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Computing Volumes of Polyhedra

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Abstract. In this note we give two simple methods for calculating the volume of any closed bounded polyhedron in $\mathbf{R}^n$ having an orientable boundary which is triangulated into a set of $(n-1)$-dimensional simplices. The formulas given require only coordinates of the vertices of the polyhedron.

1. Introduction. The purpose of this note is to give two simple methods for calculating the volume of any closed bounded polyhedron P in $\mathbf{R}^n$ having an orientable boundary ∂P which is triangulated into a set T of $(n-1)$-dimensional simplices. Following Hadwiger [2], we define a polyhedron to be the union of pairwise disjoint convex polyhedra, each of which is the convex hull of a finite number of points.

In [1] we have described an algorithm for obtaining a piecewise linear manifold which closely approximates an implicitly defined manifold. If P has been given in such a way, then the affine pieces of ∂P are in general easy to triangulate with an inherited orientation. For polyhedra P which are determined by a given system of inequalities, methods and programs for triangulating P have been given in [5], [6]. For such polyhedra, a triangulation of the boundary is not easily available, so our method is inappropriate. Of course, our approach would also be unnecessary for computing the volume of a parallelotope.

Practical applications of the methods given here may be made to the approximation of an area bounded by an implicitly defined curve or to approximation of the volume of a solid which is bounded by an implicitly defined surface.

Although some formulas for the volume of convex polyhedra in $\mathbf{R}^n$ appear in the literature (e.g., [3]), these formulas generally require the computation of the $(n-1)$-volume of the facets and additionally they involve extra computations of certain distances. The volume formulas we give here involve only the coordinates of the vertices of P.

2. Volume Formulas. Let us assume that the boundary ∂P is triangulated into $(n-1)$-simplices $\sigma \in T$ which are positively oriented relative to the outward normals to the facets in ∂P. For our purpose it is only necessary that the simplices are so oriented as to form a boundary chain (see [7]).

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Our first formula for the n-volume $V_n(P)$ of the polyhedron P derives from the classical formula for the volume of an n-simplex,

$$V_n(P) = \sum_{\sigma \in T} \frac{1}{n!} \det(v_1(\sigma) \cdots v_n(\sigma)). \tag{2.1}$$

Here the $v_i(\sigma)$ are the vertices of the $(n-1)$-simplex σ ordered according to the orientation of σ.

Each term in the sum in (2.1) represents the signed volume of an n-simplex $\tau(\sigma)$ (possibly degenerate) having one vertex at the origin and the remaining vertices being those of σ. The orientation of σ gives the same sign to the volume as that of the inner product of $b(\sigma)$, the position vector from the origin to the barycenter of σ, and $n(\sigma)$, the outward normal to σ.

Formula (2.1) is a special case of formula (17) on page 42 of Hadwiger [2], but we include its derivation for completeness. Let

$$\Sigma_+ = \left\{\sigma \in T | b(\sigma)^T n(\sigma) \geqslant 0\right\} \quad \text{and} \quad \Sigma_- = \left\{\sigma \in T | b(\sigma)^T n(\sigma) < 0\right\}.$$

Then $P = \text{closure}\{\bigcup_{\sigma \in \Sigma_+} \tau(\sigma) / \bigcup_{\sigma \in \Sigma_-} \tau(\sigma)\}$ and hence, due to the sign properties of the classes Σ_+, Σ_-,

$$V_n(P) = \sum_{\sigma \in \Sigma_+} V_n(\tau(\sigma)) + \sum_{\sigma \in \Sigma_-} V_n(\tau(\sigma))$$

and (2.1) follows.

Our second formula for $V_n(P)$ is a generalization of the trapezoidal rule for calculating the area of a polygon. We form the sum of the signed volumes of n-dimensional prisms $p(\sigma)$ each of which is bounded "above" by an $(n-1)$-simplex σ belonging to T, "below" by a coordinate plane, and laterally by the planes which are orthogonal to this coordinate plane and contain an $(n-2)$-face of σ. The resulting formula is

$$V_n(P) = (-1)^{n-1} \sum_{\sigma \in T} \left(\frac{1}{n} \sum_{i=1}^{n} v_i^n \right) \cdot \frac{1}{(n-1)!} \det \begin{pmatrix} 1 & \cdots & 1 \\ \hat{v}_1^n & & \hat{v}_n^n \end{pmatrix}, \tag{2.2}$$

where the v_i's are the ordered vertices of σ as discussed above. Here v_i^n is the nth coordinate of v_i and $\hat{v}_i^n$ is the projection of v_i into $\mathbf{R}^{n-1}$ obtained by deleting the nth coordinate from v_i.

Each term in (2.2) corresponds to the signed volume of a prism $p(\sigma)$, which is easily seen to be the product of its average height multiplied by the signed $(n-1)$-volume of its base. The orientation gives the same sign to this term as that of the inner product between $n(\sigma)$ and the unit vector orthogonal to the plane $x^n = 0$ in the direction of $b(\sigma)$.

The formula (2.2) can be verified in the same manner as (2.1), but with

$$\Sigma_+ = \left\{\sigma \in T | \left(b(\sigma)^T e_n\right)\left(n(\sigma)^T e_n\right) \geqslant 0\right\} \quad \text{and}$$

$$\Sigma_- = \left\{\sigma \in T | \left(b(\sigma)^T e_n\right)\left(n(\sigma)^T e_n\right) < 0\right\},$$

where e_n is the nth standard unit vector.

3. Computational Considerations. Before we discuss computational considerations related to Formulas (2.1) and (2.2) we interpret these formulas in the two-dimensional case where the area of a polygon is calculated. Suppose that the boundary of the polygon is the piecewise linear path formed by traversing the points $\{(x_i, y_i)\}_{i=1}^m$ in order.

Formula (2.1) is illustrated by Figure (i). The formula becomes

$$V_2(P) = \sum_{i=1}^{m} \frac{1}{2} \det\begin{vmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{vmatrix} = \frac{1}{2} \sum_{i=1}^{m} (x_i y_{i+1} - x_{i+1} y_i), \tag{3.1}$$

$$\text{where } \begin{pmatrix} x_{m+1} \\ y_{m+1} \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

Formula (2.2) is illustrated by Figure (ii); here the formula is

$$\begin{aligned} V_2(P) &= (-1) \sum_{i=1}^{m} \left(\frac{y_{i+1} + y_i}{2}\right) \det\begin{pmatrix} 1 & 1 \\ x_i & x_{i+1} \end{pmatrix} \\ &= \sum_{i=1}^{m} \left(\frac{y_{i+1} + y_i}{2}\right)(x_i - x_{i+1}). \end{aligned} \tag{3.2}$$

Notice that Formula (3.2) requires only one multiplication per term while (3.1) requires two. It is generally true that the determinant in Formula (2.2) is equivalent to one of order $(n - 1)$ while that in (2.1) is of order n. Thus, (2.2) is computationally more efficient than (2.1).

The calculations of the determinants involved in these formulas present no difficulty if n is 2. If n is large, then an efficient method for computing them is desirable. Such a method is possible if the simplices of T can be traversed so that successive $(n - 1)$-simplices share an $(n - 2)$-face. This is exactly the scheme in [1] and this volume calculation procedure could be easily added to the algorithm described there.

In case the polyhedron is described as in [1], the successive determinants differ in sign and in the entries of the column corresponding to the vertices of the $(n - 1)$-simplices which are opposite the common $(n - 2)$-face of these simplices. Thus only a rank-one change is made between the two matrices whose determinants are successively computed. If an LU factorization of this matrix is stored, then these rank-one updates may be efficiently and stably carried out by the method of Fletcher and Matthews [4]. Furthermore, the determinants in (2.2) are easy to compute from these factors.

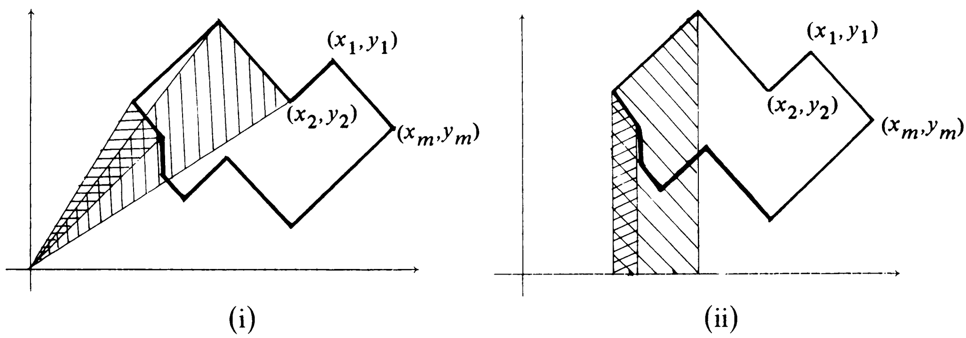

FIGURE 3.1

An estimate of the number of operations needed to calculate the volume is possible if one knows M, the number of $(n-1)$-simplices in the triangulation of the boundary. Each full LU factorization takes $\frac{1}{3}n^3$ operations while the updates each take approximately $2.6n^2$ operations [4]. If the entire boundary can be traversed by moving between $(n-1)$-simplices which share a common $(n-2)$-face, then the number of operations needed to compute the volume would be approximately $2.6Mn^2$. Even if occasional full factorizations were needed because of the inability to move to an adjacent $(n-1)$-simplex, the preceding estimate should serve, since in general $M \gg n$.

4. Concluding Remarks. Formulas (2.1)–(2.2) are not restricted to simply connected polyhedra. If ∂P consists of separated components, one merely needs to account in T for the oriented triangulations of all components of ∂P, where of course the orientations of the components must be mutually consistent. This will be the case if P itself is triangulated into consistently oriented n-simplices and the triangulation of ∂P inherits this orientation.

These formulas can be modified for the purpose of computing the centroid of P or for other geometric computations.

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