

Further Extensions of a Legendre Function Integral

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Abstract. The integral

$$\int_z^1 \left(\frac{1-t}{2}\right)^{\beta-1} \left(\frac{1-t}{1+t}\right)^{\mu/2} \ln\left(\frac{1-t}{2}\right) P_{\nu-1}^{\mu}(t) dt$$

is evaluated as a hypergeometric function for arbitrary values of “ ν ”, “ μ ”, $-1 \leq z \leq 1$, and $\operatorname{Re}(\beta) > 0$.

In this paper, we obtain expressions for integrals of the type

$$(1) \quad \int_z^1 \left(\frac{1-t}{2}\right)^{\beta-1} \left(\frac{1-t}{1+t}\right)^{\mu/2} \ln\left(\frac{1-t}{2}\right) P_{\nu}^{\mu}(t) dt,$$

where $P_{\nu}^{\mu}(z)$ is the Legendre function, $-1 \leq z \leq 1$, and $\operatorname{Re}(\beta) > 0$. The relations thus obtained include as special cases results given previously by Blue [1], Gautschi [2], Ainsworth and Liu [3], [6], Gatteschi [7], and the present author [4].

The starting point of the development is the beta-transform for the hypergeometric function ${}_pF_q(z)$ as given in [5, Art. 5.2.3]:

$$(2) \quad \frac{\Gamma(\beta + \gamma)}{\Gamma(\beta)\Gamma(\gamma)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-1} {}_pF_q\left(\begin{matrix} \alpha_p \\ \rho_q \end{matrix} \middle| \zeta t\right) dt = {}_{p+1}F_{q+1}\left(\begin{matrix} \beta, \alpha_p \\ \beta + \gamma, \rho_q \end{matrix} \middle| \zeta\right)$$

which, with an obvious change of variables, may be written

$$(3) \quad \begin{aligned} & \frac{\Gamma(\beta + \gamma)}{\Gamma(\beta)\Gamma(\gamma)} \int_z^1 (1-x)^{\beta-1}(x-z)^{\gamma-1} {}_pF_q\left(\begin{matrix} \alpha_p \\ \rho_q \end{matrix} \middle| \frac{1-x}{2}\right) dx \\ & = (1-z)^{\beta+\gamma-1} {}_{p+1}F_{q+1}\left(\begin{matrix} \beta, \alpha_p \\ \beta + \gamma, \rho_q \end{matrix} \middle| \frac{1-z}{2}\right), \end{aligned}$$

and since

$$(4) \quad \Gamma(1-\mu) P_{\nu-1}^{\mu}(x) = \left(\frac{1+x}{1-x}\right)^{\mu/2} {}_2F_1\left(\begin{matrix} \nu, 1-\nu \\ 1-\mu \end{matrix} \middle| \frac{1-x}{2}\right),$$

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(3) gives

$$(5) \quad \begin{aligned} \Gamma(1 - \mu) \int_z^1 \left(\frac{1-x}{2} \right)^{\beta-1} \left(\frac{x-z}{2} \right)^{\gamma-1} \left(\frac{1-x}{1+x} \right)^{\mu/2} P_{\nu-1}^\mu(x) dx \\ = 2 \left(\frac{1-z}{2} \right)^{\beta+\gamma-1} \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} {}_3F_2 \left(\begin{matrix} \beta, \nu, 1-\nu \\ \beta+\gamma, 1-\mu \end{matrix} \middle| \frac{1-z}{2} \right). \end{aligned}$$

Equation (5) may be specialized in a number of ways. For example, if $\beta = 1$, we get

$$(6a) \quad \begin{aligned} \Gamma(1 - \mu) \int_z^1 \left(\frac{x-z}{2} \right)^{\gamma-1} \left(\frac{1-x}{1+x} \right)^{\mu/2} P_{\nu-1}^\mu(x) dx \\ = \frac{2}{\gamma} \left(\frac{1-z}{2} \right)^\gamma {}_3F_2 \left(\begin{matrix} 1, \nu, 1-\nu \\ 1+\gamma, 1-\mu \end{matrix} \middle| \frac{1-z}{2} \right); \end{aligned}$$

$$(6b) \quad \begin{aligned} \Gamma(1 - \mu) \int_{-1}^1 \left(\frac{1+x}{2} \right)^{\gamma-1} \left(\frac{1-x}{1+x} \right)^{\mu/2} P_{\nu-1}^\mu(x) dx \\ = \frac{2}{\gamma} {}_3F_2 \left(\begin{matrix} 1, \nu, 1-\nu \\ 1+\gamma, 1-\mu \end{matrix} \middle| 1 \right), \end{aligned}$$

and, if, in addition, $\mu = 0$, the ${}_3F_2$ function reduces to an ${}_2F_1$, and we obtain the following:

$$(7a) \quad \int_z^1 \left(\frac{x-z}{2} \right)^{\gamma-1} P_{\nu-1}(x) dx = \frac{2}{\gamma} \left(\frac{1-z}{2} \right)^\gamma {}_2F_1 \left(\begin{matrix} \nu, 1-\nu \\ 1+\gamma \end{matrix} \middle| \frac{1-z}{2} \right);$$

$$(7b) \quad \begin{aligned} \int_{-1}^1 \left(\frac{x+1}{2} \right)^{\gamma-1} P_{\nu-1}(x) dx \\ = \frac{2}{\gamma} {}_2F_1 \left(\begin{matrix} \nu, 1-\nu \\ 1+\mu \end{matrix} \middle| 1 \right) = \frac{2[\Gamma(\gamma)]^2}{\Gamma(\gamma+\nu)\Gamma(\gamma-\nu+1)}, \end{aligned}$$

the latter being equivalent to Eq. (3) of [2].

Similarly, by taking $\gamma = 1$ in (5) we get

$$(8a) \quad \begin{aligned} \Gamma(1 - \mu) \int_z^1 \left(\frac{1-x}{2} \right)^{\beta-1} \left(\frac{1-x}{1+x} \right)^{\mu/2} P_{\nu-1}^\mu(x) dx \\ = \frac{2}{\beta} \left(\frac{1-z}{2} \right)^\beta {}_3F_2 \left(\begin{matrix} \beta, \nu, 1-\nu \\ 1+\beta, 1-\mu \end{matrix} \middle| \frac{1-z}{2} \right); \end{aligned}$$

$$(8b) \quad \Gamma(1 - \mu) \int_{-1}^1 \left(\frac{1-x}{2} \right)^{\beta-1} \left(\frac{1-x}{1+x} \right)^{\mu/2} P_{\nu-1}^\mu(x) dx = \frac{2}{\beta} {}_3F_2 \left(\begin{matrix} \beta, \nu, 1-\nu \\ 1+\beta, 1-\mu \end{matrix} \middle| 1 \right)$$

and it appears that, in general, (8a) and (8b) cannot be reduced or simplified except in the special case where $\mu = 0$, $z = -1$, and ν is an integer, n , in which instance, the integrals in (7b) and (8b) differ only in sign, leading to the relation

$$(9) \quad \frac{1}{\beta} {}_3F_2 \left(\begin{matrix} \beta, n, 1-n \\ 1+\beta, 1 \end{matrix} \middle| 1 \right) = \frac{(-1)^{n+1} [\Gamma(\beta)]^2}{\Gamma(\beta+n)\Gamma(\beta-n+1)},$$

a result also obtainable directly from formula 5.2.4(2) of [5].

On the other hand, if $\nu = n$ and μ is not a positive integer, the functions $P_{n-1}^\mu(x)$ and $P_{n-1}^{-\mu}(x)$ differ only by a numerical factor;

$$(10) \quad P_{n-1}^{-\mu}(-x) = (-1)^n \frac{\Gamma(n - \mu)}{\Gamma(n + \mu)} P_{n-1}^\mu(x)$$

which leads to the following, and not immediately evident, relation:

$$(11) \quad \begin{aligned} &(-1)^{n-1} [(1 - \mu)_{n-1}] {}_3F_2 \left(\begin{matrix} \beta, n, 1 - n \\ 1 + \beta, 1 - \mu \end{matrix} \middle| 1 \right) \\ &= [(1 + \mu)_{n-1}] {}_3F_2 \left(\begin{matrix} 1, n, 1 - n \\ 1 + \beta, 1 + \mu \end{matrix} \middle| 1 \right). \end{aligned}$$

The corresponding integrals involving the logarithmic term are obtained by differentiation with respect to “ β ” (or “ γ ”). However, this can, apparently, only be done explicitly in two special cases, one being that where the integral is expressible by gamma-functions, as in (7b). In this instance, the result becomes the one given by Gautschi and the present author ([2], [4]), and previously for integer “ ν ” by Ainsworth and Liu [3]. (Cf. [2, Eq. (1)].) The second case where explicit expressions are obtainable for the logarithmic integrals occurs when $\gamma = 1$ (Eqs. (8a), (8b)), since then we have

$$(12) \quad \begin{aligned} &-\beta^2 \frac{\partial}{\partial \beta} \left\{ \frac{1}{\beta} {}_3F_2 \left(\begin{matrix} \beta, \nu, 1 - \nu \\ 1 + \beta, 1 - \mu \end{matrix} \middle| \frac{1 - z}{2} \right) \right\} \\ &= {}_4F_3 \left(\begin{matrix} \beta, \beta, \nu, 1 - \nu \\ 1 + \beta, 1 + \beta, 1 - \mu \end{matrix} \middle| \frac{1 - z}{2} \right), \end{aligned}$$

the result being a generalization of one previously found by Ainsworth [6] for $z = -1, \mu = 0$, and integer values of ν and β . Equation (12) also provides an explicit expression for the ${}_4F_3$ function when $z = -1, \mu = 0$. Further, the above idea can be extended to the case where γ is a positive integer, m , by noting that expansion of the term $(1 - t)^{m-1}$ in the beta-transform (2) by the binomial theorem leads to the result**

$$(13) \quad \begin{aligned} &\frac{1}{\beta} {}_3F_2 \left(\begin{matrix} \beta, a, b \\ \beta + m, c \end{matrix} \middle| z \right) \\ &= \frac{\Gamma(\beta + m)}{\Gamma(\beta + 1)} \sum_{n=0}^{m-1} (-1)^n \binom{m-1}{n} \left(\frac{1}{\beta + n} \right) {}_3F_2 \left(\begin{matrix} \beta + n, a, b \\ \beta + n + 1, c \end{matrix} \middle| z \right). \end{aligned}$$

Additional observations are that, although (5), (6) and (8) were originally restricted to noninteger values of “ μ ”, they are easily modified to include integer values of this parameter by applying the limiting expression given by Eq. 6.3(12) of [5] to the hypergeometric function in (5), and that with the aid of the relation

$$(14) \quad (\sin \nu \pi) Q_{\nu-1}^\mu(x) = \frac{\pi}{2} \cos(\mu - \nu) \pi \frac{\Gamma(\nu + \mu)}{\Gamma(\nu - \mu)} P_{\nu-1}^{-\mu}(x) + \cos(\mu \pi) P_{\nu-1}^\mu(-x),$$

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(6b) and (8b) may be combined to give, for noninteger “ ν ” and “ μ ”, the following:

$$\begin{aligned}
 & \sin(\nu\pi) \int_{-1}^1 \left(\frac{1-t}{2}\right)^{\beta-1} \left(\frac{1-t}{1+t}\right)^{\mu/2} Q_{\nu-1}^{\mu}(t) dt \\
 (15) \quad & = \frac{\pi}{\beta} \cos(\mu + \nu) \pi \left\{ \frac{\Gamma(\nu + \mu)}{\Gamma(1 + \mu)\Gamma(\nu - \mu)} {}_3F_2 \left(\begin{matrix} 1, \nu, 1 - \nu \\ 1 + \beta, 1 + \mu \end{matrix} \middle| 1 \right) \right\} \\
 & + \frac{\cos(\mu\pi)}{\Gamma(1 - \mu)} \left\{ {}_3F_2 \left(\begin{matrix} \beta, \nu, 1 - \nu \\ 1 + \beta, 1 - \mu \end{matrix} \middle| 1 \right) \right\}.
 \end{aligned}$$

Finally, it may be noted that, since

$$(16) \quad P_n^{(p,q)}(2z - 1) = (-1)^n \frac{(q+1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, p + q + n + 1 \\ q + 1 \end{matrix} \middle| z \right),$$

the beta-transform (3) also provides a simple means of obtaining results relating to the Jacobi polynomials $P_n^{(p,q)}(z)$ which are more general than some of those given in [7].** For example, Eq. (1.3) (loc. cit.) may be generalized in this way to

$$\begin{aligned}
 (17) \quad & (-1)^n \frac{n!}{(q+1)_n} \int_{-1}^z (z-t)^{\alpha} (1+t)^{\beta+u} P_n^{(p,q)}(2t-1) dt \\
 & = (1+z)^{\alpha+\beta-1} {}_3F_2 \left(\begin{matrix} -n, p + q + n + 1, \beta + u + 1 \\ q + 1, \alpha + \beta + u + 2 \end{matrix} \middle| \frac{1+z}{2} \right)
 \end{aligned}$$

which, in the special case where $z = 1$, $p = \alpha$, $q = \beta$, can, with the aid of formula 5.2.4(2) of [5], be reduced to Eq. (1.5) of the first cited paper.

1. J. L. BLUE, “A Legendre polynomial integral,” *Math. Comp.*, v. 33, 1979, pp. 739–741.
2. W. GAUTSCHI, “On the preceding paper ‘A Legendre polynomial integral’ by James L. Blue”, *Math. Comp.*, v. 33, 1979, pp. 742–743.
3. O. R. AINSWORTH & C. K. LIU, “An application of Legendre functions in the inversion of a Hilbert matrix,” *J. Franklin Inst.*, v. 299, No. 4, 1975, pp. 297–299.
4. H. E. FETTIS, “Extension of a result of Ainsworth & Liu,” *J. Franklin Inst.*, v. 316, No. 2, 1983, p. 191.
5. Y. L. LUKE, *Mathematical Functions and Their Approximations*, Academic Press, New York, 1975.
6. O. R. AINSWORTH, private communication, 1982.
7. L. GATTESCHI, “On some orthogonal polynomial integrals,” *Math. Comp.*, v. 35, 1980, pp. 1291–1298.