Odd Triperfect Numbers Are Divisible By Eleven Distinct Prime Factors

By Masao Kishore

Abstract. We prove that an odd triperfect number has at least eleven distinct prime factors.

1. Introduction. A positive number \( N \) is called a triperfect number if \( \sigma(N) = 3N \) where \( \sigma(N) \) is the sum of the positive divisors of \( N \). Six even triperfect numbers are known:

\[
\begin{align*}
2^{14} \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151, \\
2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127, \\
2^9 \cdot 3 \cdot 11 \cdot 31, \\
2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73, \\
2^5 \cdot 3 \cdot 7, \\
2^3 \cdot 3 \cdot 5.
\end{align*}
\]

However, the existence of an odd triperfect (OT) number is an open question. McDaniel [4] and Cohen [2] proved that an OT number has at least nine distinct prime factors; the author proved that it has at least ten prime factors [3], and Beck and Najar [1] showed that it exceeds \( 10^{50} \).

In this paper we prove

**Theorem.** If \( N \) is OT, \( N \) has at least eleven distinct prime factors.

2. Proof of Theorem. Throughout this paper we let

\[
N = \prod_{i=1}^{10} p_i^{a_i},
\]

where \( p_i \)'s are odd primes, \( p_1 < \cdots < p_{10} \) and \( a_i \)'s are positive integers. We call \( p_i^{a_i} \) a component of \( N \) and write \( p_i^{a_i} \mid N \).

The following lemmas are easy to prove:

**Lemma 1.** If \( N \) is OT, \( a_i \)'s are even for \( 1 \leq i \leq 10 \).

**Lemma 2.** If \( N \) is OT and \( q \) is a prime factor of \( \sigma(p_i^{a_i}) \) for some \( i \), then \( q = 3 \) or \( q = p_j \) for some \( j \), \( 1 \leq j \leq 10 \).

The following lemmas are stated in [5].
Lemma 3. Suppose $q$ is a prime, $q \geq 2$ and $a \geq 1$. Then $\sigma(q^a)$ has a prime factor $p$ such that $a + 1$ is the order of $q$ modulo $p$ except for $q = 2$ and $a = 5$ and for $q = a$ a Mersenne prime and $a = 1$. In particular $a + 1 \mid p - 1$.

Lemma 4. Suppose $p$ is a Fermat prime (3, 5, 17, etc.), $q$ is an odd prime and $a$ is even. If $p^b \mid \sigma(q^a)$, then $q \equiv 1 \pmod{p}$, $p^b \mid a + 1$, and $\sigma(q^a)$ has $b$ distinct prime factors congruent to 1 modulo $p$.

Lemma 5. If $N$ is OT, $17 \nmid N$.

Proof. Suppose $N$ is OT. Since the three smallest primes $\equiv 1 \pmod{17}$ are 103, 137, and 239 and

$$\frac{3 5 7 11 13 17 19 103 137 239}{2 4 6 10 12 16 18 102 136 238} < 3,$$

$N$ has at most two primes $\equiv 1 \pmod{17}$. Suppose $p^a$ and $q^b$ are components of $N$ and $p = q = 1 \pmod{17}$. If $17 \mid N$ and $c \geq 4$, then $17^2 \mid \sigma(p^a)$ or $17^2 \mid \sigma(q^b)$, and, by Lemma 4, $N$ would have two more primes $\equiv 1 \pmod{17}$, a contradiction. Hence $17^4 \nmid N$. Suppose $17^2 \parallel N$. Then $N$ has a component $307^d$ because $\sigma(17^2) = 307$. Then $17 \nmid \sigma(307^d)$ because $16661 \cdot 36857 \mid \sigma(307^d)$ and

$$\frac{3 5 7 11 13 17 19 307 16661 36857}{2 4 6 10 12 16 18 306 16660 36856} < 3.$$

Hence $N$ has another component $p^b$ such that $17^2 \mid \sigma(p^b)$. Then we get a contradiction again. Hence $17 \nmid N$. Q.E.D.

The proof of the following lemma is easy.

Lemma 6. If $N$ is OT, $p_9 \leq 283$.

Lemma 7. If $N$ is OT and $5^a \mid N$, then $a = 2$, $5^2 \mid \sigma(P_{10}^{a(0)})$ and $p_{10} \geq 311$.

Proof. Suppose $N$ is OT, $p^b$ is a component of $N$ and $5 \mid \sigma(p^b)$. By Lemma 4, $p \equiv 1 \pmod{5}$, $5 \mid b + 1$ and $\sigma(p^4) \mid \sigma(p^b)$. If $61 \leq p \leq 281$, then $\sigma(p^4)$ has a prime factor $q$ such that

$$\frac{3 5 7 11 13 19 23 29 p}{2 4 6 10 12 18 22 28} \frac{q}{p - 1} \frac{q}{q - 1} < 3,$$

or $\sigma(p^4)$ has prime factors $q$ and $r$ such that

$$\frac{3 5 7 11 13 19 23 p}{2 4 6 10 12 18 22} \frac{q}{p - 1} \frac{r}{q - 1} \frac{r}{r - 1} < 3.$$

Hence $p = 11, 31$ or $41$ or $p \geq 311$.

Suppose $p = 11, 31$ or $41$. If $5^2 \mid \sigma(p^b)$, $5^2 \mid b + 1$ by Lemma 4. Then $\sigma(p^{2a}) \mid \sigma(p^b)$ and $\sigma(p^{24})$ has two distinct prime factors $> 283$, contradicting Lemma 6. Hence $5^2 \mid \sigma(p^b)$. Since $3221 \mid \sigma(11^4)$, $17351 \mid \sigma(31^4)$ and $579281 \mid \sigma(41^4)$, $5^2 \mid \sigma(\prod_{i=1}^{5} p_i^{a_i})$ and $p_{10} = 3221, 17351$ or $579281$ and $5 \mid \sigma(p_{10}^{a(0)})$. However, $\sigma(p_{10}^{a(0)})$ has a prime factor $> 283$, contradicting Lemma 6. Hence $p \geq 311$ and $5^a \mid \sigma(p^b)$.

If $a \geq 4$, then by Lemma 4, $N$ would have four more primes $\equiv 1 \pmod{5}$, which is a contradiction because

$$\frac{3 5 7 11 13 19 31 41 61 311}{2 4 6 10 12 18 30 40 60 310} < 3.$$

Hence $a = 2$. Q.E.D.
Lemma 8. If \( N \) is OT, \( p_9 \leq 71 \).

Proof. By Lemma 6, \( 31 = \sigma(5^2) \mid N \). Since

\[
\begin{array}{cccccccccc}
3 & \sigma(5^2) & 7 & 11 & 13 & 19 & 23 & 31 & 73 & 311 \\
2 & 5^2 & 6 & 10 & 12 & 18 & 22 & 30 & 72 & 310
\end{array}
\]

\( p_9 \leq 71 \). Q.E.D.

Proof of Theorem. If \( N \) is OT, then by Lemmas 4 and 7, \( 5^2 \mid \sigma(p_{10}^{10}) \), \( 5^2 \mid a_{10} + 1 \) and \( \sigma(p_{10}^{10}) \mid (p_{10}^{10}) \). By Lemma 3, \( \sigma(p_{10}^{10}) \) has a prime factor \( q \) such that \( 25 \mid q - 1 \). Hence \( q = 25b + 1 \) for some \( b \). Since \( q \) is a prime, \( b \neq 1 \) or 2. Then \( q > 71 \) and \( q \neq p_{10} \), contradicting Lemma 8. Q.E.D.

Department of Mathematics
East Carolina University
Greenville, North Carolina 27834