Supplement to
Linear Multistep Methods for Volterra Integral
and Integro-Differential Equations

By P. J. van der Houwen and H. J. J. te Riele

In these appendices we present, successively,
I conditions for the existence of a unique solution of (1.1) and (1.2);
II three tables of coefficients of forward differentiation formulas, and
of two common LM formulas for ODEs, viz., backward differentiation
formulas and Adams-Moulton formulas;
III two lemmas which are needed in;
IV proofs of the main results of this paper, as far as they are non-
trivial (in the opinion of the authors).

APPENDIX I

Conditions for the existence of a unique solution $y(t) \in C(I)$ of (1.1) with
$\theta = 1$
- $K(t,r,y)$ is continuous with respect to $t$ and $r$, for all $(t,r) \in S$;
- $K$ satisfies a (uniform) Lipschitz condition with respect to $y$, i.e.,
  $|K(t,r,y) - K(t,r,z)| \leq L_1 |y-z|$, for all $(t,r) \in S$, for all finite
  $y,z \in \mathbb{R}$;
- $g(t) \in C(I)$.

Conditions for the existence of a unique solution $y(t) \in C(I)$ of (1.1) with
$\theta = 0$
- $K(t,r,y) \in C^1(S\times\mathbb{R})$;
- for $t = t_0$ the derivative $\partial K/\partial y$ is bounded away from zero:
  $|\partial K(t,t_0,y)/\partial y| \geq r_0 > 0$ for all $t \in I$, $y \in \mathbb{R}$;
- $\partial^2 K(t,t_0,y)/\partial y \partial t$ satisfies a (uniform) Lipschitz condition with respect to $y$
  on $S = \mathbb{R}$;
- $g(t) \in C^1(I)$ with $g(t_0) = 0$.

Conditions for the existence of a unique solution $y(t) \in C^1(I)$ of (1.2),
for given initial value $y(t_0) = y_0$
The following three (uniform) Lipschitz conditions:
- $|f(t,y_1,x) - f(t,y_2,x)| \leq L_2 |y_1-y_2|$, for all $t \in I$, for all finite
  $x,y_1,y_2 \in \mathbb{R}$;
Table 1  Coefficients of forward differentiation formulas

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Table 2  Coefficients of the backward differentiation formulas

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Table 3  Coefficients of the Adams-Moulton formulas

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APPENDIX II

**Lemma A.1.** Let \( x_n \geq 0 \) for \( n = 0, 1, \ldots, N \), and suppose that
\[
 x_n = h C_i \sum_{i=0}^{n-1} x_i + C_2, \quad n = k, k+1, \ldots, N,
\]
where \( k > 0, h > 0 \) and \( C_1 > 0 \) (\( i=1,2 \)). Suppose, moreover, that \( x_j \leq z/k \)
for \( j = 0, 1, \ldots, k-1 \). Then
\[
x_n \leq (h C_2 + C_2) (1 + h C_1)^{-k}, \quad n = k, k+1, \ldots, N.
\]

**Proof.** See [?].

**Lemma A.2.** Consider the linear inhomogeneous difference equation with constant coefficients \( \xi_j \):
\[
 (A.1) \quad \xi_0 x_{n+k} + \xi_1 x_{n+k-1} + \cdots + \xi_{N} x_{n} = \eta_{n+k}, \quad n \geq 0,
\]
where \( \{\eta_n\} \) is a given sequence, independent of the \( x_n \).
\((i) \text{ If } \{x_n\} \) is simple von Neumann (cf. Section 2.3) then the solution of (A.1) satisfies the inequality

\[
\]
\[ |y_n| \leq C \max_{j \leq k} |y_j| + k \max_{j \leq k} |g_j|, \quad n \geq k, \]
\[ 0 \leq g_j \leq 1 \]
where \( C \) is independent of \( n \).

(iii) If \( f(x) \) is Schur (cf. Section 2.3) then the solution of (A.1) satisfies the inequality
\[ |y_n| \leq C \max_{j \leq k} |y_j| + \max_{j \leq k} |g_j|, \quad n \geq k, \]
where \( C \) is independent of \( n \).

Proof. See [7].

Appendix IV

Proof of Theorem 2.2.1. Taylor expansion of \( Y(t_{n+1}, t_{n-1}) \) around \( (t_n, t_n) \)
yields
\[ \begin{align*}
\mathcal{L}_{n}(Y) &= \sum_{i=0}^{k} \left( \frac{1}{q_0} \right)^{i} \left( \frac{1}{q_0} \right)^{\frac{3}{38}} \hat{h}^{\left(-i - \frac{3}{38} - \frac{3}{38}\right)} Y(t, s) \\
&+ \sum_{j=k}^{p} \hat{h}^{\left(j - \frac{3}{38}ight)} \sum_{q=0}^{j-k} \left( \frac{1}{q_0} \right)^{\frac{3}{38}} \hat{h}^{\left(j - \frac{3}{38}ight)} Y(t, s) (t_n, t_n) + O(h^{p+1}) \\
&= O(h^{p+1}) \text{ as } h \to 0.
\end{align*} \]
Writing this formula in the form
\[ \mathcal{L}_{n}(Y) = \sum_{q=0}^{p} \frac{1}{q_0} \hat{h}^{\left(q - \frac{3}{38}\right)} \hat{h}^{\left(q - \frac{3}{38}\right)} Y(t, s) + O(h^{p+1}) \]
and expanding the differential operator \( D_q \) by the binomial theorem we find
\[ D_q = \sum_{i=0}^{k} \left( \frac{1}{q_0} \right)^{i} \left( \frac{1}{q_0} \right)^{\frac{3}{38}} \hat{h}^{\left(-i - \frac{3}{38} - \frac{3}{38}\right)} + \sum_{j=k}^{p} \left( \frac{1}{q_0} \right)^{\frac{3}{38}} \hat{h}^{\left(j - \frac{3}{38}\right)} \hat{h}^{\left(j - \frac{3}{38}\right)} \hat{h}^{\left(q - \frac{3}{38} - \frac{3}{38}\right)} \]
\[ = \sum_{i=0}^{k} \left( \frac{1}{q_0} \right)^{i} \left( \frac{1}{q_0} \right)^{\frac{3}{38}} \hat{h}^{\left(-i - \frac{3}{38} - \frac{3}{38}\right)} \hat{h}^{\left(i + q - \frac{3}{38}\right)} + \sum_{j=k}^{p} \left( \frac{1}{q_0} \right)^{\frac{3}{38}} \hat{h}^{\left(j - \frac{3}{38}\right)} \hat{h}^{\left(j - \frac{3}{38}\right)} \hat{h}^{\left(q - \frac{3}{38} - \frac{3}{38}\right)} \]
where \((-i)^{\ell - 1} \ell \) is assumed to be zero for \( i = \ell \). Equating to zero all terms in the above yields the order equations (2.2.3) and at the same time
\[ \mathcal{L}_{n}(Y) = O(h^{p+1}) \text{ as } h \to 0. \]

Proof of Theorem 2.2.2. Taylor expansion of \( Y(t_{n+1}, t_{n-1}) \) around \( (t_n, t_n) \)
yields
\[ \begin{align*}
Y(t_{n+1}, t_{n-1}) &= \sum_{q=0}^{p} \frac{1}{q_0} \hat{h}^{\left(q - \frac{3}{38}\right)} \hat{h}^{\left(q - \frac{3}{38}\right)} Y(t, s) (t_n, t_n) + O(h^{p+1}) \\
&= O(h^{p+1}) \text{ as } h \to 0.
\end{align*} \]
In order to exploit the fact that \( Y(t, t) \equiv 0 \) (see definition 2.2.1), we introduce the variables \( u = t + s \) and \( v = t - s \) and write
\[ Y(t, s) = Y(t_{u}^{2}, t_{v}^{2}) = Z(u, v). \]
The identity \( Y(t, t) \equiv 0 \) implies that \( Z \) and all its derivatives with respect to \( u \) vanish for \( u = 2t \) and \( v = 0 \). In the following we use the notation
\[ Z_{(n,m)} := \frac{d^{n+m}}{2^{u} 3^{v}} Z(2t, 0). \]
By means of the binomial theorem we have
\[ \begin{align*}
Y(t_{n+1}, t_{n-1}) &= \sum_{q=0}^{p} \frac{1}{q_0} \hat{h}^{\left(q - \frac{3}{38}\right)} \hat{h}^{\left(q - \frac{3}{38}\right)} Y(t, s) (t_n, t_n) + O(h^{p+1}) \\
&= \sum_{q=0}^{p} \frac{1}{q_0} \hat{h}^{\left(q - \frac{3}{38}\right)} \hat{h}^{\left(q - \frac{3}{38}\right)} \hat{h}^{\left(q - \frac{3}{38}\right)} Y(t_n, t_n) + O(h^{p+1}) \text{ as } h \to 0.
\end{align*} \]
From these expansions it is immediate that the VLM formula (2.1.4) satisfies the relation
\[ h^\tau \gamma(t_{n}, s) = \sum_{i=0}^{k} \frac{\partial f_n}{\partial x_i} \gamma(t_{n}, s) - N_{n}^t(t_{n}, s) + O(h^{n+1}) \]

where \( A_n \) and \( C_n \) are defined by (2.3.2) and (2.2.3), respectively. Under the conditions of the theorem it is easily verified that this equation leads to (2.3.3). Furthermore, (2.3.3) is obviously the \( m \)-times differentiated form of equation (1.1).

**PROOF OF THEOREM 2.3.1.**

**Proof.** Taylor expansion in a fixed point \( t = t_n \) yields, respectively,
\[
y(t_{n+1}) = h^\tau N_{n+1}^t(t_n) + O(h^{n+1}),
\]
\[
Y_{n+1}(t_{n+1}) = Y(t_{n+1}; t_n) - h^\tau N_{n+1}^t(t_n),
\]
\[
K_{n+1}(t_{n+1}) = K(t_{n+1}; t_n) - h^\tau Y(t_{n+1}; t_n) + O(h^{n+1}),
\]
\[
\begin{align*}
\frac{d}{dt} & \left( Y(t_{n+1}; t_n) \right) = \frac{1}{h^\tau} - \frac{d}{dt} Y(t_{n+1}; t_n) + O(h^{n+1}), \\
\frac{d}{dt} & \left( K(t_{n+1}; t_n) \right) = \frac{1}{h^\tau} - \frac{d}{dt} K(t_{n+1}; t_n) + O(h^{n+1}).
\end{align*}
\]

Substitution of the functions \( Y(t, s) \) and \( Y_n(t) \) using (2.1.3) and (2.3.6b) leads to
\[
\begin{align*}
Y(t_{n+1}; t_n) & = \sum_{i=0}^{k} \frac{\partial f_n}{\partial x_i} Y(t_{n+1}; t_n) - N_{n}^t(t_{n+1}, t_n) + O(h^{n+1}), \\
K(t_{n+1}; t_n) & = \sum_{i=0}^{k} \frac{\partial f_n}{\partial x_i} K(t_{n+1}; t_n) - Y(t_{n+1}; t_n) + O(h^{n+1}).
\end{align*}
\]
(A.5) \[ l_n(Y) = \frac{h}{i = 0} \left\{ y_{i+1} - y_{i} \right\} + \frac{h}{j = k} \left[ \frac{h}{k = 0} \sum_{i = 0}^{n-1} \frac{\partial^2 Y(t_{n+j} + \tau_Y(t_{n+j} + \tau_{n+j}))}{\partial t^2} \right] \]

Thus, we have found for the errors \( e_n \) the relation

(A.6) \[ \sum_{i = 0}^{k} a_i e_{n-i} = v_n, \quad n \geq k^*, \text{ where} \]

\[ v_n = l_n(Y) - \sum_{i = 0}^{k} \sum_{j = k}^{k} \left( \frac{h^2}{k = 0} \sum_{i = 0}^{n-1} \frac{\partial^2 Y(t_{n+j} + \tau_{n+j})}{\partial t^2} \right) \]

We now proceed with the two cases (a) and (b) separately.

(a) \( a(\xi) \equiv a_0 \xi^k, \quad a_0 \neq 0. \)

We want to apply the discrete Gronwall inequality stated in Lemma A.1 in order to derive an upper bound for the solution of this linear difference equation, and therefore we need an upper bound for \( |v_n| \). A straightforward calculation yields

(A.7) \[ |v_n| \leq T(h) + \frac{h}{i = 0} \sum_{j = k}^{k} \left( \frac{h^2}{k = 0} \sum_{i = 0}^{n-1} |c_{i+j}| + C_1 E(h) \right) \]

so that for \( h \) sufficiently small

\[ |e_n| \leq \frac{1}{T_0} \left[ \frac{d}{k = 0} \sum_{k = 0}^{n-1} |e_k| + C_1 E(h) + T(h) \right] \]

Application of Lemma A.1 (with \( k = k^*(h) \)) yields

\[ |e_n| \leq (1 + C_2) \frac{n}{k^*} \left( C_2 h C_2 + C_3 E(h) + T(h) \right), \]

since \( n h \leq T - T_0 \), part (a) of the theorem is immediate.

(b) \( a(\xi) \) is simple von Neumann, \( \beta(\xi) \equiv 0. \)

Instead of directly applying Lemma A.1 to the inequality (obtained from (A.6))

\[ \frac{h}{i = 0} \sum_{j = k}^{k} |a_i| |e_{n-i}| \leq |v_n|, \]

we first apply Lemma A.2 (i) to obtain the "sharper" inequality

(A.8) \[ |e_n| \leq C_0 (A(h) + \frac{h}{j = 0} |v_j|), \quad n \geq k^*. \]

Unfortunately, if we use the upper bound (A.7) for \( |v_j| \) and then apply Lemma A.1, we cannot prove convergence. However, by using the property \( \beta(\xi) \equiv 0 \), that is \( a_i = 0, i = k^* \), \( e_k = 0 \), a sharper upper bound than (A.7) can be derived. To that end we write
Since \( nh \leq T = t_0 \) we find for \( h \) sufficiently small

\[
|c_n| \leq C_2 h \sum_{i=0}^{n-1} |c_i| + C_2 h^{-1}(b_{1}(h) + \Delta E(h) + T(h))
\]

Finally, by applying Lemma A.1 we arrive at the estimate

\[
|c_n| \leq (1 + C_2 h)^{n-k} \left( k h C_2 \epsilon(h) + C_2 h^{-1}(b_{1}(h) + \Delta E(h) + T(h)) \right)
\]

from which part (b) of the theorem follows. \( \square \)

**PROOF OF THEOREM 2.3.4.** Following the first lines of the proof of Theorem 2.3.2 we obtain the following relation, analogous to (A.5), where

\[
K_{\epsilon} = K(\epsilon, \epsilon, \epsilon)
\]

\[
|c_n| \leq C_2 \left( (h) + h \sum_{i=0}^{n-1} |c_i| + h \sum_{i=0}^{n-1} |c_i| h^{-1} \Delta E(h) + h^{-1} T(h) \right)
\]

Substitution into (A.8) yields the inequality

\[
|c_n| \leq C_2 \left( (h) + h \sum_{i=0}^{n-1} |c_i| + h^{-1} \Delta E(h) + h^{-1} T(h) \right)
\]

It is easily verified that

\[
\sum_{i=0}^{n-1} |c_i| = (n+1) \sum_{i=0}^{n-1} |c_i|
\]

Hence,

\[
|c_n| \leq C_2 \left( (h) + h \sum_{i=0}^{n-1} |c_i| + nh^{-1} \Delta E(h) + nh^{-1} T(h) \right)
\]

where

\[
K_{\epsilon, \epsilon, \epsilon} = K(\epsilon, \epsilon, \epsilon)
\]

\[
h \sum_{i=0}^{n-1} |c_i| + nh^{-1} \Delta E(h) + nh^{-1} T(h)
\]

\[
h \sum_{i=0}^{n-1} |c_i| + nh^{-1} \Delta E(h) + nh^{-1} T(h)
\]

\[
h \sum_{i=0}^{n-1} |c_i| + nh^{-1} \Delta E(h) + nh^{-1} T(h)
\]

\[
h \sum_{i=0}^{n-1} |c_i| + nh^{-1} \Delta E(h) + nh^{-1} T(h)
\]
Since \( \gamma(z) \) is Schur, we may apply Lemma A.2 (ii) to (A.11) and find

\[
(A.12) \quad |c_n| \leq C(4h) + \max_{k \leq j \leq n} |v_j|, \quad n \geq k^*.
\]

where \( C \) (and all subsequent \( C_k \)) is independent of \( h \) and \( n \). So we have to find bounds on \( |v_j| \). Using the conditions of the theorem, we find

\[
|v_r| \leq C_1 \left( \sum_{i,j} |v_{ij}|(j+i)|\epsilon_{r-j-i}| \right) \left( \sum_{i,j} \left| \sum_{k \in \mathbb{C}} \frac{1}{\lambda_i \lambda_j} \sum_{\tau = 1}^{k} \epsilon_{r-j-i} \right| \epsilon_r \right) + C_2 \left( \sum_{i,j} |v_{ij}| \sum_{\tau = 1}^{k} \left| \sum_{k \in \mathbb{C}} \frac{1}{\lambda_i \lambda_j} \sum_{\tau = 1}^{k} \epsilon_r \right| \epsilon_r \right),
\]

where \( r \geq k^* \).

Now we use the condition \( \delta(z) = 0 \), i.e., \( \delta = 0 \), and (2.3,6a) to obtain (cf. the derivation of (A.9) in the proof of Theorem 2.3.2)

\[
|v_r| \leq C_3 \left( \sum_{i,j} \left| \sum_{\tau = 1}^{k} \epsilon_r \right| \right) + C_4 \left( \sum_{i,j} \left| \sum_{\tau = 1}^{k} \epsilon_r \right| \right), \quad r \geq k^*.
\]

Substituting this into (A.12) we find, for \( h \) sufficiently small,

\[
|c_n| \leq C_5 \left( \sum_{i,j} \left| \sum_{\tau = 1}^{k} \epsilon_r \right| \right) + C_6 \left( \sum_{i,j} \left| \sum_{\tau = 1}^{k} \epsilon_r \right| \right),
\]

and application of Lemma A.1 yields the result of the theorem. \( \square \)

**PROOF OF THEOREM 3.3.1.** Proceeding as in the proof of Theorem 2.3.2 we derive the relations

\[
\begin{align*}
|\epsilon_n| & \leq C_7 \left( \sum_{r=1}^{n-k} |\epsilon_r| \right) + C_8 \left( \sum_{r=1}^{n-k} |\epsilon_r| \right), \\
|\epsilon_n| & \leq C_9 \left( \sum_{r=1}^{n-k} |\epsilon_r| \right) + C_{10} \left( \sum_{r=1}^{n-k} |\epsilon_r| \right),
\end{align*}
\]

and applying Lemma A.2 yields the result of the theorem. \( \square \)
for some constant \( C_1 \). Substitution into (A.15) yields

\[
|\varepsilon_n| \leq C_2 \left\{ h \sum_{j=0}^{n} |\varepsilon_j| + h \sum_{j=0}^{n} |\varepsilon_j| + \sum_{j=0}^{n} T_j(h) + \sum_{j=0}^{n} \delta(h) + h^\delta(h) \right\} + T_0(h)
\]

\[
\leq C_3 \left\{ h \sum_{j=0}^{n} |\varepsilon_j| + E_n(h) + T_n(h) + h^{-1}T_n(h) + \delta(h) + h^\delta(h) \right\}
\]

where we have used that \( nh \leq T - t_0 \). From Lemma A.1, part (a) of the theorem easily follows.

(b) Since \( a(z) \) is simple von Neumann, we apply Lemma A.2 (i) to (A.16) and use (A.9) (since \( d(z); 0 \) to find

\[
|\varepsilon_n| \leq C_4 \left\{ \delta(h) + h \sum_{j=0}^{n} \left( |e_j| + |e_{j-1}| + h^\delta(h) + T_0(h) \right) \right\}
\]

\[
\leq C_5 \left\{ h \sum_{j=0}^{n} |\varepsilon_j| + \delta(h) + h^{-1}E_n(h) + h^{-1}T_n(h) \right\}.
\]

Substitution into (A.15) and applying Lemma A.1 leads to part (b) of the theorem.

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