A Generalization of Swan’s Theorem

By Harold M. Fredricksen, Alfred W. Hales and Melvin M. Sweet

Abstract. Let \( f \) and \( g \) denote polynomials over the two-element field. In this paper we show that the parity of the number of irreducible factors of \( x^n f + g \) is a periodic function of \( n \), with period dividing eight times the period of the polynomial \( f^2 (x(g/f) - n(g/f)) \). This can be considered a generalization of Swan’s trinomial theorem [3].

1. Introduction. Let \( f \) and \( g \) denote polynomials in \( x \) over the two-element field \( \mathbb{F}_2 \), i.e., members of \( \mathbb{F}_2[x] \). Let \( r = r_n \) denote the number of irreducible factors of the polynomial \( x^n f + g \). In this paper we show that, for fixed \( f \) and \( g \), the parity of \( r_n \) is an eventually periodic function of \( n \). For fixed parity of \( n \) this period is a divisor of \( 8\pi \), where \( \pi \) is the period (in the usual sense) of the polynomial

\[
h = f^2 (x(g/f) - n(g/f)),
\]

i.e., the least \( \pi \) so that the polynomial \( h \) divides (a power of \( x \) times) \( x^n - 1 \). Our result can be considered a generalization of Swan’s theorem [3] concerning the number of irreducible factors of a trinomial \( x^n + x^k + 1 \) (by taking \( f = 1 \) and \( g = x^k + 1 \)).

We also investigate initial tail effects and some observed antiperiodicity properties of (the parity of) \( r_n \). The paper concludes with tables of values of \( r_n \) for various \( f \), \( g \), and \( n \).

We wish to thank Lloyd Welch for providing us with a polynomial factoring program. This program was an immense help in checking and refining our results.

2. Background. We begin by recalling various properties of the resultant and discriminant for polynomials \( F, G \) with integer coefficients [1]. If

\[
F(x) = a \prod_{i=1}^{n} (x - \alpha_i) \quad \text{and} \quad G(x) = b \prod_{j=1}^{m} (x - \beta_j),
\]

then the resultant \( R(F, G) \) is an integer given by any one of the following equal expressions:

\[
R(F, G) = a^m b^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\alpha_i - \beta_j),
\]

\[
R(F, G) = a^m \prod_{i=1}^{n} G(\alpha_i),
\]

\[
R(F, G) = (-1)^{mn} b^n \prod_{j=1}^{m} F(\beta_j).
\]
$R(F, G)$ is also the value of the determinant of the following $(m + n) \times (m + n)$ matrix where

\[
F(x) = ax^n + a_1x^{n-1} + \cdots + a_n \quad \text{and} \quad G(x) = bx^m + b_1x^{m-1} + \cdots + b_m:
\]

\[
\begin{array}{ccccccccc}
a & a_1 & \cdots & a_n & 0 & 0 & \cdots & 0 \\
0 & a & a_1 & \cdots & a_n & 0 & \cdots & 0 \\
0 & 0 & a & a_1 & \cdots & a_n & \cdots & 0 \\
\vdots & & & & & & \ddots & & \vdots \\
0 & 0 & \cdots & \cdots & a_{n-2} & a_{n-1} & a_n \\
b & b_1 & \cdots & 0 & 0 & \cdots & 0 \\
0 & b & b_1 & \cdots & 0 & \cdots & 0 \\
0 & 0 & b & b_1 & \cdots & \cdots & 0 \\
\vdots & & & & & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & b_{m-2} & b_{m-1} & b_m
\end{array}
\]

From the above it is easy to deduce the following properties of $R$:

(4) $R(G, F) = (-1)^m R(F, G)$,

(5) $R(F, G, G_2) = R(F, G_1)R(F, G_2)$, \quad $R(F_1F_2, G) = R(F_1, G)R(F_2, G)$,

(6) $R(F, G) = a^{m-\deg(G-FH)}R(F, G-FH)$ for any $H$.

The discriminant $D(F)$ of a monic polynomial $\prod_{i=1}^n (x - \alpha_i)$ is given by

(7) $D(F) = \prod_{i<j} (\alpha_i - \alpha_j)^2$,

which can also be written

(8) $D(F) = (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^n F'(\alpha_i)$,

(9) $D(F) = (-1)^{\frac{n(n-1)}{2}} R(F, F')$.

Our main tool will be Swan's version of Stickelberger's theorem ([1], [3]). Suppose $F$ is a monic polynomial of degree $n$ with integral coefficients and that $F$, reduced modulo 2 (which we denote by $\bar{F}$ or $f$), has $r$ irreducible factors. Then

(a) $D(F) \equiv 1 \pmod{8}$ implies $r \equiv n \pmod{2}$,

(b) $D(F) \equiv 5 \pmod{8}$ implies $r \not\equiv n \pmod{2}$,

(c) $D(F) \not\equiv 1, 5 \pmod{8}$ implies $f$ has repeated factors.

Hence, the value of $D(F) \pmod{8}$ determines the parity of $r$ if $f$ has no repeated factors and the parity of $n$ is known.

3. Theoretical Results. We consider first a special case. Let $g$ be a polynomial of degree $k$ over the two-element field $\mathbb{F}_2$ with $g(0) = 1$, and let $G$ be a polynomial with integer coefficients of the same degree with $\bar{G} = g$. (Take, say, all coefficients of $G$ to be 0 or 1.) Consider the family $\{p_n\}$ of polynomials over $\mathbb{F}_2$ given by $p_n = x^n + g(x)$ and the associated family $\{P_n\}$ with $P_n = x^n + G(x)$. We have (considering only cases with $n > k$)

\[
D(P_n) = (-1)^{\frac{n(n-1)}{2}} R(P_n, P_n')
\]
However, the assumption $G(0) = 1$ implies $R(P_n, x) = (-1)^n$, so

$$R(P_n, xP'_n) = (-1)^n R(P_n, P'_n),$$

and

$$D(P_n) = (-1)^{(n-1)/2 + n} R(P_n, xP'_n) = (-1)^{(n+1)/2} R(P_n, xP'_n - nP_n),$$

using property (6) of Section 2.

Let $H_n = xP'_n - nP_n$, and $h_n = H_n$. Then we have $H_n = xG' - nG$ since the contributions from $x^n$ cancel, and $h_n = xg' - ng$ only depends on the parity of $n$. Hence, if the parity of $n$ is fixed, $h_n$ does not depend on $n$.

For fixed parity of $n$, let $\pi$ denote the period of $h_n$, i.e., the least positive integer such that $h_n$ divides $x^n - 1$ in $F_2[x]$. (If $h_n$ has zero constant term, let $\pi$ denote the period of $k_n$ where $h_n = x'k_n$ and $k_n(0) = 1$.) Then we have

**Theorem 1.** Let $r_n$ denote the number of irreducible factors of $p_n = x^n + g$. Then if $p_n$ has no repeated factors (and $n$ is sufficiently large) we have

$$r_n \equiv r_n + \text{LCM}(8, 4\pi) \pmod 2,$$

where $\pi$ is the period of $h_n = xg' - ng$.

**Proof.** Using the Stickelberger-Swan theorem it suffices to prove that

$$D(P_n) \equiv D(P_n + \text{LCM}(8, 4\pi)) \pmod 8.$$

Now we have shown

$$D(P_n) = (-1)^{(n+1)/2} R(P_n, H_n)$$

and $(-1)^{(n+1)/2}$ has period 4, so it suffices to show that $R(P_n, H_n)$ is congruent to $R(P_n + \text{LCM}(8, 4\pi), H_n + \text{LCM}(8, 4\pi)) \pmod 8$.

Clearly $H_n$ and $H_{n+8}$ are congruent (coefficient by coefficient) modulo 8 and have the same degree, so from the determinant definition of the resultant we need only show that

$$R(P_n, H_n) \equiv R(P_{n+4\pi}, H_n) \pmod 8.$$

Since $h_n$ divides $x^n - 1$, we know that $H_n$ divides $x^n - 1 \pmod 2$, i.e., $x^n \equiv 1 + 2K \pmod {H_n}$. Therefore

$$x^{4n} \equiv (1 + 2K)^4 \equiv 1 + 8L \pmod {H_n}.$$  

(If $h_n$ divides $x'(x^n - 1)$, then $x^{4n+4L} \equiv x^{4L} + 8L \pmod {H_n}$.)

**Case 1.** Suppose $n - k$ is odd. Then $H_n$ and $h_n$ have the same degree. This means that in the congruence $x^{4n} \equiv 1 + 8L \pmod {H_n}$ we can take the degree of $L$ to be less than $4\pi$. In other words we have $x^{4n} \equiv 1 + H_nM \pmod 8$ with the degree of $1 + H_nM$ equal to $4\pi$. This gives $x^{n+4\pi} \equiv x^n + x^nH_nM \pmod 8$.

Now we have

$$R(P_{n+4\pi}, H_n) = R(x^{n+4\pi} + G, H_n)$$

$$\equiv R(x^n + x^nH_nM + G, H_n) \pmod 8,$$

*$^*$Note: $h_n$ will always be a square or $x$ times a square, so $\pi$ will be even unless $\pi = 1$, and hence $\text{LCM}(8, 4\pi) = 4\pi$ unless $\pi = 1$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
using the determinant definition of $R$ and the fact that $x^n + x^a H_n M$ has degree $n + 4 \pi$. Hence,

$$R(P_{n+4\pi}, H_n) \equiv (k - n)^4 R(x^n + G, H_n) \pmod{8},$$

where $(k - n)$ is the leading coefficient of $H_n$ and we are using properties (6) and (4) of Section 2. Since $(k - n)^4 \equiv 1 \pmod{8}$ we conclude

$$R(P_{n+4\pi}, H_n) \equiv R(P_n, H_n) \pmod{8}.$$

This completes Case 1.

Although we have only given the details when $h_n(0) = 1$, the argument is similar for $t > 0$ and shows that periodicity will hold as soon as $n$ is at least $4t$ (and of course greater than $k$).

**Case 2.** Suppose $n - k$ is even. Then $h_n$ has degree $l < k$ and we write $u = k - l$. By applying Hensel’s lemma [2, p. 275] we can write $H_n = H_n^{(1)} H_n^{(2)}$, where $H_n^{(1)}$, $H_n^{(2)}$ have 2-adic coefficients

$$H_n^{(1)} = h_n, \quad H_n^{(2)} = 1,$$

$H_n^{(1)}$ has degree $l$, and $H_n^{(2)}$ has degree $u$. By property (5) of Section 2 we need only that

$$R(P_{n+4\pi}, H_n^{(1)}) \equiv R(P_n, H_n^{(1)}) \pmod{8}$$

and

$$R(P_{n+4\pi}, H_n^{(2)}) \equiv R(P_n, H_n^{(2)}) \pmod{8}.$$

The former follows immediately from Case 1. For the latter we proceed as follows. Since $H_n^{(2)} \equiv 1 \pmod{2}$ we can write $x \equiv 1 + 2Q \pmod{H_n^{(2)}}$, where the degree of $Q$ is $u + 1$. Hence $x^4 \equiv 1 + 8R \pmod{H_n^{(2)}}$, where the degree of $R$ is $4u + 4$, or $x^4 \equiv 1 + H_n^{(2)} S \pmod{8}$ with the degree of $H_n^{(2)} S$ equal to $4u + 4$. Now for any nonzero $x^a$ present in $G$ we have

$$x^{4+a} \equiv x^a + x^a H_n^{(2)} S \pmod{8}.$$ 

Suppose $n$ is larger than $k + 4u$, so that $n + 4$ is larger than $k + 4u + 4$. Then the degree of $x^a H_n^{(2)} S$ will be less than $n + 4$. Hence

$$R(x^{n+4} + G, H_n^{(2)}) \equiv R(x^{n+4} + x^4 G - H_n^{(2)} G, H_n^{(2)}) \pmod{8},$$

using the determinant definition of $R$. Hence, applying properties (6) and (4) of Section 2, we have

$$R(x^{n+4} + G, H_n^{(2)}) \equiv R(x^{n+4} + x^4 G, H_n^{(2)}) \pmod{8}.$$ 

But we have

$$R(x^4(x^n + G), H_n^{(2)}) = R(x^4, H_n^{(2)}) R(x^n + G, H_n^{(2)})$$

$$= (H_n^{(2)}(0))^4 R(x^n + G, H_n^{(2)})$$

$$\equiv R(x^n + G, H_n^{(2)}) \pmod{8},$$

using properties (5) and (2) of Section 2. Hence, we have

$$R(P_{n+4\pi}, H_n^{(2)}) \equiv R(P_n, H_n^{(2)}) \pmod{8}$$

and, upon iterating,

$$R(P_{n+4\pi}, H_n^{(2)}) \equiv R(P_n, H_n^{(2)}) \pmod{8}.$$ 

This completes Case 2 and hence the proof of Theorem 1.
In Case 2, periodicity will hold as soon as \( n \geq 4t \) and \( n > k + 4u \).

Now let us consider the more general case of a family of polynomials \( \{ p_n \} \) with 
\[
p_n = x^n f + g,
\]
where \( f, g \) are coprime polynomials over \( F_2 \) with \( g(0) = f(0) = 1 \) (consider only \( n > k = \text{degree } g \)). Suppose \( F, G \) are polynomials with integer coefficients with \( \overline{F} = f, \overline{G} = g \), degree \( F = \text{degree } f \), degree \( G = \text{degree } g \). (As before, we take all coefficients of \( F, G \) to be 0 or 1.) Let \( P_n = x^n F + G \). We have
\[
R(P_n, P_n') = R(x^n F + G, n x^n F') + G')
= R(x^n F + G, nx^n F + x^n F' + x'G')
= (-1)^{n + \deg F} R(x^n F + G, x^n F' + G' - n G)
= (-1)^{n + \deg F} R(G, F) F + (x^n F, x^n F' + x'G' - n F G')

using various properties from Section 2. Letting \( H_n = x F G' - x F' G - n F G \), and hence obtaining
\[
\delta = H_n = x f g' - x f' g - n f g = f^2 \left( (x^g/f)' - n (g/f) \right),
\]
we have

**Theorem 2.** Let \( r_n \) denote the number of irreducible factors of \( p_n = x^n f + g \). If \( p_n \) has no repeated factors (and \( n \) is sufficiently large), then \( r_n \equiv r_n + \text{LCM}(8,4n) \pmod{2} \), where \( \pi \) is the period of \( h_n = f^2 ((x/g)' - n (g/f)). 

From the above calculations it clearly suffices (for the proof of Theorem 2) to show that
\[
R(P_n, H_n) \equiv R(P_{n+4n}, H_n) \pmod{8}.
\]
We omit the details, since they are very similar to those of the proof of Theorem 1, i.e., the case \( f = 1 \). Periodicity will again hold as soon as \( n \geq 4t \) and \( n > k + 4u \), where \( x' \) exactly divides \( h_n \), \( k = \text{degree } g \), and \( u = \text{degree } H_n - \text{degree } h_n \).

**4. Further Comments.** Although our results appear to be best possible in general, there are many special cases in which the actual period of the function \( r_n \) is less than the period predicted by our theorems. One such case is that of trinomials \( x^n + x^k + 1 \) with \( n \) odd and \( k \) even, where the period is 8 rather than 4k.

Theorems 1 and 2 do not address the case of repeated factors. Certainly, if \( p_n \) has repeated factors so will \( p_n + \text{LCM}(8,4n) \), since this is detected by the parity of \( D(P_n) \). Unfortunately, the Stickelberger-Swan theorem does not give information about the parity of \( r_n \) in this case. However, any repeated factors of \( p_n \) must divide \( h_n \). For given \( h_n \), these can be divided out of \( p_n \) at the start, giving a new family of polynomials parameterized by \( n \) in a more complicated way than that of our Theorems 1 and 2. Our techniques can be used to extend our results to cover this situation also, and hence to extend Theorems 1 and 2 to the repeated factor case. We omit the (relatively messy) details.

Finally, consider a family of the form \( p_n = x^n + g(x) \), where \( n \) is odd and \( g(x) = u(x)^2 \). Then \( h_n = g \). Suppose further that \( u(x) \) has odd period \( \pi' \). Then \( u(x) \) and \( (x^{\pi} - 1)/u(x) \) are coprime, so we can find (by Hensel’s lemma) 2-adic

**Note:** As in Theorem 1, either \( \text{LCM}(8,4n) = 4n \) or \( n = 1 \).
polynomials $U(x), V(x)$ with $\bar{U} = u, UV = x^{\pi'} - 1$, and degree $U = \deg u$. This gives $x^{2^{\pi'}} - 1 = U(x^2)V(x^2)$, where $\bar{U}(x^2) = u(x)^2 = g(x)$. By appropriately choosing $G$ (i.e., no longer with 0,1 coefficients) with degree $G = \deg g$, we can guarantee that $H_n = xG' - nG$ is congruent to $U(x^2)$ (mod 8). (Adding two to the coefficient of a term $x^a$ of $G$ adds $2(a-n)$ to the coefficients of $x^a$ in $H_n$.) Hence $x^{2^{\pi'}} \equiv 1 + H_nM \pmod{8}$. From this, as in Case 1 of Theorem 1, we deduce

$$R(P_{n+4^{\pi'}}, H_n) \equiv R(P_n, H_n) \pmod{8}.\]  

On the other hand, if we compare $R(P_{n+4^{\pi'}}, H_n)$ and $R(P_{n+4^{\pi'}}, H_{n+4^{\pi'}})$ via the determinant definition of $R$, using the fact that the nonzero coefficients of $H_{n+4^{\pi'}}$ are each $4^{\pi'}$ smaller than those of $H_n$, and the fact that for any odd integer $N$ we have $N - 4 \equiv (-3)N \pmod{8}$, we find that

$$R(P_{n+4^{\pi'}}, H_{n+4^{\pi'}}) \equiv (-3)^{(n+4^{\pi'})} R(P_{n+4^{\pi'}}, H_n) \pmod{8}.\]  

But $(-3)^{(n+4^{\pi'})} \equiv (-3) \pmod{8}$, so we conclude that $r_n$ is antiperiodic with antiperiod $4^{\pi'}$. (Note that the predicted period of $r_n$ is $4\pi = 8\pi'$, which this result implies.)

### 5. Experimental Results

In this section, we give tables of values of $r_n$ for various $f$, $g$, and $n$. The cases where Theorem 1 applies ($f = 1$) are listed first. The cases of odd and even $n$ are listed separately. The parity (0 or 1) of $r_n$ is also given.

After each table, we give the polynomial $h = h_n$; the predicted period $\text{LCM}(8, 4\pi')$ of $r_n$; the observed period (if it is different); and the antiperiod for those cases covered in Section 4.

#### I. $f = 1$; $g = x^3 + x + 1$.

| $n$  | 4   | 6   | 8   | 10  | 12  | 14  | 16  | 18  | 20  | 22  | 24  | 26  | 28  | 30  | 32  | 34  |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $r_n$ | 3   | 4   | 3   | 5   | 3   | 4   | 5   | 5   | 6   | 3   | 5   | 5   | 4   | 3   | 7   |
| parity | 1   | 0   | 1   | 1   | 0   | 1   | 1   | 0   | 1   | 1   | 0   | 1   | 0   | 1   | 1   |

$h = x^3 + x$; period = 8.

#### II. $f = 1$; $g = x^4 + x^2 + 1$.

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$h = x^4 + x^2 + 1$; period = 24; antiperiod = 12.

#### III. $f = 1$; $g = x^5 + x + 1$.

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$h = x^5 + x$; period = 16.

#### IV. $f = 1$; $g = x^6 + x^2 + 1$.

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$h = x^6 + x^2 + 1$; period = 56; antiperiod = 28.
A GENERALIZATION OF SWAN'S THEOREM

\[ h = x^n + x; \text{ period } = 24. \]

VI. \( f = 1; \quad g = x^8 + x^4 + 1. \)

\[
\begin{array}{cccccccccccccccccccc}
r_n & 3 & 2 & 4 & 3 & 3 & 2 & 2 & 3 & 3 & 4 & 2 & 3 & 3 & 2 & 4 & 3 & 5 & 4 & 4 \\
parity & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[ h = x^8 + x^4 + 1; \text{ predicted period } = 48; \text{ observed period } = 8. \]

VII. \( f = 1; \quad g = x^9 + x + 1. \)

\[
\begin{array}{cccccccccccccccccccc}
 n & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 & 40 \\
r_n & 4 & 6 & 4 & 10 & 4 & 6 & 4 & 10 & 5 & 8 & 5 & 10 & 5 & 6 & 5 & 12 \\
parity & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\]

\[ h = x^9 + x; \text{ period } = 32. \]

VIII. \( f = 1; \quad g = x^9 + x + 1. \)

\[
\begin{array}{cccccccccccccccccccc}
r_n & 3 & 4 & 2 & 3 & 5 & 4 & 3 & 3 & 4 & 5 & 3 & 3 \\
parity & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
\]

\[ h = 1; \text{ period } = 8. \]

IX. \( f = 1; \quad g = x^{17} + x + 1. \)

\[
\begin{array}{cccccccccccccccccccc}
 n & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 & 40 & 42 & 44 & 46 & 48 \\
r_n & 4 & 6 & 4 & 10 & 3 & 6 & 4 & 18 & 6 & 6 & 5 & 12 & 4 & 6 & 4 & 20 \\
parity & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[ h = x^{17} + x; \text{ period } = 64. \]

X. \( f = 1; \quad g = x^5 + x^3 + x + 1. \)

\[
\begin{array}{cccccccccccccccccccc}
 n & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 \\
r_n & 3 & 1 & 2 & 4 & 1 & 1 & 4 & 2 & 1 & 5 & 2 & 2 & 5 & 3 & 2 & 4 \\
parity & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

\[ h = x^5 + x^3 + x; \text{ predicted period } = 24; \text{ observed period } = 8. \]
XI. $f = 1; g = x^7 + x^3 + x + 1$.

\[ n \quad 9 \quad 11 \quad 13 \quad 15 \quad 17 \quad 19 \quad 21 \quad 23 \quad 25 \quad 27 \quad 29 \quad 31 \quad 33 \quad 35 \quad 37 \quad 39 \]
\[ r_n \quad 2 \quad 2 \quad 1 \quad 3 \quad 2 \quad 2 \quad 3 \quad 1 \quad 3 \quad 2 \quad 4 \quad 1 \quad 3 \quad 4 \quad 2 \quad 3 \]
\[ \text{parity} \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \]

\[ n \quad 41 \quad 43 \quad 45 \quad 47 \quad 49 \quad 51 \quad 53 \quad 55 \quad 57 \quad 59 \]
\[ r_n \quad 1 \quad 2 \quad 2 \quad 1 \quad 3 \quad 2 \quad 4 \quad 3 \quad 3 \quad 4 \]
\[ \text{parity} \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \]

$h = 1; \text{period} = 8$.

XII. $f = 1; g = x^7 + x^3 + x + 1$.

\[ n \quad 8 \quad 10 \quad 12 \quad 14 \quad 16 \quad 18 \quad 20 \quad 22 \quad 24 \quad 26 \quad 28 \quad 30 \quad 32 \quad 34 \quad 36 \]
\[ r_n \quad 1 \quad 1 \quad 1 \quad 4 \quad 2 \quad 2 \quad 2 \quad 1 \quad 3 \quad 3 \quad 4 \quad 1 \quad 3 \quad 3 \quad 2 \]
\[ \text{parity} \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \]

\[ n \quad 38 \quad 40 \quad 42 \quad 44 \quad 46 \quad 48 \quad 50 \quad 52 \quad 54 \quad 56 \quad 58 \quad 60 \quad 62 \quad 64 \quad 66 \quad 68 \quad 70 \]
\[ r_n \quad 2 \quad 4 \quad 3 \quad 5 \quad 5 \quad 3 \quad 2 \quad 2 \quad 4 \quad 6 \quad 2 \quad 4 \quad 2 \quad 3 \quad 3 \quad 3 \quad 4 \]
\[ \text{parity} \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \]

\[ n \quad 72 \quad 74 \quad 76 \quad 78 \quad 80 \quad 82 \quad 84 \quad 86 \quad 88 \quad 90 \quad 92 \quad 94 \quad 96 \quad 98 \quad 100 \quad 102 \quad 104 \]
\[ r_n \quad 4 \quad 4 \quad 2 \quad 3 \quad 3 \quad 3 \quad 8 \quad 3 \quad 3 \quad 3 \quad 4 \quad 2 \quad 4 \quad 3 \quad 3 \quad 5 \quad 5 \]
\[ \text{parity} \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \]

$h = x^7 + x^3 + x; \text{period} = 56$.

XIII. $f = 1; g = x^8 + x^7 + x + 1$.

\[ n \quad 10 \quad 12 \quad 14 \quad 16 \quad 18 \quad 20 \quad 22 \quad 24 \quad 26 \quad 28 \quad 30 \quad 32 \]
\[ r_n \quad 4 \quad 1 \quad 2 \quad 3 \quad 3 \quad 1 \quad 4 \quad 2 \quad 1 \quad 4 \quad 2 \quad 2 \]
\[ \text{parity} \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \]

\[ n \quad 34 \quad 36 \quad 38 \quad 40 \quad 42 \quad 44 \quad 46 \quad 48 \quad 50 \quad 52 \quad 54 \quad 56 \quad 58 \quad 60 \]
\[ r_n \quad 7 \quad 1 \quad 2 \quad 7 \quad 3 \quad 3 \quad 4 \quad 2 \quad 3 \quad 6 \quad 2 \quad 4 \quad 7 \quad 5 \]
\[ \text{parity} \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \]

$h = x^7 + x; \text{period} = 24$.

XIV. $f = 1; g = x^9 + x^7 + x + 1$.

\[ n \quad 10 \quad 12 \quad 14 \quad 16 \quad 18 \quad 20 \quad 22 \quad 24 \quad 26 \quad 28 \quad 30 \quad 32 \quad 34 \quad 36 \quad 38 \quad 40 \quad 42 \quad 44 \quad 46 \quad 48 \]
\[ r_n \quad 1 \quad 5 \quad 2 \quad 2 \quad 3 \quad 1 \quad 1 \quad 6 \quad 2 \quad 1 \quad 4 \quad 2 \quad 1 \quad 5 \quad 2 \quad 2 \quad 4 \quad 3 \quad 3 \quad 6 \]
\[ \text{parity} \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \]

\[ n \quad 50 \quad 52 \quad 54 \quad 56 \quad 58 \quad 60 \quad 62 \quad 64 \quad 66 \quad 68 \quad 70 \quad 72 \quad 74 \quad 76 \quad 78 \quad 80 \quad 82 \quad 84 \quad 86 \quad 88 \]
\[ r_n \quad 2 \quad 3 \quad 5 \quad 4 \quad 5 \quad 7 \quad 2 \quad 2 \quad 5 \quad 3 \quad 3 \quad 8 \quad 2 \quad 3 \quad 6 \quad 2 \quad 3 \quad 7 \quad 4 \quad 2 \]
\[ \text{parity} \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \]

$h = x^9 + x^5 + x; \text{period} = 48$.

XV. $f = x + 1; g = x^2 + x + 1$.

\[ n \quad 3 \quad 5 \quad 7 \quad 9 \quad 11 \quad 13 \quad 15 \quad 17 \quad 19 \quad 21 \quad 23 \quad 25 \quad 27 \quad 29 \quad 31 \quad 33 \]
\[ r_n \quad 1 \quad 1 \quad 1 \quad 2 \quad 1 \quad 3 \quad 2 \quad 2 \quad 1 \quad 3 \quad 4 \quad 2 \quad 3 \quad 1 \quad 2 \quad 2 \]
\[ \text{parity} \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \]

\[ n \quad 35 \quad 37 \quad 39 \quad 41 \quad 43 \quad 45 \quad 47 \quad 49 \]
\[ r_n \quad 3 \quad 3 \quad 2 \quad 4 \quad 3 \quad 3 \quad 2 \quad 2 \]
\[ \text{parity} \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \]

$h = 1; \text{period} = 8$. 

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XVI. \( f = x + 1; \ g = x^2 + x + 1. \)
\[
\begin{array}{cccccccccccccccc}
 n & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 \\
r_n & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 3 & 1 & 2 & 2 & 3 & 3 & 2 & 2 \\
\text{parity} & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
 n & 36 & 38 & 40 & 42 & 44 & 46 & 48 & \\
r_n & 3 & 3 & 2 & 2 & 3 & 3 & 4 & \\
\text{parity} & 1 & 1 & 0 & 0 & 1 & 1 & 0 &
\end{array}
\]
\( h = x^3; \ \text{period} = 8. \)

XVII. \( f = x + 1; \ g = x^3 + x + 1. \)
\[
\begin{array}{cccccccccccccccc}
 n & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 \\
r_n & 1 & 1 & 3 & 1 & 2 & 2 & 1 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 4 \\
\text{parity} & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

XVIII. \( f = x + 1; \ g = x^3 + x + 1. \)
\[
\begin{array}{cccccccccccccccc}
r_n & 3 & 1 & 2 & 3 & 1 & 2 & 4 & 2 & 1 & 5 & 2 & 3 & 7 & 1 & 4 \\
\text{parity} & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
 n & 35 & 37 & 39 & 41 & 43 & 45 & 47 & 49 & 51 & 53 & 55 \\
r_n & 5 & 1 & 2 & 4 & 4 & 3 & 5 & 2 & 1 & 5 & 1 \\
\text{parity} & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
\end{array}
\]

\( h = x^4 + x^2 + 1; \ \text{period} = 24. \)

XIX. \( f = x + 1; \ g = x^3 + x^2 + 1. \)
\[
\begin{array}{cccccccccccccccc}
 n & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 \\
r_n & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 3 & 2 & 2 & 5 & 1 & 2 & 2 & 3 & 1 \\
\text{parity} & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
 n & 36 & 38 & 40 & 42 & 44 & 46 & 48 & 50 \\
r_n & 4 & 2 & 1 & 5 & 2 & 2 & 3 & 3 \\
\text{parity} & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
\end{array}
\]

XX. \( f = x + 1; \ g = x^3 + x^2 + 1. \)
\[
\begin{array}{cccccccccccccccc}
r_n & 1 & 1 & 4 & 1 & 1 & 4 & 2 & 2 & 5 & 2 & 2 & 4 & 5 & 1 & 4 \\
\text{parity} & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
 n & 35 & 37 & 39 & 41 & 43 & 45 & 47 & 49 & 51 \\
r_n & 3 & 3 & 4 & 2 & 2 & 5 & 2 & 2 & 4 \\
\text{parity} & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\]

XX. \( f = x + 1; \ g = x^3 + x^2 + 1. \)
\[
\begin{array}{cccccccccccccccc}
r_n & 1 & 2 & 1 & 2 & 1 & 1 & 3 & 3 & 2 & 2 & 3 & 2 & 4 & 2 & 4 & 1 \\
\text{parity} & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
 n & 35 & 37 & 39 & 41 & 43 & 45 & 47 & 49 & 51 \\
r_n & 4 & 3 & 1 & 3 & 1 & 4 & 5 & 4 & 2 \\
\text{parity} & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
\end{array}
\]

\( h = x^4 + x^2 + 1; \ \text{period} = 24. \)
XXII. $f = x + 1; \ g = x^4 + x^2 + 1.$

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$h = x^5 + x^3 + x; \ period = 24.$

XXIII. $f = x^2 + 1; \ g = x^4 + x^2 + 1.$

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$h = x^6 + 1; \ period = 24.$

XXIV. $f = x^2 + x + 1; \ g = x^3 + x + 1.$

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$h = x^5; \ period = 8.$

XXV. $f = x^2 + x + 1; \ g = x^3 + x + 1.$

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$h = x^4 + 1; \ period = 16.$

XXVI. $f = x^2 + x + 1; \ g = x^3 + x^2 + 1.$

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$h = x^5 + x; \ period = 16.$

XXVII. $f = x^2 + x + 1; \ g = x^3 + x^2 + 1.$

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$h = 1; \ period = 8.$
A GENERALIZATION OF SWAN'S THEOREM

XXVIII. $f = x^3 + x^2 + 1; \ g = x^3 + x + 1.$

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$h = x^5 + x^3 + x; \ predicted\ period = 24; \ observed\ period = 8.$

XXIX. $f = x^3 + x^2 + 1; \ g = x^3 + x + 1.$

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</table>

$h = x^6 + x^4 + x^2 + 1; \ predicted\ period = 32; \ observed\ period = 16.$

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