On the Differential-Difference Properties of the Extended Jacobi Polynomials

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Abstract. We discuss differential-difference properties of the extended Jacobi polynomials

\[ P_n(x) = \,_{p+2} F_q \left( \begin{array}{c} -n, n + \lambda, a_p \\ b_q \end{array} \right) x \quad (n = 0, 1, \ldots). \]

The point of departure is a corrected and reformulated version of a differential-difference equation satisfied by the polynomials \( P_n(x) \), which was derived by Wimp (Math. Comp., v. 29, 1975, pp. 577–581).

1. Introduction. Here we are concerned with the properties of the extended Jacobi polynomials [3, Vol. 1, Section 7.4],

\[
P_n(x) = \frac{(-n)_k (n + \lambda)_k (a_p)_k}{k! (b_q)_k} x^k \quad (n = 0, 1, \ldots),
\]

where

\[ (\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} \]

is Pochhammer’s symbol and the above contracted notation will be used throughout the paper:

\[ f(a_p) = \prod_{i=1}^{p} f(a_i), \quad f(b_q) = \prod_{j=1}^{q} f(b_j). \]

\( f \) being a given function. We assume that no \( a_i \) equals any \( b_j \) (\( i = 1, 2, \ldots, p; j = 1, 2, \ldots, q \)) and set

\[ b_j = 1 \quad \text{for} \quad j = q + 1. \]

Let

\[ \sigma = \max \{ p + 1, q \}. \]

Wimp ([5]; see also [3, Vol. 2, Section 12.2]) has proved the following.
Theorem 1.1. Let \( \lambda, a_i, b_j (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q + 1) \) be complex constants such that none of the quantities \( \lambda, \lambda + 1 - b_j (j = 1, 2, \ldots, q) \) are negative integers or zero. Then the polynomials \( P_n(x) \), see (1), satisfy the difference equation

\[
\sum_{m=0}^{\sigma+1} \left[ C_m(n; \sigma + 1) + xD_m(n; \sigma + 1) \right] P_{n-m}(x) = 0 \quad (n \geq \sigma + 1),
\]

where

\[
C_m(n; \sigma) = \frac{(n - m + 1)_m(2n + \lambda - 2m)_m(n - m - 1 + b_{q+1})}{m!(n + \lambda - m)_m(2n + \lambda - m - t)_m(n - 1 + b_{q+1})}
\times {}^3 F_2 \left( \begin{array}{c} -m, 2n + \lambda - m - t, n - m + b_{q+1} \\ 2n + \lambda - 2m + 1, n - m - 1 + b_{q+1} \end{array} \right) 1 
\]

\[
D_m(n; \sigma) = \frac{(n - m + 1)_m(2n + \lambda - 2m)_m(n - m + a_p)}{\Gamma(m)(n + \lambda - m)_m(2n + \lambda - m - t + 1)_m(n - 1 + b_{q+1})}
\times {}^2 F_1 \left( \begin{array}{c} 1 - m, 2n + \lambda - m - t + 1, n - m + 1 + a_p \\ 2n + \lambda - 2m + 1, n - m + a_p \end{array} \right) 1 
\]

Moreover, these polynomials do not satisfy a nontrivial equation of the form (4) of lower order than \( \sigma + 1 \).

The next theorem supplements Theorem 1.1 in a certain particular case.

Theorem 1.2 [2]. If \( p + 1 = q \) and \( x = 1 \) in (1) then the sequence \( P_n(1) \) satisfies the recurrence relation

\[
\sum_{m=0}^{\sigma} \left[ C_m(n; \sigma) + D_m(n; \sigma) \right] P_{n-m}(1) = 0 \quad (n \geq \sigma).
\]

The above result contains as a special case the recurrence relation given by Bailey [1] for

\[
{}^3 F_2 \left( \begin{array}{c} -n, n + \lambda, a_1 \\ b_1, b_2 \end{array} \right) 1.
\]

In this paper, our attention is focussed on another result of Wimp which is contained in the following theorem.

Theorem 1.3 ([6]; see also [4, Section 5.13]). Let \( \lambda \neq 1, 2, \ldots \) and let the assumptions of Theorem 1.1 be satisfied. Then the polynomials \( P_n(x) \) satisfy the differential-difference equation

\[
x(\delta x - \epsilon) \frac{d}{dx} P_n(x) = \sum_{m=0}^{\sigma} \left[ A_m(n) + xB_m(n) \right] P_{n-m}(x),
\]
where

\[
\delta = \begin{cases} 
1 & (p + 1 \geq q), \\
0 & (p + 1 < q), 
\end{cases} \quad \epsilon = \begin{cases} 
0 & (p + 1 > q), \\
1 & (p + 1 \leq q), 
\end{cases}
\]

and

\[
A_m(n) = \begin{cases} 
\alpha_m(n) \left[ \frac{1}{m!} \sum_{k=0}^{m} \frac{(-m)_k (n-k-1+b_{q+1})}{k!(2n+\lambda-k-\sigma)_{\sigma+1-m}} - \epsilon \right] & (m > 0), \\
\omega(n) - \epsilon \alpha_0(n) & (m = 0), 
\end{cases}
\]

\[
B_m(n) = \begin{cases} 
\alpha_m(n) \left[ \frac{1}{\Gamma(m)} \sum_{k=0}^{m-1} \frac{(1-m)_k (n-k-1+a_p)}{k!(2n+\lambda-k-\sigma)_{\sigma-m}} \right] & (m > 0), \\
\delta \alpha_0(n) & (m = 0), 
\end{cases}
\]

Moreover, no equation of the type (9) of lower order \( \sigma' < \sigma \) exists.

Note that the formulae (5) and (6) of the paper [6], defining the coefficients \( A_m \) and \( B_m \), respectively, are not correct as can be seen by considering the case of \( p + 1 = q = 2 \) and \( x = 1 \) and observing that the resulting second-order (pure) difference equation for quantities (8) disagrees with that given by Bailey [1]. An inspection of Wimp's proof of the equation (9) (see [6, esp. the last paragraph of Section II]) reveals how the formulae should be corrected.

Wimp's theorem was reproduced in the book [4] (see Theorem 2 in Section 5.13.2). Unfortunately, in the Russian edition of that book (Mir Publs., Moscow, 1980) the list of errors was increased by six other misprints.

2. Alternative Forms for \( A_m \) and \( B_m \). In Theorem 2.1, below, we give some alternative forms for the coefficients \( A_m \) and \( B_m \), see (1.11) and (1.12), respectively. We need the following lemma.

**Lemma 2.1.** Let \( m, r, s \) be integers \( \geq 0 \). Let none of the complex constants \( \gamma, c_i \) \((i = 1, 2, \ldots, r)\) be integers. Then the identity

\[
\sum_{k=0}^{m} \frac{(-m)_k (c_r - k)}{k!(\gamma - s - k)_{s+1-m}} = (-1)^m \frac{(\gamma - 2m + 1)(c_r - m)}{(\gamma - s - m)_{s+m}}
\]

holds.
Proof. Making use of some properties of Pochhammer's symbol [3, Vol. 1, Section 2.1], we obtain

\begin{align*}
(-1)^m & \frac{(\gamma - s - m)_{s+m}}{(\gamma - 2m + 1)_{2m-1}(c_r - m)} \sum_{k=0}^{m} \frac{(-m)_k (c_r - k)}{k! (\gamma - s - k)_{s+1-m}} \\
& = (-1)^m \sum_{k=0}^{m} \frac{(-m)_k (\gamma - s - m)_{m-k} (c_r - k)}{k! (\gamma - 2m + 1)_{m-k}(c_r - m)} \\
& = (-1)^m \sum_{k=0}^{m} \frac{(-m)_{m-k} (\gamma - s - m)_k (c_r - m + k)}{(m-k)! (\gamma - 2m + 1)_k (c_r - m)} \\
& = \sum_{k=0}^{m} \frac{(-m)_k (\gamma - s - m)_k (c_r - m + 1)_k}{k! (\gamma - 2m + 1)_k (c_r - m)} \\
& = \sum_{r+2} F_{r+1}^r \left( \begin{array}{c}
-m, \gamma - s - m, c_r - m + 1 \\
\gamma - 2m + 1, c_r - m
\end{array} \right). \quad \square
\end{align*}

**Theorem 2.1.** The equations (1.11) and (1.12) can be rewritten in the form

\begin{align*}
(2) & \quad A_m(n) = \omega(n) C_m(n; \sigma) - \epsilon \alpha_m(n), \\
(3) & \quad B_m(n) = \omega(n) D_m(n; \sigma) + \delta \alpha_m(n)
\end{align*}

respectively. Here the notation is that of (1.3), (1.5), (1.6), (1.10), (1.13) and (1.14).

Proof. Putting \( r = q + 1, \gamma = 2n + \lambda, s = \sigma, c_i = n - 1 + b_i (i = 1, 2, \ldots, q + 1) \) in (1), we obtain

\begin{align*}
\sum_{k=0}^{m} \frac{(-m)_k (n - k - 1 - b_{q+1})}{k! (2n + \lambda - \sigma - k)_{\sigma+1-m}} = (-1)^m \frac{(2n + \lambda - 2m + 1)_{2m-1}(n - m - 1 + b_{q+1})}{(2n + \lambda - \sigma - m)_{\sigma+m}} \\
\times \sum_{q+2}^r F_{q+1}^r \left( \begin{array}{c}
-m, 2n + \lambda - m - \sigma, n - m + b_{q+1} \\
2n + \lambda - 2m + 1, n - m - 1 + b_{q+1}
\end{array} \right) \\
= \frac{m! \omega(n)}{\alpha_n(n)} C_m(n; \sigma).
\end{align*}

Now, it readily follows that the first part of (1.11) may be written in the form (2). Obviously, the second part of (1.11) can be written as

\[ A_0(n) = \omega(n) C_0(n; \sigma) - \epsilon \alpha_0(n) \]

because \( C_0(n; \sigma) = 1 \). Proceeding in a similar fashion, one arrives at (3). \( \square \)

Note that if \( p + 1 = q \) and \( x = 1 \) then the equation (1.9) takes the form (1.7) as, in view of (2) and (3), we have

\[ A_m(n) + B_m(n) = \omega(n) [C_m(n; \sigma) + D_m(n; \sigma)]. \]

3. **Further Differential-Difference Equations.** With the aid of Theorems 1.3 and 2.1 we derive further differential-difference equations satisfied by the polynomials \( P_s(x) \). We require one more lemma.
Lemma 3.1. We have

\begin{enumerate}
\item \(C_m(n; \sigma) + \theta(n) C_{m-1}(n-1; \sigma) = C_m(n; \sigma + 1)\) \((m = 1, 2, \ldots, \sigma + 1)\),
\item \(C_{\sigma+1}(n; \sigma) = e\alpha_{\sigma+1}(n)/\omega(n)\),
\item \(D_m(n; \sigma) + \theta(n) D_{m-1}(n-1; \sigma) = D_m(n; \sigma + 1)\) \((m = 1, 2, \ldots, \sigma + 1)\),
\item \(D_{\sigma+1}(n; \sigma) = -\delta\alpha_{\sigma+1}(n)/\omega(n)\),
\end{enumerate}

where

\begin{equation}
\theta(n) = \frac{n\omega(n-1)}{(n+\lambda-1)\omega(n)},
\end{equation}

and the notation used is that of (1.3), (1.5), (1.6), (1.10), (1.13) and (1.14).

Proof. Equations (1) and (3) can be checked by a straightforward calculation, using the definitions (1.5) and (1.6), respectively, and (5).

We prove the formula (2). We have

\[
C_{\sigma+1}(n; \sigma) = \frac{(n-\sigma)_{\sigma+1}(2n+\lambda-2\sigma-2)_{2\sigma+2}(n-\sigma-2+bq+1)}{(\sigma+1)!} \frac{(n+\lambda-\sigma-1)_{\sigma+1}(2n+\lambda-2\sigma-1)_{\sigma+1}(n-1+bq+1)}{f},
\]

where

\[
f = {}_{q+2}F_{q+1} \left( \begin{array}{c}
-n-\sigma-1+bq+1 \\
n-\sigma-2+bq+1
\end{array} \right)
\]

(see (1.5)). Now, observe that

\[
f = \begin{cases} 
\frac{(-1)^{\sigma+1}(\sigma+1)!}{(n-\sigma-2+bq+1)} & (\sigma = q \geq p + 1), \\
0 & (\sigma = p + 1 > q)
\end{cases}
\]

(cf. [3, Vol. 1, Section 2.9]). Thus, for arbitrary \(p\) and \(q\),

\[
C_{\sigma+1}(n; \sigma) = e(-1)^{\sigma+1} \frac{(n-\sigma)_{\sigma+1}(2n+\lambda-\sigma)_{\sigma}(2n+\lambda-2-\sigma)}{(n+\lambda-\sigma-1)_{\sigma+1}(n-1+bq+1)}
\]

\[
= e\alpha_{\sigma+1}(n)/\omega(n).
\]

Equation (4) can be proved in a similar way. \(\Box\)

Theorem 3.1. Under the assumptions of Theorem 1.3, the polynomials \(P_n(x)\) satisfy the equations

\begin{enumerate}
\item \[x \left[ \frac{dP_n(x)}{dx} + \frac{n}{n+\lambda-1} \frac{dP_{n-1}(x)}{dx} \right] = n [P_n(x) - P_{n-1}(x)],\]
\item \[x \frac{dP_n(x)}{dx} = \sum_{k=1}^{n} (-1)^{k}(n-k) (2n+\lambda-2k) \frac{P_{n-k}(x) - nP_{n}(x)}{(1-\lambda-n)_k}.
\]
\end{enumerate}
Proof. First observe that Eq. (1.9) can, by virtue of Theorem 2.1, be written in the form

\[
\begin{aligned}
&x(8x - \varepsilon)^4 = (8x - \varepsilon) \sum_{m=0}^{\sigma} \alpha_m(n) P_{n-m}(x) \\
&\quad + \omega(n) \sum_{m=0}^{\sigma} \left[ C_m(n; \sigma) + xD_m(n; \sigma) \right] P_{n-m}(x).
\end{aligned}
\]

(8)

Now, replace in the above equation \( n \) by \( n - 1 \), multiply the resulting equation by \( n/(n + \lambda - 1) \) and add to (8). The result is

\[
\begin{aligned}
x(8x - \varepsilon) &\left[ \frac{dP_n(x)}{dx} + \frac{n}{n + \lambda - 1} \frac{dP_{n-1}(x)}{dx} \right] \\
&= (8x - \varepsilon) \left\{ \sum_{m=0}^{\sigma} \alpha_m(n) P_{n-m}(x) + \frac{n}{n + \lambda - 1} \sum_{m=1}^{\sigma+1} \alpha_{m-1}(n-1) P_{n-m}(x) \right\} \\
&\quad + \omega(n) \left\{ \sum_{m=0}^{\sigma} \left[ C_m(n; \sigma) + xD_m(n; \sigma) \right] P_{n-m}(x) \\
&\quad + \theta(n) \sum_{m=1}^{\sigma+1} \left[ C_{m-1}(n-1; \sigma) + xD_{m-1}(n-1; \sigma) \right] P_{n-m}(x) \right\}.
\end{aligned}
\]

(9)

Using Lemma 3.1 and considering that \( C_0(n; \sigma) = 1, \ D_0(n; \sigma) = 0 \), we write the right-hand side of (9) in the form

\[
(8x - \varepsilon) \sum_{m=0}^{\sigma+1} \beta_m(n) P_{n-m}(x)
\]

(10)

in which

\[
\beta_m(n) = \begin{cases} 
\alpha_m(n) + \frac{n}{n + \lambda - 1} \alpha_{m-1}(n-1) & (m > 0), \\
\alpha_0(n) & (m = 0),
\end{cases}
\]

or, in view of (1.13),

\[
\beta_m(n) = \begin{cases} 
n & (m = 0), \\
n & (m = 1), \\
0 & (m > 1).
\end{cases}
\]

By virtue of Theorem 1.1, the second sum of (10) is zero, therefore the right-hand side of (9) reduces to

\[
(8x - \varepsilon)n \left[ P_n(x) - P_{n-1}(x) \right],
\]

and Eq. (6) follows.
Formula (6) is a first-order difference equation with respect to $U_n(x) = xP'_n(x)$. Using the well-known formula for the general solution of such an equation and remembering that $U_0(x) = 0$ (see (1.1)), we get (7). □

Of course, Eqs. (6) and (7) provide a generalization of the classical differential-difference formulae for the Jacobi polynomials, see [7, p. 262].

Introducing the notation

$$Q_n(x) = (-1)^n(n\lambda)_nP_n(x)/n!$$

yields somewhat simpler forms of (6) and (7):

\begin{align*}
  x[Q'_n(x) - Q'_{n-1}(x)] &= nQ_n(x) + (n + \lambda - 1)Q_{n-1}(x),
  \\
  xQ'_n(x) &= \sum_{k=0}^{n-1} (2k + \lambda)Q_k(x) + nQ_n(x).
\end{align*}

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