Finite Element Methods of Optimal Order for Problems with Singular Data

By Kenneth Eriksson

Abstract. An adapted finite element method is proposed for a class of elliptic problems with singular data. The idea is to subtract the main singularity from the solution and to solve for the remainder using suitable mesh-refinements. Optimal order error estimates are proved.

1. Introduction and Results. Let Ω be a bounded domain in \( R^N \) with smooth boundary \( Γ \) and consider the following problem: Given \( x_0 \in Ω \) find \( u = u(x) \) such that

\[
\begin{align*}
Lu(x) & = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^{N} a_i(x) \frac{\partial u}{\partial x_i} + a(x)u \\
& = \delta(x - x_0) \quad \text{in } Ω,
\end{align*}
\]

\[
(1.1a)
\]

\[
lu(x) \equiv \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u}{\partial x_i} n_j(x) = 0 \quad \text{on } Γ,
\]

\[
(1.1b)
\]

where \( \delta \) is the Dirac distribution (unit impulse), \( n = (n_j) \) is the outward unit normal to \( Γ \), and \( a_{ij}, a_i, \) and \( a \) are smooth (\( C^\infty \) regular) functions on \( \overline{Ω} \), with \( a_{ij} = a_{ji} \), and such that the associated bilinear form

\[
A(v, w) = \int_{Ω} \left( \sum_{i,j} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} + \sum_{i} a_i \frac{\partial v}{\partial x_i} w + aw \right) dx
\]

satisfies the ellipticity-coercivity condition

\[
A(v, v) \geq c \|v\|_{1,Ω}^2 \quad \text{for all } v \in H^1(Ω),
\]

where \( c \) is a positive constant and \( \| \cdot \|_{1,Ω} \) the usual norm in \( H^1(Ω) \), the space of functions with square-integrable first-order derivatives in \( Ω \). It is well known (cf., e.g., [8]) that problem (1.1) admits a unique (distributional) solution \( u \), which is also determined by the corresponding variational equations

\[
A(u, ψ) = ψ(x_0) \quad \text{for all } ψ \in W^1_\infty(Ω),
\]

where \( W^1_\infty(Ω) \) is the space of functions with bounded first-order derivatives on \( \overline{Ω} \).

Received January 10, 1983; revised July 27, 1984.
1980 Mathematics Subject Classification. Primary 65N15, 65N30.
Key words and phrases. Neumann problem, Green's function, adapted finite element methods, mesh-refinement, optimal order, error estimate.

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0025-5718/85 $1.00 + $.25 per page
In general, problem (1.1) (or (1.3)) cannot be solved exactly, so we are faced with
the problem of finding an accurate approximate solution. We shall consider the
following standard approach for this (cf., e.g., [1], [7], [11]): Given a finite-dimen-
sional space $S_h \subset H^1(\Omega) \cap C(\overline{\Omega})$ find $u_h \in S_h$ such that
\begin{equation}
A(u_h, \chi) = \chi(x_0) \quad \text{for all } \chi \in S_h.
\end{equation}
By the coercivity of $A(\cdot, \cdot)$, there exists a unique such $u_h$ determined by the linear
system of equations
\begin{equation}
\sum_{i=1}^{M} U_i A(\chi_i, \chi) = \chi(x_0) \quad \text{for } j = 1, \ldots, M,
\end{equation}
where $u_h = \sum_{i=1}^{M} U_i \chi_i$, and $(\chi_j)_{j=1}^{M}$ is an arbitrary basis for $S_h$. For appropriate
finite element spaces $S_h$, where $h$ is the associated mesh-size parameter, there exist a
priori estimates for the error $u_h - u$ in terms of $h$. One problem in deriving these
estimates is that the singularity of $u$ at $x_0$, which is of order $\log|x - x_0|^{-1}$ for $N = 2,$
and $|x - x_0|^{-N+2}$ for $N \neq 2$, frustrates the usual type of error analysis. For $N = 1$
this is a minor problem since then the singularity is concentrated at $x_0$ and $u$ is
continuous. Choosing $x_0$ as one of the nodal points and using continuous piecewise
polynomials for $S_h$, the usual analysis carries through, and, for example, for
polynomials of degree $r - 1$ we have $\|u_h - u\|_{1,0} \leq C h^{r-1}$. For $N > 1$, however, the
standard method of analysis fails since then the solution does not even belong to
$H^1(\Omega)$. Nevertheless, Babuška [1] was able to show, for $N = 2$, $L = -\Delta + I$ (minus
Laplacian plus identity), and finite element spaces $S_h$ possessing standard approxi-
mation and inverse properties (such as piecewise linear on a quasi-uniform triangu-
lation of $\Omega$), that
\begin{equation}
\|u_h - u\|_{0,\Omega} \leq C h^{1-\epsilon},
\end{equation}
where $\| \cdot \|_{0,\Omega}$ is the $L_2(\Omega)$-norm and $\epsilon > 0$ is arbitrary. Later, Scott [11] improved
and generalized this result by showing that for elliptic operators of order $2m$, normal
covering boundary conditions (cf., e.g., [9]), and dimension $N \geq 2$, one has
\begin{equation}
\|u_h - u\|_{s,\Omega} \leq C(x_0) h^{2m-s-N/2} \quad \text{for } 2m - r \leq s < 2m - N/2,
\end{equation}
where $C(x_0)$ tends to infinity as $x_0$ approaches $\Gamma$, $r \geq 2m$ is the order of approxima-
tion of $S_h \subset H^r(\Omega)$, and the Sobolev norm index $s$ may also be negative (cf., e.g., [9]
for the definition). Despite its generality, Scott’s result falls short in a certain respect.
For instance, if $m = 1$, $r = 2$ (piecewise linears), and $N = 2$ or 3, we obtain no
information about $\nabla(u_h - u)$, and for $N \geq 4$, no information whatsoever.

Recently, in [7], we proposed the use of adapted finite element spaces for the given
problem. Denoted by $S_h(x_0, \alpha, r)$ where $h \in (0, \frac{1}{2}]$, $\alpha \in [0,1)$, and $r \geq 2$ are
parameters, these spaces can be described as follows: Given positive constants $c$ and
$C$ let $\Omega$ be divided into elements $\tau$ such that
\begin{equation}
c \text{diam}(\tau) \leq h \left( \text{dist}(x_0, \tau) \right)^{\alpha} + h^{1/(1-\alpha)} \leq C \text{diam}(\tau) \quad \text{for all } \tau,
\end{equation}
where each $\tau$ is the restriction to $\Omega$ of the interior of a $N$-simplex $\hat{\tau}$ (cf. [4]), with
\begin{equation}
\left( \text{diam}(\hat{\tau}) \right)^N \leq C \int_{\hat{\tau}} dx.
\end{equation}
and the intersection of any two such simplices is either a face of both, or of lower dimension. For \( S_h = S_h(x_0, \alpha, r) \) we take the space of all continuous functions on \( \bar{\Omega} \) which reduces to polynomials of degree at most \( r - 1 \) on each \( \tau \subset \Omega \). By (1.5) the mesh is refined (graded) around \( x_0 \) in such a way that elements at distance \( d \) from \( x_0 \) have diameters of order \( hd^\alpha \), but a minimum diameter of order \( h^{1/(1-\alpha)} \). Hence \( h \) is a parameter for the maximal global mesh-size, and \( \alpha \) determines the degree of refinement. Of course, such a mesh-refinement enables a better approximation in \( S_h \) of any given function which is irregular near \( x_0 \). The condition (1.6) is used to derive local inverse estimates.

The following results were obtained in [7] for \( u_h \in S_h(x_0, \alpha, r) \) being the solution of (1.4). For \( \alpha > (r - 2)/(r - 1) \),

\[
\| \nabla (u_h - u) \|_{L^1(\Omega)} \leq C h^{r-1},
\]

and for \( \alpha > (r - 2)/r \),

\[
\| u_h - u \|_{L^1(\Omega)} \leq C h^r,
\]

where \( C \) is a constant independent of \( h \) and \( x_0 \). Further, if \( \alpha > (r - 1)/r \), and if \( d = |x - x_0| \geq ch^{1/(1-\alpha)} \) and \( \text{dist}(x, \Gamma) \geq d \) for a suitable \( c > 0 \), then

\[
|u_h(x) - u(x)| \leq Ch' h \ln(1/h) d^{-N},
\]

where \( \tilde{r} = 1 \) if \( r = 2 \), \( \tilde{r} = 0 \) if \( r > 2 \).

In this paper, we analyze a method to approximate the solution of (1.1) to the accuracy (1.7), (1.8), and (1.9) which requires a lesser degree of mesh-refinement than the one in [7]. One reason for introducing such a method is that the computational effects of strong mesh-refinements are not yet very well known. Recall that the condition number for the stiffness matrix \( (A(\chi_i, \chi_j)) \) depends on the mesh-size.

In order to describe the method we first note that the solution of (1.1) can be written in the form

\[
u = u_0 + v,
\]

where \( u_0 \) is the fundamental singularity of \( u \) defined by (in the sequel we only consider the case \( N \geq 2 \))

\[
u_0(x) = \begin{cases} 
\frac{|\det(Q)|}{2\pi} \ln\left(|\hat{x} - \hat{x}_0|^{-1}\right) & \text{if } N = 2, \\
\frac{|\det(Q)|}{(N - 2)\sigma_N} |\hat{x} - \hat{x}_0|^{-N+2} & \text{if } N > 2,
\end{cases}
\]

where \( Q \) is the inverse of \( A^{1/2} \), the positive square root of \( A \equiv (a_{ij}(x_0)) \), \( \hat{x} - \hat{x}_0 = Q(x - x_0) \), and \( \sigma_N \) is the surface area of the unit ball \( B_1(0) \subset R^N \). For example, if \( L = -\Delta + 1 \) we can take \( Q = I \) and thus obtain

\[
u_0(x) = \frac{1}{(N - 2)\sigma_N} |x - x_0|^{-N+2} \quad (N > 2),
\]

which we recognize as the fundamental solution of \( -\Delta \). It is a matter of straightforward calculation to verify that in the general case \( u_0 \) satisfies

\[
- \sum_{i,j=1}^N a_{ij}(x_0) \frac{\partial^2 u_0}{\partial x_j \partial x_i} = \delta(x - x_0) \quad \text{in } \Omega.
\]
In view of (1.10) we are led to seek an approximate solution of (1.1) in the form
\[(1.13)\quad \tilde{u}_h = u_0 + v_h, \quad \text{with} \quad v_h \in S_h = S_h(x_0, \alpha, r),\]
and such that
\[(1.14)\quad A(\tilde{u}_h, \chi) = \chi(x_0) \quad \text{for all} \quad \chi \in S_h.\]
Again, by the coercivity of \(A(\cdot, \cdot)\), there is a unique such \(\tilde{u}_h\). In fact, to seek \(\tilde{u}_h\) is to seek \(v_h = \sum_{j=1}^{M} V_j \chi_j \in S_h\) such that
\[(1.15)\quad \sum_{j=1}^{M} V_j A(\chi_j, \chi_j) = \chi_j(x_0) - A(u_0, \chi_j) \quad \text{for} \quad j = 1, \ldots, M,\]
where \(\{\chi_j\}_{j=1}^{M}\) is a basis for \(S_h\).

We shall prove the following error estimates for this method:

**Theorem 1.** Let \(u\) be the solution of (1.1) and \(\tilde{u}_h = u_0 + v_h\) that of (1.14), with \(v_h \in S_h(x_0, \alpha, r)\). Then, for \(\alpha > (r - 3)/(r - 1)\),
\[(1.16)\quad \|\nabla \tilde{u}_h - \nabla u\|_{L_1(\Omega)} \leq C h^{r-1},\]
and for \(\alpha > (r - 3)/r\),
\[(1.17)\quad \|\tilde{u}_h - u\|_{L_1(\Omega)} \leq C h^{r},\]
where \(C\) may depend on the given problem as well as on \(\alpha, r\), and the constants in (1.5) and (1.6), but not on \(h\).

Further, we have the following pointwise error estimate:

**Theorem 2.** Let \(u\) and \(\tilde{u}_h\) be as in Theorem 1 with \(\alpha > (r - 3)/r\). Then
\[
|\tilde{u}_h(x) - u(x)| \leq C \left( \frac{1}{h} \right) \left( \ln(|x - x_0|) \right)^{-N} \left( |x - x_0| \right)^{-N} \quad \text{for} \quad x \in \Omega, \; x \neq x_0,
\]
where \(C\) is independent of \(x\) and \(h\), and \(\bar{r} = 1\) if \(r = 2\), \(\bar{r} = 0\) if \(r > 2\).

**Remark.** The constants \(C\) in Theorems 1 and 2 become infinite as \(x_0\) approaches \(\Gamma\). In order to have a method which is effective also when \(x_0\) is close to (or even on) the boundary \(\Gamma\) one can modify the definition of \(u_0\) according to
\[
u_0(x) = \begin{cases} \frac{|\det(Q)|}{2\pi} \left( \ln(|\hat{x} - \hat{x}_0|^{-1}) + \ln\left(|\hat{x} - \hat{x}_0^*|^{-1}\right) \right) & \text{if} \ N = 2, \\ \frac{|\det(Q)|}{(N - 2)\sigma_N} \left(|\hat{x} - \hat{x}_0|^{-N+2} + |\hat{x} - \hat{x}_0^*|^{-N+2}\right) & \text{if} \ N > 2, \end{cases}
\]
where \(\hat{x}_0^*\) is the "\(Q\)-reflection" of \(\hat{x}_0\) in \(\Gamma\) defined by \(\hat{x}_0^* = Q x_0^*, \quad x_0^* = 2z - x_0\), and \(z\) minimizes \(|Q(y - x_0)|, \quad y \in \Gamma\). For this modification of the method the estimates of Theorems 1 and 2 hold with constants \(C\) independent of \(x_0\).

The proofs of Theorems 1 and 2 are given in Sections 4 and 5, respectively. Sections 2 and 3 are devoted to preparatory work.

**2. Preliminaries.** Throughout this paper we shall denote by \(c\) and \(C\) various positive constants which are independent of \(h\) (but which may depend on the data of the given problem (1.1), the constants in (1.5) and (1.6), and on the parameters \(\alpha\) and \(r\)). Similarly, \(C_1\) and \(C_\ast\) will denote two specific such constants.
Besides the usual $L^p$-norms
\[
\|v\|_{L^p(\Omega')} = \left( \int_{\Omega'} |v(x)|^p \, dx \right)^{1/p}
\]
for $p = 1$ and $p = 2$,
and
\[
\|v\|_{L^\infty(\Omega')} = \operatorname{ess \, sup}_{x \in \Omega'} |v(x)|,
\]
we shall use the Sobolev norms
\[
\|v\|_{k, \Omega'} = \left( \sum_{|\beta| \leq k} \|D^\beta v\|_{L^2(\Omega')}^2 \right)^{1/2},
\]
where $|\beta| = \beta_1 + \cdots + \beta_N$ is the length of the multi-index $\beta = (\beta_1, \ldots, \beta_N)$, and
\[
D^\beta = \left( \frac{\partial}{\partial x_1} \right)^{\beta_1} \cdots \left( \frac{\partial}{\partial x_N} \right)^{\beta_N}.
\]
In particular, $\| \cdot \|_{0, \Omega'}$ denotes the usual $L^2(\Omega')$-norm. The Sobolev space $H^k(\Omega')$ is the space of all functions $w$ such that $\|w\|_{k, \Omega'}$ is finite.

In the proofs we shall consider subdomains of $\Omega$ defined by
\[
D_j = \left\{ x \in \Omega : 2^{-j+1} < |x - x_0| < 2^{-j} \right\},
\]
and
\[
\Omega_j = \left\{ x \in \Omega : |x - x_0| < 2^{-j} \right\}
\]
for $j$ integer, and set $d_j = 2^{-j}$ and $h_j = hd_j^\alpha$. Accordingly, $d_j$ is proportional to the diameter of $D_j$ and $\Omega_j$, and as long as $j$ is not too large so that $h_j$ is smaller than the minimal mesh-size $h^{1/(1-\alpha)}$, $h_j$ is proportional to the maximal mesh-size on $D_j$ and $\Omega_j$. We shall frequently use the obvious facts that $d_j \leq C d_{j+1}$ and $h_j \leq C h_{j+1}$, and we also note that since $\Omega$ is bounded, there is an integer $j_1$ such that $D_j$ is empty for $j < j_1$.

Due to the variable mesh-size a typical interpolant in $S_h(x_0, \alpha, r)$ approximates a given function with variable degree of accuracy over $\Omega$. The following three results are quoted from [7].

**Lemma 1.** Given $w$ there is an interpolant $w_j \in S_h(x_0, \alpha, r)$ of $w$ such that for $j \leq J_1$,
\[
\|w - w_j\|_{1, \Omega_j} \leq C h_j \|w\|_{2, \Omega_j-1},
\]
and
\[
\|w - w_j\|_{1, D_j} \leq C h_j^{m-1} \|w\|_{m, D_j} \quad \text{for } 2 \leq m \leq r,
\]
where $J_1$ is determined by $2^{-J_1} = C_1 h^{1/(1-\alpha)}$ for a suitable sufficiently large constant $C_1$, and $D_j^1 = \Omega_{j-1} \setminus \overline{\Omega}_{j+2}$.

The next lemma shows a similar property for the Galerkin approximation.

**Lemma 2.** If $j \leq J_1$, and if $v_h \in S_h(x_0, \alpha, r)$ and $v$ satisfy
\[
A(v_h - v, \chi) = 0 \quad \text{for all } \chi \in S_h(x_0, \alpha, r) \text{ with support in } \overline{D_j^1},
\]
then
\[
\|v_h - v\|_{1, D_j} \leq C h_j^{r-1} \|v\|_{r, D_j} + C d_j^{-1} \|v_h - v\|_{0, D_j}.
\]
From (1.6) we have the following inverse property:

**Lemma 3.** For any \( \tau \) as in (1.6) and any polynomial \( p \) of degree at most \( r - 1 \), we have

\[
\|p\|_{L^\infty(\tau)} \leq C(\text{diam}(\tau))^{-N/q}\|p\|_{L^q(\tau)} \quad \text{for} \ q \in [1, \infty).
\]

For the proofs of these results we refer to [7].

3. A Step of Reduction. Recall that by definition \( u = u_0 + v \) and \( \tilde{u}_h = u_0 + v_h \). Hence, we can estimate \( \tilde{u}_h - u \) by estimating \( v_h - v \). We shall do this by estimating, each individually, \( v_h - \tilde{v} \) and \( \tilde{v} - v \), where \( \tilde{v} \) is an appropriate approximation of \( v \). In this section we introduce such a \( \tilde{v} \) and derive estimates for \( \tilde{v} - v \). For the analysis of \( v_h - \tilde{v} \) we also investigate the regularity of \( \tilde{v} \).

Setting

\[
\Phi(x) = \sum_{i,j=1}^N \frac{\partial}{\partial x_j} \left( (a_{ij}(x) - a_{ij}(x_0)) \frac{\partial u_0}{\partial x_i} \right) - \sum_{i=1}^N a_i(x) \frac{\partial u_0}{\partial x_i} - a(x) u_0 \quad \text{in} \ \Omega,
\]

and

\[
\phi(x) = - \sum_{i,j=1}^N a_{ij}(x) \frac{\partial u_0}{\partial x_i} n_j(x) \quad \text{on} \ \Gamma,
\]

we see from (1.1) and (1.12) that

\[
\begin{align*}
(3.1a) & \quad L\phi = \Phi \quad \text{in} \ \Omega, \\
(3.1b) & \quad \phi = \phi \quad \text{on} \ \Gamma.
\end{align*}
\]

We shall use the following facts:

**Lemma 4.** Let \( \Phi, \phi, \) and \( v \) be defined as above. Then \( \phi \) is a smooth function (with degree of smoothness depending on \( \text{dist}(x_0, \Gamma) \)), and the following estimates hold for \( \Phi \) and \( v \) (for \( |\beta| \leq r \), say):

\[
\begin{align*}
(3.2) & \quad \left| D^\beta \Phi(x) \right| \leq C|x - x_0|^{-N+1-|\beta|}, \\
(3.3) & \quad \left| D^\beta v(x) \right| \leq \begin{cases} 
C(\ln(|x - x_0|^{-1}) + 1) & \text{if } -N + 3 - |\beta| = 0, \\
C(|x - x_0|^{-N+3-|\beta|} + 1) & \text{if } -N + 3 - |\beta| \neq 0.
\end{cases}
\end{align*}
\]

**Proof.** The smoothness of \( \phi \) and the estimate (3.2) follows at once from the definitions. The estimate (3.3) can be obtained, implicitly, from [8]. For completeness, we show in an appendix that the result follows easily from the properties of the Green’s function.

We now introduce \( \tilde{v} \), requiring that \( \tilde{v} \) be close to \( v \), that \( \tilde{v} \in H^2(\Omega) \), that \( \tilde{v} \) possess at least \( r \) derivatives away from \( x_0 \), and that \( v_h \) (which is the Galerkin approximation of \( v \)) be also the Galerkin approximation of \( \tilde{v} \). Therefore, let \( \tilde{v} \) be the solution of

\[
\begin{align*}
(3.4a) & \quad L\tilde{v} = \Phi \quad \text{in} \ \Omega, \\
(3.4b) & \quad \tilde{v} = \phi \quad \text{on} \ \Gamma,
\end{align*}
\]

where \( \Phi \) is defined as follows: Set \( \varepsilon = h^{1/(1-\alpha)} \) and let \( \Omega_{h\varepsilon} \) be the smallest mesh-domain covering \( B_\varepsilon(x_0) \cap \Omega \). \( B_\varepsilon(x_0) = \{ x : |x - x_0| < \varepsilon \} \). By a mesh-domain we mean...
(the interior of the closure of) a union of elements. Set

$$\Phi(x) = \begin{cases} \pi_r \Phi & \text{on } \Omega_h, \\ \Phi & \text{outside } \Omega_h, \end{cases}$$

where $\pi_r$ is the local $L_2$-projection onto $P_{r-1}(\tau)$, the space of polynomials of degree at most $r - 1$ restricted to $\tau$.

It is well known that problem (3.4) admits a (unique) solution, and the following estimates hold for $\Phi$ and $\Phi$:

**Lemma 5.** Let $\Phi$ be defined by (3.5) and let $\Phi$ be the solution of (3.4). Then (with $\varepsilon = h^{1/(1-\alpha)}$)

$$\|\Phi\|_{L_2(\Omega)} \leq C \varepsilon^{-N/2+1}(\ln 1/\varepsilon)^{N/2},$$

$$\|\Phi\|_{L_1(\Omega_h)} \leq C \varepsilon,$$

$$\|\Phi\|_{L_2(\Omega_h)} \leq C \varepsilon^{-N/2+1}(\ln 1/\varepsilon)^{N/2},$$

where $N = 1$ if $N = 2$, $N = 0$ if $N > 2$, and for $j < J$ and $r \geq 2$ (cf. Lemma 1),

$$\|\Phi\|_{L_2(\tau)} \leq C d_j^{-N/2+3-r}.$$

**Proof.** We have first

$$\|\Phi\|_{L_2(\Omega)} \leq \sum_{\tau \subset \Omega_h} \|\Phi\|_{L_2(\tau)},$$

and by (3.2), with $R = |x - x_0|$,  

$$\|\Phi\|_{L_2(\Omega \setminus \Omega_h)}^2 \leq \int_{\tau \subset \Omega_h} \left( R^{-N+1} \right)^2 R^{N-1} dR \leq C \varepsilon^{-N+2}(\ln 1/\varepsilon)^N.$$  

Using Lemma 3, we obtain

$$\|\Phi\|_{L_2(\tau)}^2 = \int_\tau \Phi \Phi \, dx = \int_\tau \Phi \Phi \, dx \leq \|\Phi\|_{L_1(\tau)} \|\Phi\|_{L_2(\tau)},$$

and hence, using (3.2),

$$\sum_{\tau \subset \Omega_h} \|\Phi\|_{L_2(\tau)} \leq C \varepsilon^{-N/2} \|\Phi\|_{L_1(\Omega_h)} \leq C \varepsilon^{-N/2+1}.$$  

Together our estimates now prove (3.6).

The estimate (3.7) follows at once from the proof of (3.6), and (3.8) follows from (3.6) by the standard $H^2$-regularity estimate, since $\phi$ is smooth (cf., e.g., [9]).

For the proof of (3.9) we note that such an estimate holds for $v$, because of (3.3). Hence,

$$\|\Phi\|_{r,D_j} \leq \|\Phi - v\|_{r,D_j} + C d_j^{-N/2+3-r}.$$  

Further, we have

$$\Phi(x) = \int_{\Omega_h} g(x,y)(\Phi(y) - \Phi(y)) \, dy.$$
where \( g \) is the associated Green's function; i.e., \( g(x, y) \) is the solution of
\[
A(\psi, g(x, \cdot)) = \psi(x) \quad \text{for all } \psi \in W^1_\infty(\Omega).
\]

It is known (cf., e.g., [8]) that such a \( g \) exists and that
\[
|D_\beta D_\gamma g(x, y)| \leq \begin{cases} 
C \ln(1 + |x - y|) & \text{if } -N + 2 - |\beta| - |\gamma| = 0, \\
C |x - y|^{-N + 2 - |\beta| - |\gamma|} & \text{if } -N + 2 - |\beta| - |\gamma| < 0.
\end{cases}
\]

Hence, for \( x \in D_\beta \) and \(|\beta| \leq r\),
\[
|D_\beta (\tilde{v} - v)(x)| \leq \sup_{y \in \Omega_h} |D_\beta g(x, y)| \left( \|\Phi\|_{L_1(\Omega_h)} + \|\Phi\|_{L_1(\Omega_h)} \right) \leq Cd^{-N + 2 - |\beta| - |\gamma|},
\]
where we have also used (3.2) and (3.7) in the last step. Together our estimates show (3.9) which completes the proof of Lemma 5.

We shall now see that \( \tilde{v} \) is appropriately close to \( v \).

**Lemma 6.** Let \( \tilde{v} \) be the solution of (3.4) and \( v \) that of (3.1), or, equivalently, set \( v = u - u_0 \). Then
\[
\|\nabla (\tilde{v} - v)\|_{L_1(\Omega)} \leq C \varepsilon^2,
\]
and
\[
\|\tilde{v} - v\|_{L_1(\Omega)} \leq C \varepsilon^3 \ln 1/\varepsilon.
\]

**Proof.** Let \( B_{2\varepsilon}(x_0) \) be the ball of smallest radius such that \( \Omega_h \subset B_{2\varepsilon}(x_0) \) and set \( B = B_{2\varepsilon}(x_0) \cap \Omega \). We have at once that
\[
\|\nabla^i (\tilde{v} - v)\|_{L_1(\Omega)} \leq \|\nabla^i (\tilde{v} - v)\|_{L_1(B)} + \|\nabla^i (\tilde{v} - v)\|_{L_1(\Omega \setminus B)} \quad \text{for } i = 0, 1,
\]
and, by (3.11) and a change of order of integration,
\[
\|\nabla^i (\tilde{v} - v)\|_{L_1(B)} \leq \sup_{y \in \Omega_h} \|\nabla^i g(\cdot, y)\|_{L_1(B)} \|\Phi - \Phi\|_{L_1(\Omega_h)} \leq C \varepsilon^{2-i} \ln(1/\varepsilon)^{N(1-i)} \left( \|\Phi\|_{L_1(\Omega_h)} + \|\Phi\|_{L_1(\Omega_h)} \right) \leq C \varepsilon^{3-i} \ln(1/\varepsilon)^{N(1-i)} \quad \text{for } i = 0, 1.
\]

Replacing \( g(x, \cdot) \) by its expansion
\[
g(x, y) = g(x, x_0) + (y - x_0) \nabla_y g(x, x_0) + \frac{1}{2} (y - x_0)^2 \nabla_y^2 g(x, \eta)(y - x_0),
\]
where \( \eta = \theta x_0 + (1 - \theta)y \), \( 0 \leq \theta \leq 1 \), and \( t \) denotes transpose, and using the fact that \( \Phi - \Phi \) is orthogonal on \( \Omega_h \) to the linear part of the expansion, we see from (3.10) that
\[
(\tilde{v} - v)(x) = \int_{\Omega_h} (y - x_0) \nabla_y^2 g(x, \eta)(y - x_0)^t (\Phi(y) - \Phi(y)) \, dy.
\]

Hence, again by (3.11),
\[
|\nabla^i (\tilde{v} - v)(x)| \leq C \varepsilon^2 \|\nabla_x^i \nabla_y^2 g(x, \cdot)\|_{L_\infty(\Omega_h)} \left( \|\Phi\|_{L_1(\Omega_h)} + \|\Phi\|_{L_1(\Omega_h)} \right) \leq C \varepsilon^3 |x - x_0|^{-N-i} \quad \text{for } x \in \Omega \setminus B, i = 0, 1.
\]
Integration over $\Omega \setminus B$ shows that
\[ \| \nabla' (\bar{v} - v) \|_{L^i(\Omega \setminus B)} \leq C e^{3i \ln(1/e)} \] for $i = 0, 1$.

This completes the proof of the lemma.

We close this section by noting that also the final claim on $\bar{v}$ is satisfied, namely that its Galerkin approximation is $v_h$. For by (3.1), (3.4), and the definition of $\bar{v}$, we have
\[ A(\bar{v} - v, \chi) = (L(\bar{v} - v), \chi) = (\bar{v} - v, \chi) = 0 \quad \text{for all } \chi \in S_h, \]
and by (1.3) and (1.4),
\[ A(v_h - v, \chi) = 0 \quad \text{for all } \chi \in S_h, \]
so that
\[ (3.14) \quad A(v_h - \bar{v}, \chi) = 0 \quad \text{for all } \chi \in S_h. \]

4. Proof of Theorem 1. In view of Lemma 6 and the fact that $\bar{u}_h - u = v_h - v$, it is sufficient to show that for the appropriate $\alpha$'s
\[ (4.1) \quad \| \nabla (v_h - \bar{v}) \|_{L^1(\Omega)} \leq C h^{r-1}, \]
and
\[ (4.2) \quad \| v_h - \bar{v} \|_{L^1(\Omega)} \leq C h^r, \]
respectively.

Given a positive constant $C_*$ let $J$ be determined by
\[ C_*/C_1 < h^1/(1-a) < 2C_*/C_1. \]
Thus, $h_j, d_j$, and $h^{1/(1-a)}$ are of the same order; but, by choosing $C_*$ sufficiently large, $h_j d_j^{-1}$ is suitably small, since
\[ (4.3) \quad h_j d_j^{-1} = h d_j^{a-1} \leq 1/C_1^{1-a}. \]
The constant $C_*$ will be determined later. For the moment we only require that the results of Lemma 1, Lemma 2, and (3.9) apply for $j \leq J$, i.e., that $C_* \geq C_1$.

In order to prove (4.1) we first use Schwarz's inequality to change from the $L^1$-norm to a weighted $L^2$-norm. Setting $e = v_h - \bar{v}$ and
\[ S = \sum_{j \leq J} d_j^{N/2} \| e \|_{1,D_j}, \]
we have
\[ \| \nabla e \|_{L^1(\Omega)} = \sum_{j \leq J} \| \nabla e \|_{L^1(D_j)} + \| \nabla e \|_{L^1(\Omega_{j+1})} \leq C S + d_j^{N/2} \| e \|_{1,D_{j+1}}. \]
We shall show that for $\alpha > (r-3)/(r-1)$ and a suitable choice of $C_*$,
\[ (4.4) \quad S \leq \frac{1}{2} S + C d_j^{N/2} \| e \|_{1,\Omega_j} + C h^{r-1}, \]
and
\[ (4.5) \quad d_j^{N/2} \| e \|_{1,\Omega} \leq C h^{r-1}. \]
Obviously, the desired result then follows.
By Lemma 2, we have
\begin{equation}
S \leq C \sum_{j < J} d_j^{N/2} \left( h_j^{-1} \| e \|_{r,D_j} + d_j^{r-1} \| e \|_{0,D_j} \right) + d_j^{N/2} \| e \|_{1,D_j}
\end{equation}
and by Lemma 5 and our assumption \( \alpha > (r - 3)/(r - 1) \),
\begin{equation}
\sum_{j < J} d_j^{N/2} h_j^{-1} \| e \|_{r,D_j} \leq C \sum_{j < J} h_j^{-1} d_j^{3-r} = Ch_j^{-1} \sum_{j < J} d_j^{(r-1)+3-r} \leq Ch_j^{-1}.
\end{equation}

In order to estimate \( \| e \|_{0,D_j} \), we use duality. Let \( e_j \) equal \( e/\| e \|_{0,D_j} \) on \( D_j \) and vanish outside \( D_j \), and let \( w \) solve
\begin{equation}
A(\psi, w) = (\psi, e_j) \quad \text{for all } \psi \in H^1(\Omega).
\end{equation}
Hence, \( \| e \|_{0,D_j} = (e, e_j) = A(e, w) \), and by (3.14) and Lemma 1,
\[ \| e \|_{0,D_j} = A(e, w - w_t) \leq C \sum_{i < J} \| e \|_{1,D_i} h_i \| w \|_{2,D_i} + C \| e \|_{1,A_j}, h_j \| w \|_{2,A_j}. \]
It is well known that problem (4.8) admits a unique solution \( w \) such that
\begin{equation}
\| w \|_{2,\Omega} \leq C |e_j|_{0,\Omega},
\end{equation}
and with the representation
\begin{equation}
w(x) = \int_\Omega g^*(x, y) e_j(y) \, dy,
\end{equation}
with a \( g^* \) (the Green's function for the adjoint problem) such that
\begin{equation}
|D^\beta g^*(x, y)| \leq C |x - y|^{-N} \quad \text{for } |\beta| \leq 2.
\end{equation}
Hence, for \( w \) we have the estimates
\[ \| w \|_{2,D_i} \leq Cd_i^{-N/2} d_j^{N/2} \quad \text{for } i \leq j, \]
\[ \| w \|_{2,A_i} \leq Cd_i^{-N/2} d_j^{N/2} \quad \text{for } i \geq j. \]
For \( i = j - 1, j \), and \( j + 1 \) this follows from (4.9), since \( e_j \) has \( L_2 \)-norm equal to one, and for the other \( i \)'s from the representation (4.10) and the estimate (4.11). We have thus the following estimate:
\begin{equation}
\| e \|_{0,D_j} \leq C d_j^{N/2} \sum_{i < J} \| e \|_{1,D_i} h_i d_i^{-N/2} + C d_j^{N/2} \sum_{j < i < J} \| e \|_{1,D_i} h_i d_i^{N/2}
\end{equation}
\[ + C \| e \|_{1,A_j}, h_j d_j^{N/2} d_j^{-N/2}. \]
Using obvious arguments, we obtain
\[ \| e \|_{0,D_j} \leq C d_j^{N/2} \max_i \left( h_i d_i^{-N} \right) \sum_{i < J} \| e \|_{1,D_i} d_i^{N/2} \]
\[ + Ch_j d_j^{N/2} \sum_{j < i < J} \| e \|_{1,D_i} d_i^{N/2} + C \| e \|_{1,A_j}, h_j d_j^{N/2} d_j^{-N/2} \]
\[ \leq Ch_j d_j^{-N/2} S + C \| e \|_{1,A_j}, h_j d_j^{N/2} d_j^{-N/2}, \]
and hence, using (4.3),
\[ C \sum_{j \in J} d_j^{N/2 - 1} \|e\|_{0, D_j} \leq CS \sum_{j \in J} h_j d_j^{-1} + C\|e\|_1, \Omega_j, h_j d_j^{N/2} \sum_{j \in J} d_j^{-1} \]
\[ \leq CSH_j d_j^{-1} + C\|e\|_1, \Omega_j, h_j d_j^{N/2} d_j^{-1} \leq \left( C/C_*^{-\alpha} \right) S + Cd_j^{N/2}\|e\|_1, \Omega_j, \]
For a suitable choice of \( C_* \) and together with (4.6) and (4.7) this shows (4.4).

We now prove (4.5). By the coercivity and the continuity of \( A(\cdot, \cdot) \), and by (3.14), we have
\[ \|e\|_1^2, \Omega \leq CA(e, e) = CA(e, \tilde{\varphi} - \chi) \leq C\|\tilde{\varphi} - \chi\|^2_1, \Omega \quad \text{for all} \ \chi \in S_h. \]
Lemma 1 then shows
\[ \|e\|_1^2, \Omega \leq C \sum_{j \in J} h_j^{2(r-1)}\|\tilde{\varphi}\|_{r, D_j}^2 + Ch_j^2\|\tilde{\varphi}\|_{2, \Omega_j, 1}^2, \]
and by Lemma 5,
\[ \|e\|_1^2, \Omega \leq C \sum_{j \in J} h_j^{2(r-1)}d_j^{2(-N/2 + 3 - r)} + Ch_j^2h_j^{(-N+2)/(1-\alpha)}(\ln 1/h_j) \frac{N}{h_j}, \]
since \( \varepsilon = h_1/(1-\alpha) \). But for \( \alpha > (r - 3)/(r - 1) \),
\[ \sum_{j_1 < j < J} h_j^{2(r-1)}d_j^{2(-N/2 + 3 - r)} \leq d_j^{-\alpha}h_j^{2(r-1)}d_j^{2\alpha(r-1) + 2(3-r)} \leq Cd_j^{-\alpha}h_j^{2(r-1)}, \]
and
\[ h_j^{2h_j^{(-N+2)/(1-\alpha)}(\ln 1/h_j) \frac{N}{h_j}} \leq C(C_*)d_j^{-\alpha}h_j^{4(1-\alpha)}(\ln 1/h_j) \frac{N}{h_j} \leq C(C_*)d_j^{-\alpha}h_j^{2(r-1)}. \]
Together these estimates show (4.5) which completes the proof of (4.1).

We now turn to the proof of (4.2). We have at once
\[ \|e\|_{L_1(\Omega)} \leq C \sum_{j \in J} d_j^{N/2}\|e\|_{0, D_j} + d_j^{N/2}\|e\|_{2, \Omega_j, 1}. \]
Applying (4.12) and changing order of summation we obtain
\[ \sum_{j \in J} d_j^{N/2}\|e\|_{0, D_j} \leq C \sum_{i \in J} \|e\|_{1, D_i, h_i, d_i^{-N/2}} \sum_{i \leq j \in J} d_i^{N} \]
\[ + C \sum_{i \in J} \|e\|_{1, D_i, h_i, d_i^{-N/2}} \sum_{j_1 < j < i} 1 + C\|e\|_{1, \Omega_j, h_j d_j^{N/2}} \sum_{j_1 < j \in J} 1, \]
and hence (for convenience we now assume that \( j_1 \geq 1 \)),
\[ \sum_{j \in J} d_j^{N/2}\|e\|_{0, D_j} \leq C \sum_{i \in J} ih_i d_i^{N/2}\|e\|_{1, D_i} + CJh_j d_j^{N/2}\|e\|_{1, \Omega_j, 1}. \]
We shall show that the single term in (4.14) can be estimated in the same way.
Repeating the arguments used to derive (4.12), we obtain
\[ \|e\|_{0, \Omega_j, 1} \leq C \sum_{i \leq J} \|e\|_{1, D_i, h_i, w_{2, D_i}} + C\|e\|_{1, \Omega_j, h_j, w_{2, \Omega_j}}, \]
where \( w \) now is the solution of the problem
\[ A(\psi, w) = (\psi, e_{j+1}) \quad \text{for all} \ \psi \in H^1(\Omega), \]
for an appropriate \( e_{j+1} \) with \( L^2 \)-norm equal to one. Hence by (4.9),
\[ d_j^{N/2}\|e\|_{0, \Omega_j, 1} \leq Cd_j^{N/2} \left( \sum_{i \in J} h_i\|e\|_{1, D_i} + h_j\|e\|_{1, \Omega_j, 1} \right), \]
which is an even better estimate than we required. We have thus shown that
\[ \|e\|_{L^1(\Omega)} \leq C \sum_{j \leq J} jh_j d_j^{N/2} \|e\|_{1,D_j} + CJh_j d_j^{N/2} \|e\|_{1,\Omega_j}. \]

Now, set
\[ S' = \sum_{j \leq J} jh_j d_j^{N/2} \|e\|_{1,D_j}. \]

We shall show that for \( \alpha > (r - 3)/r \) and a sufficiently large \( C_* \),
\[ S' \leq \frac{1}{2} S' + CJh_j d_j^{N/2} \|e\|_{1,\Omega_j} + Ch', \]
and
\[ Jh_j d_j^{N/2} \|e\|_{1,\Omega} \leq Ch'. \]
Clearly the desired result then follows.

By Lemma 2,
\[ S' \leq C \sum_{j \leq J} jh_j d_j^{N/2} \|\bar{\theta}\|_{r,D_j} + C \sum_{j \leq J} jh_j d_j^{N/2-1} \|e\|_{0,D_j} + Jh_j d_j^{N/2} \|e\|_{1,\Omega_j}, \]
and by Lemma 5 and our assumption \( \alpha > (r - 3)/r \),
\[ \sum_{j \leq J} jh_j d_j^{N/2} \|\bar{\theta}\|_{r,D_j} \leq Ch' \sum_{j_1 \leq j} \sum_{j \leq J} jh_j d_j^{N/2} \|e\|_{1,\Omega_j} \leq Ch'. \]
Using (4.12), we have
\[
\sum_{j \leq J} jh_j d_j^{N/2-1} \|e\|_{0,D_j} \leq C \sum_{i \leq J} \|e\|_{1,D_i} h_i d_i^{N-1} \sum_{i \leq J} jh_j d_j^{N-1} \\
+ C \sum_{i \leq J} \|e\|_{1,D_i} h_i d_i^{N/2} \sum_{j_1 \leq j < i} jh_j d_j^{N-1} \\
+ C \|e\|_{1,\Omega_j} h_j d_j^{N/2} \sum_{j_1 \leq j < i} jh_j d_j^{N-1},
\]
which shows that
\[ \sum_{j \leq J} jh_j d_j^{N/2-1} \|e\|_{0,D_j} \leq C \sum_{i \leq J} \|e\|_{1,D_i} h_i d_i^{N/2-1} + C \|e\|_{1,\Omega_j} h_j d_j^{N/2-1}, \]
and, hence, that
\[ C \sum_{j \leq J} jh_j d_j^{N/2-1} \|e\|_{0,D_j} \leq \left( C/C_*^{1-\alpha} \right) S' + CJh_j d_j^{N/2} \|e\|_{1,\Omega_j}. \]
For \( C_* \) sufficiently large our estimates now together show (4.15).
It remains to prove (4.16). Since for \( j \leq J \)
\[ Jh_j d_j^{N/2} \leq Jh_j d_j^{N/2}, \]
we obtain from (4.13) that
\[ J^2h_j^2 d_j^N \|e\|_{1,\Omega}^2 \leq C \sum_{j_1 \leq j \leq J} j^2h_j^2 d_j^{2(3-r)} + CJ^2h_j^2 d_j^N h^{(-N+2)/(1-\alpha)} \left( \ln \frac{1}{h} \right)^N, \]
so for $\alpha > (r - 3)/r$,

$$J^2 h^2 d_j^2 \|e\|_{1, \Omega}^2 \leq C h^{2r} \sum_{j < i} j^2 d_{ij}^{2\alpha + 2(3-r)} + C(C_*) J^2 h^{6/(1-\alpha)} \left(\ln \frac{1}{h}\right)^N \leq C h^{2r}.$$

This completes the proof of (4.2) and hence of Theorem 1.

5. Proof of Theorem 2. We shall show that for $\alpha > (r - 3)/r$,

$$|\tilde{u}_h(x) - u(x)| \leq C h^\alpha (\ln 1/h)^\beta |x - x_0|^{-N}.$$

In view of Theorem 1 and the fact that $\tilde{u}_h - u$ equals $v_h - v$, it is sufficient to show that

$$|v_h(x) - v(x)| \leq C h^\alpha (\ln 1/h)^\beta d^{-N} + Cd^{-N}\|v_h - v\|_{L^1(\Omega)}, \tag{5.1}$$

where $d = |x - x_0|$.

We first consider the case when $d > c h^{1/(1-\alpha)}$ and $c$ is a sufficiently large constant. Let $B$ be the intersection of $\Omega$ and $B_d(x_0)$, i.e., set

$$B = \{x \in \Omega : |x - x_0| < \frac{1}{2}d\}.$$

Following the arguments in [10] (cf. Corollary 5.1 of [10] and the remark below), we deduce that

$$|v_h(x) - v(x)| \leq C(h^\alpha)^\alpha \left(\ln \frac{1}{h}\right)^\beta \max_{|B| < \tau} \|D^p v\|_{L^1(B)} + Cd^{-N}\|v_h - v\|_{L^1(B)}. \tag{5.2}$$

Hence (5.1) follows from (3.3) and our assumption $\alpha > (r - 3)/r$.

In the case $d \leq c h^{1/(1-\alpha)}$ we first use Lemma 3. Thinking of $v(x)$ as a constant on $\tau \equiv x$ we obtain

$$|v_h(x) - v(x)| \leq C h^{-N/(1-\alpha)}\|v_h - v\|_{L^1(\tau)} \leq C h^{-N/(1-\alpha)} \left(\|v_h - v\|_{L^1(\tau)} + \|v - v(x)\|_{L^1(\tau)}\right).$$

It remains to show that

$$\|v - v(x)\|_{L^1(\tau)} \leq C h^{N/(1-\alpha)} h^d. \tag{5.3}$$

For $N > 2$ this follows at once from (3.3), since

$$\|v\|_{L^1(\tau)} + h^{N/(1-\alpha)}|v(x)| \leq C h^{N/(1-\alpha)} d^{-N} h^\tau.$$

For $N = 2$ we first note that for $y \in \tau$ and a suitable curve $S$ we have, using (3.3),

$$|v(y) - v(x)| = \left|\int_S v'(s) \, ds\right| \leq C h^{1/(1-\alpha)} \max\left(\frac{1}{d}, \ln \frac{1}{|y - x_0|}\right),$$

and hence

$$\|v - v(x)\|_{L^1(\tau)} \leq C h^{3/(1-\alpha)}\ln 1/d \leq C h^{2/(1-\alpha)} h^d d^{-2}.$$

This completes the proof.

Remark. For $x$ bounded away from $\Gamma$ the local estimate (5.2) follows at once from Corollary 5.1 in [10]. Following the arguments in [10] and using cut-off functions which satisfy an appropriate boundary condition (the vanishing of the conormal
Appendix.

Proof of (3.3): Recall that by definition \( v = u - u_0, L v = \Phi, \) and \( L \psi = \phi. \) We shall show that for, say, \( |\beta| \leq r \)

\[
|D^\beta v(x)| \leq \begin{cases} 
C \left( \left| x - x_0 \right|^{N+3-|\beta|} + 1 \right) & \text{if } |\beta| \neq 3 - N, \\
C \ln \left( \left| x - x_0 \right|^{-1} + 1 \right) & \text{if } |\beta| = 3 - N. 
\end{cases}
\]

Let \( x \neq x_0 \) be given and set \( d = |x - x_0|. \) Since \( v = u - u_0, \) and both \( u \) and \( u_0 \) are smooth functions away from \( x_0, \) it is sufficient to consider the case when \( d \) is suitably small. Let \( w \) be a smooth cut-off function such that

\[
\omega = 1 \text{ in } A_1, \quad \text{supp}(\omega) \subset \overline{A}_2, \quad \text{and} \quad \|D^\gamma \omega\|_{L^\infty(\Omega)} \leq C d^{-|\gamma|},
\]

where \( A_i = \{ y \in \Omega: (i + 1)^{-1}d < |y - x_0| < (i + 1)d \}. \) Let \( g(y, z) \) be the Green's function for \( L \) and \( I; \) i.e., let \( g \) be the solution of \( L^*_\gamma g(y, z) = \delta(z - y) \) in \( \Omega, \)

\[
l^*_\gamma g(y, z) = 0 \text{ on } \Gamma \text{ in } \Omega. \]

Using Green's formula and a splitting of \( \Phi \) we can write \( v \) as the sum of three terms

\[
v = \int_\Omega g \omega \Phi \, dx + \int_\Omega g(1 - \omega) \Phi \, dz + \int_\Gamma g \psi d\Gamma(z) = v_1 + v_2 + v_3,
\]

where the latter identity defines \( v_1, v_2, \) and \( v_3. \) For \( d \) sufficiently small, \( x \) is bounded away both from \( \Gamma \) and the support of \( (1 - \omega) \Phi, \) and thus, in a neighborhood of \( x, \)

we can differentiate \( v_2 \) and \( v_3 \) under the integral signs. Using (3.2), (3.11), and straightforward calculations, we obtain

\[
|D^\beta v_2(x)| \leq \int_\Omega |D^\beta g(x, z)| \left| (1 - \omega(z)) \Phi(z) \right| dz
\]

\[
\leq C \int_{\Omega \setminus A_1} \max \left( \ln|x - z|^{-1}, |x - z|^{N+2-|\beta|} \right)|z - x_0|^{-N+1} \, dz
\]

\[
\leq \begin{cases} 
C \left( d^{-N+3-|\beta|} + 1 \right) & \text{if } |\beta| \neq 3 - N, \\
C \ln 1/d & \text{if } |\beta| = 3 - N,
\end{cases}
\]

and

\[
|D^\beta v_3(x)| \leq \int_\Gamma |D^\beta g(x, z)| \left| \phi(z) \right| d\Gamma(z) \leq C.
\]

Similarly, we obtain for \( y \in \Omega \setminus A_3, \)

\[
|D^\nu v_1(y)| \leq \begin{cases} 
Cd^{-N+3-|\gamma|}(\ln 1/d)^\mu \quad & \text{if } |y - x_0| < d/4, \\
C|y - x_0|^{-N+2-|\gamma|} \left( \ln \left( |y - x_0|^{-1} + 1 \right) \right)^\mu \quad & \text{if } |y - x_0| > 4d,
\end{cases}
\]

where \( \mu = 1 \) if \( |\gamma| = 2 - N \) and \( \mu = 0 \) otherwise. In order to estimate \( D^\beta v_1(x) \) we proceed as follows: For \( |\beta| \leq 1 \) we have at once that

\[
|D^\beta v_1(x)| \leq \|D^\beta g(x, \cdot)\|_{L^1(A_2)} \|\omega \Phi\|_{L^\infty(A_2)}
\]

\[
\leq Cd^{-|\beta|}(\ln 1/d)^\mu (1 - |\beta|) d^{-N+1} \leq C \left( d^{-N+3-|\beta|} + 1 \right).
\]
For $|\beta| > 1$, set $D^{\beta} = D^{\sigma}D^{\mu}$ and $w = D^{\sigma}v_1$ for some $\sigma$ and $\mu$ such that $|\sigma| = 1$. Since $Lw = D^{\mu}(\omega \Phi) + \eta$ for some $\eta = \sum_{|\gamma| < |\beta|} b_{\gamma}D^{\gamma}v_1$, where $b_{\gamma}$ are certain derivatives of the coefficients of $L$, we have

$$w = \int_{\Omega} gD^{\mu}(\omega \Phi) \, dx + \int_{\Omega} g\eta \, dz + \int_{\Gamma} glw \, d\Gamma(z) = w_1 + w_2 + w_3.$$  

Here

$$|D^{\sigma}w_1(x)| \leq \|D^{\sigma}g(x, \cdot)\|_{L^1(\Omega)}\|D^{\mu}(\omega \Phi)\|_{L^\infty(\Omega)} \leq Cd^{-N+1-|\mu|} = Cd^{-N+3-|\beta|},$$

and

$$|D^{\sigma}w_2(x)| \leq \int_{\Gamma} |D^{\sigma}g(x, z)| |lw(z)| \, d\Gamma(z) \leq C,$$

since $lw$ is smooth. In order to estimate $D^{\sigma}w_3(x)$ we first use (6.1) to obtain

$$\int_{\Omega \setminus A_3} D^{\sigma}g(x, z) \eta(z) \, dz \leq Cd^{-N+4-|\beta|},$$

and hence

$$|D^{\sigma}w_2(x)| \leq Cd^{-N+4-|\beta|} + Cd \sup_{|\gamma| < |\beta|, y \in A_3} |D^{\gamma}v_1(y)| .$$

We have thus shown

$$|D^{\beta}v_1(x)| = |D^{\sigma}w(x)| \leq Cd^{-N+3-|\beta|} + Cd \sup_{|\gamma| < |\beta|} |D^{\gamma}v_1(y)| .$$

Since we may as well assume that the supremum is attained for $\gamma = \beta$ and $y = x$, and since $d$ is small, it follows that

$$|D^{\beta}v_1(x)| \leq Cd^{-N+3-|\beta|}.$$  

This completes the proof.

Department of Mathematics
Chalmers University of Technology and
The University of Göteborg
S-412 96 Göteborg, Sweden


