A Note on the Diophantine Equation

\[ x^3 + y^3 + z^3 = 3 \]

By J. W. S. Cassels

Abstract. Any integral solution of the title equation has \( x = y = z \).

The report of Scarowsky and Boyarsky [3] that an extensive computer search has failed to turn up any further integral solutions of the title equation prompts me to give the proof of a result which I noted many years ago and which might be of use in further work (cf. footnote on p. 505 of [2]).

Theorem. Any integral solution of

(1) \[ x^3 + y^3 + z^3 = 3 \]
has

(2) \[ x = y = z \].

Proof. Trivially,

(3) \[ x^3 + y^3 + z^3 = 3 \].

We work in the ring \( \mathbb{Z}[\rho] \) of Eisenstein integers, where \( \rho \) is a cube root of unity. If \( \alpha \in \mathbb{Z}[\rho] \) is prime to 3, then there is precisely one unit \( \varepsilon = \pm \rho^j \) (\( j = 0, 1, 2 \)) such that \( e\alpha \equiv 1 \). The supplement [1] to the law of cubic reciprocity states that if \( \pi \in \mathbb{Z}[\rho] \) is prime, \( \pi \equiv 1 \), then 3 is a cubic residue of \( \pi \) in \( \mathbb{Z}[\rho] \) precisely when \( \pi = a \) for some \( a \in \mathbb{Z} \). It follows that if \( \alpha \in \mathbb{Z}[\rho] \), \( \alpha \equiv 1 \) and if 3 is congruent to a cube modulo \( \alpha \), then \( \alpha \equiv b \) for some \( b \in \mathbb{Z} \).

Put

\[ \alpha = -\rho^2x - \rho y, \]

so

\[ \alpha = x + (x - y)\rho \equiv 1 \]

by (3). By (1) we have \( z^3 \equiv 3 \), so the preceding remarks apply. Hence \( x - y \equiv 0 \) (9). Finally, (2) follows by symmetry.

Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
16 Mill Lane
Cambridge CB2 1SB, England

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