( p_9 \leq 71 \). Q.E.D.

Proof of Theorem. If \( N \) is OT, then by Lemmas 4 and 7, \( 5^2 \mid \sigma(p_9^{10}) \), \( 5^2 \mid a_{10} + 1 \) and \( \sigma(p_9^{24}) \mid (p_9^{24}) \). By Lemma 3, \( \sigma(p_9^{24}) \) has a prime factor \( q \) such that \( 25 \mid q - 1 \). Hence \( q = 25b + 1 \) for some \( b \). Since \( q \) is a prime, \( b \neq 1 \) or 2. Then \( q > 71 \) and \( q \neq p_{10} \), contradicting Lemma 8. Q.E.D.

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