A Note on the Diophantine Equation

\[ x^3 + y^3 + z^3 = 3 \]

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Abstract. Any integral solution of the title equation has \( x = y = z \).

The report of Scarowsky and Boyarsky \cite{3} that an extensive computer search has failed to turn up any further integral solutions of the title equation prompts me to give the proof of a result which I noted many years ago and which might be of use in further work (cf. footnote on p. 505 of \cite{2}).

Theorem. Any integral solution of

\[ x^3 + y^3 + z^3 = 3 \]

has

\[ x = y = z \]

Proof. Trivially,

\[ x = y = z \]

We work in the ring \( \mathbb{Z}[\rho] \) of Eisenstein integers, where \( \rho \) is a cube root of unity. If \( \alpha \in \mathbb{Z}[\rho] \) is prime to 3, then there is precisely one unit \( \epsilon = \pm \rho^j \) \( (j = 0, 1, 2) \) such that \( e\alpha \equiv 1 \). The supplement \cite{1} to the law of cubic reciprocity states that if \( \pi \in \mathbb{Z}[\rho] \) is prime, \( \pi \equiv 1 \), then 3 is a cubic residue of \( \pi \) in \( \mathbb{Z}[\rho] \) precisely when \( \pi = a \) \( (9) \) for some \( a \in \mathbb{Z} \). It follows that if \( \alpha \in \mathbb{Z}[\rho] \), \( \alpha \equiv 1 \) and if 3 is congruent to a cube modulo \( \alpha \), then \( \alpha \equiv b \) \( (9) \) for some \( b \in \mathbb{Z} \).

Put

\[ \alpha = -\rho^2 x - \rho y, \]

so

\[ \alpha = x + (x - y)\rho \equiv 1 \]

by (3). By (1) we have \( z^3 \equiv 3 \) \( (\alpha) \), so the preceding remarks apply. Hence \( x - y \equiv 0 \) \( (9) \). Finally, (2) follows by symmetry.

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