On Weighted Chebyshev-Type Quadrature Formulas

By Klaus-Jürgen Förster and Georg-Peter Ostermeyer

Abstract. A weighted quadrature formula is of Chebyshev type if it has equal coefficients and real (but not necessarily distinct) nodes. For a given weight function we study the set \( T(n, d) \) consisting of all Chebyshev-type formulas with \( n \) nodes and at least degree \( d \). It is shown that in nonempty \( T(n, d) \) there exist two special formulas having "extremal" properties. This result is used to prove uniqueness and further results for \( E \)-optimal Chebyshev-type formulas. For the weight function \( w = 1 \), numerical investigations are carried out for \( n \leq 25 \).

1. Introduction. Let \( w \) be a nonnegative weight function on the interval \((a, b)\), \(-\infty \leq a < b \leq \infty\), admitting moments \( m_j \) of all order

\[
m_j = \int_a^b x^j w(x) \, dx, \quad j = 0, 1, 2, \ldots, \quad m_0 > 0.
\]

We consider (weighted) Chebyshev-type quadrature formulas \( Q_n \). These are quadrature formulas \( Q_n \) with equal coefficients and real (but not necessarily distinct) nodes:

\[
Q_n[f] := c \sum_{i=1}^n f(x_i), \quad -\infty < x_1 \leq x_2 \leq \cdots \leq x_n < \infty,
\]

\[
\int_a^b f(x)w(x) \, dx = Q_n[f] + R_n[f].
\]

By this definition, it is possible that some nodes \( x_i \) are not contained in the interval \((a, b)\). \( Q_n \) has at least degree \( d \) (of exactness) if

\[
R_n[p_i] = 0, \quad i = 0, 1, \ldots, d,
\]

where \( p_i \), here and throughout this paper, denotes the monomial \( p_i(x) := x^i \). If \( d \geq 0 \), the coefficient \( c \) in (1.2) is determined by (1.3):

\[
c = m_0/n.
\]

The maximal possible degree of a Chebyshev-type quadrature formula with \( n \) nodes is denoted by \( d_n \).

Let, in the following, \( T(n, d) \) be the set of all Chebyshev-type quadrature formulas with \( n \) nodes and at least degree \( d \). One has \( T(n, d + 1) \subseteq T(n, d) \) and \( T(n, d) \subseteq T(kn, d) \) for every \( k \in \mathbb{N} \). For \( n > 2 \), a simple calculation shows that
each set $T(n, 2)$ contains an infinite number of elements. In case of $d_n \geq n$ the set $T(n, d_n)$ contains only one element, the so-called "Chebyshev quadrature formula in the strict sense". Weight functions $w$ which allow such formulas for every $n \in \mathbb{N}$ are rare [7, p. 109]. In case of $d_n < n$ the set $T(n, d_n)$ possibly contains more than one element. To select some of these formulas several criteria can be found in the literature.

From a historical point of view it may be obvious to consider such quadrature formulas $Q_{n}^{\text{opt}} \in T(n, d_n)$, which minimize $|R_n[p_{d_n+1}]|$ among all $Q_n \in T(n, d_n)$. Such quadrature formulas are called $E$-optimal [9], [2], [7]. In case of weight functions $w$, which are symmetric with respect to the (finite) interval $(a, b)$, several authors have distinguished between symmetric, i.e., $x_i = a = b = x_{n-i+1}$ for $i = 1, \ldots, n$, and unsymmetric formulas with regard to $E$-optimality [10], [9], [2]. There is computational evidence that $E$-optimal formulas for symmetric weight functions are indeed symmetric [7, p. 113]. Gautschi and Yanagiwara [9] have shown that symmetry would follow, if $E$-optimal formulas are unique in $T(n, d_n)$. One aim of this paper is to prove the uniqueness of $E$-optimal formulas in general.

Several authors have proposed other criteria to select special Chebyshev-type formulas—necessarily not contained in $T(n, d_n)$ (see, e.g., [7]). Therefore, it may be of interest to study the set $T(n, d)$ in general. If $T(n, d)$ contains more than one element, we show that there exists in $T(n, d)$ an infinite number of formulas which have pairwise distinct nodes. In this case, there also exists in $T(n, d)$ an infinite number of interpolatory quadrature formulas (for definition, see, e.g., [3]). Among these interpolatory quadrature formulas there are two unique formulas which have several "extremal" properties with respect to all other formulas in $T(n, d)$. By proving that the $E$-optimal formula $Q_{n}^{\text{opt}}$ is one of the two extremal formulas in $T(n, d)$, we can show various properties of $E$-optimal formulas.

The proofs of all theorems can be found in the supplements section of this issue.

2. $E$-Extremal Formulas. We call a (Chebyshev-type quadrature) formula $Q_n \in T(n, d)$ $E$-minimal in $T(n, d)$ and denote it by $Q_{n,d}^{\text{min}}$ if

$$ R_{n,d}^{\text{min}}[p_{d+1}] = \min\{R_n[p_{d+1}]|Q_n \in T(n, d)\}. $$

Correspondingly, we define $E$-maximal formulas $Q_{n,d}^{\text{max}} \in T(n, d)$ by

$$ R_{n,d}^{\text{max}}[p_{d+1}] = \max\{R_n[p_{d+1}]|Q_n \in T(n, d)\}. $$

Therefore, the following inequalities are valid for every $Q_n \in T(n, d)$:

$$ Q_{n,d}^{\text{max}}[p_{d+1}] \leq Q_n[p_{d+1}] \leq Q_{n,d}^{\text{min}}[p_{d+1}]. $$

Formulas with property (2.1) or (2.2) we call $E$-extremal. According to the arguments of Gautschi and Yanagiwara [9] for the existence of $E$-optimal formulas there exist $E$-minimal and $E$-maximal formulas in $T(n, d)$ for all $d$ with

$$ 1 < d \leq d_n. $$

Remark. In the following we require for $d$ the validity of (2.3), unless noted otherwise.

Our first result is the uniqueness of $E$-extremal formulas in $T(n, d)$. 
Theorem 1. In $T(n,d)$ there exists only one $E$-minimal formula $Q_{n,d}^{\text{min}}$ and only one $E$-maximal formula $Q_{n,d}^{\text{max}}$.

Definition (1.2) allows the possibility that some of the nodes coincide. It can be shown that $E$-extremal formulas have multiple nodes. Moreover, the two $E$-extremal formulas can be characterized by a special arrangement of these multiple nodes. To describe this arrangement we define for every Chebyshev-type quadrature formula $Q_n$ the sequence $S(Q_n) := (s_i(Q_n))_{i=1}^{n-1}$ as follows:

$$s_i(Q_n) = \begin{cases} 0, & \text{if } x_{n+1-i} \neq x_{n-i}, \\ 1, & \text{if } x_{n+1-i} = x_{n-i} \text{ and } i \text{ odd}, \\ -1, & \text{if } x_{n+1-i} = x_{n-i} \text{ and } i \text{ even}. \end{cases}$$

We speak of a change of sign of the sequence $S(Q_n)$ (between $s_i(Q_n)$ and $s_{i+1}(Q_n)$) if

$$\text{sign}(s_i(Q_n)) = -\text{sign}(s_{i+1}(Q_n)) \neq 0,$$

$$s_{i+1}(Q_n) = s_{i+2}(Q_n) = \cdots = s_{i+1}(Q_n) = 0.$$

Theorem 2. Let $Q_n$ be $E$-extremal in $T(n,d)$. Then $Q_n$ has at most $d$ distinct nodes. Moreover,

(i) Let $d < n - 1$. A formula $Q_n$ is $E$-extremal in $T(n,d)$ if and only if $S(Q_n)$ has at least $(n - d - 1)$ changes of sign. In this case the following holds: If the first nonzero term of $S(Q_n)$ is negative, then $Q_n$ is $E$-minimal. If this term is positive, then $Q_n$ is $E$-maximal. If $S(Q_n)$ has more than $(n - d - 1)$ changes of sign, then $Q_n$ is $E$-minimal as well as $E$-maximal and $T(n,d)$ contains only $Q_n$.

(ii) Let $d = n - 1$. A formula $Q_n$ is $E$-extremal in $T(n,d)$ if and only if $S(Q_n)$ has at least one nonzero term. If this term is negative, then $Q_n$ is $E$-minimal. If this term is positive, then $Q_n$ is $E$-minimal as well as $E$-maximal and $T(n,d)$ contains only $Q_n$.

Theorem 2 shows that $E$-extremal formulas are interpolatory quadrature formulas (for definition see, e.g., [3]). The following theorem answers the question for other interpolatory quadrature formulas in $T(n,d)$ and for formulas with pairwise distinct nodes.

Theorem 3. Let $Q_{n,d}^{\text{min}}$ and $Q_{n,d}^{\text{max}}$ be the $E$-minimal and the $E$-maximal formula in $T(n,d)$. Let

$$r \in \{ R_{n,d}^{\text{min}}[p_{d+1}], R_{n,d}^{\text{max}}[p_{d+1}] \}.$$

Then there exist formulas $\tilde{Q}_n$ and $\overline{Q}_n$ in $T(n,d)$ with $\tilde{R}_n[p_{d+1}] = \overline{R}_n[p_{d+1}] = r$ and

(i) $\tilde{Q}_n$ has pairwise distinct nodes,

(ii) $\overline{Q}_n$ has at most $(d + 1)$ distinct nodes.

In the case of $d < n - 1$, there exists for each such $r$ even an infinite number of formulas with property (i). In the case of $d = n - 1$ there exists for each such $r$ only
one formula $Q_n \in T(n, d)$ with $R_n[p_{d+1}] = r$ and this formula has pairwise distinct nodes.

A first justification for the consideration of $E$-extremal formulas is the fact that their first and their $n$th node have extremal properties with respect to all $Q_n \in T(n, d)$.

**Theorem 4.** Let $x_i$ be the nodes of a formula $Q_n \in T(n, d)$, which is not $E$-extremal. Let $x^\min_i$ and $x^\max_i$ be the nodes of the $E$-extremal formulas $Q^\min_n$ and $Q^\max_n$ in $T(n, d)$. Then

(i) $x^\min_i > x^\max_i$,
(ii) $(-1)^{d} x^\min_i > (-1)^{d} x^\max_i$.

Therefore, it is also possible to characterize the $E$-minimal ($E$-maximal) formula in $T(n, d)$ to be that formula, whose $n$th node has the largest (smallest) value.

Furthermore, Theorem 4 may be helpful for the investigation of the question of whether all nodes of a formula $Q_n \in T(n, d)$ are contained in the interval $[a, b]$.

The formulas $Q^\min_n$ and $Q^\max_n$ are defined by the extremal property (2.1) and (2.2) of their remainder with respect to only one function, the monomial $p_{d+1}$. The following theorem shows that these extremal properties remain valid for a wide class of functions, which contains especially all monomials $p_{d+1} + k$ for all $k \in \mathbb{N}$.

**Theorem 5.** Let $Q^\min_n$ and $Q^\max_n$ be the $E$-minimal and the $E$-maximal formula in $T(n, d)$. Then, for all $f \in C^{d+1}$, $f^{(d+1)} > 0$, there hold

(i) $R^\min_n[f] = \min\{R_n[f] | Q_n \in T(n, d)\}$,
(ii) $R^\max_n[f] = \max\{R_n[f] | Q_n \in T(n, d)\}$.

Another interpretation of Theorem 5 may be of interest:

Let $K_{d+1}$ denote the Peano kernel of degree $d+1$ ([7, p. 112], [3, p. 39]), of a formula $Q_n$ in $T(n, d)$ and $K^\min_{d+1}$, resp. $K^\max_{d+1}$, the Peano kernels of the same degree of the $E$-extremal formulas $Q^\min_n$ or $Q^\max_n$ in $T(n, d)$. Theorem 5 implies the inequalities

$$K^\min_{d+1}(x) \leq K_{d+1}(x) \leq K^\max_{d+1}(x)$$

for all $x \in \mathbb{R}$.

3. **E-Optimal Formulas.** Our basic result for $E$-optimal formulas is given in the following theorem.

**Theorem 6.** Let $n \in \mathbb{N}$ and let $Q_n^{\text{opt}}$ be $E$-optimal. Then $Q_n^{\text{opt}}$ is $E$-extremal in $T(n, d_n)$.

An $E$-optimal formula is therefore $E$-minimal or $E$-maximal in $T(n, d_n)$ and has the corresponding properties given in Section 2.

The first part of Theorem 2 has been proven for $E$-optimal formulas by Anderson and Gautschi [2]. The second part of Theorem 2 reduces all the remaining cases to only two formulas, characterized also by the value of the $n$th node according to
Theorem 4. Theorem 3 shows, in particular, the impossibility that different $E$-extremal formulas in $T(n, d_n)$ are both $E$-optimal. This answers the question of the uniqueness of $E$-optimal formulas [2], [7].

**Corollary 1.** Let $n \in \mathbb{N}$. Then there exists one and only one $E$-optimal formula $Q_n^{\text{opt}}$.

Therefore, by the result of Gautschi and Yanagiwara [9] mentioned above, it follows from Corollary 1 that

**Corollary 2.** Let the weight function $w$ be symmetric with respect to $(a, b)$ and let $n \in \mathbb{N}$. Then $d_n$ is odd and the $E$-optimal formula is symmetric.

Rabinowitz and Richter [11] have shown that $E$-optimal rules minimize $|R_n|$ for formulas (1.2) in special function spaces. With the help of Theorem 5, a different justification for the consideration of $E$-optimal formulas is given by the following theorem.

**Theorem 7.** Let $n \in \mathbb{N}$ and $Q_n^{\text{opt}}$ be the $E$-optimal formula and let $f \in C^{d_n+1}$, $f^{(d_n+1)} \geq 0$. If

$$\text{sign}(R_n^{\text{opt}}[p_{d_n+1}]) = \text{sign}(R_n^{\text{opt}}[f]),$$

then

$$|R_n^{\text{opt}}[f]| = \min \{|R_n[f]| \mid Q_n \in T(n, d_n)\}.$$

**4. Numerical Results for the Weight Function** $w \equiv 1$. For the weight function $w \equiv 1$ there exist Chebyshev formulas in the strict sense for $n = 1, 2, \ldots, 7$ and $n = 9$—see, e.g., [7]. The $E$-optimal formulas have been computed by Gautschi and Yanagiwara [9] for $n = 8, 10, 11, 13$ and by Anderson and Gautschi [2] for $n = 12, 14, 15, 16, 17$. Anderson [1] has shown that these formulas, except for $n = 12$, are definite, i.e., there exists a representation of their remainder term of the form

$$(4.1) \quad R_n[f] = \frac{R_n[p_{d_n+1}]}{(d_n + 1)!} f^{(d_n+1)}(\xi)$$

for every $f \in C^{d_n+1}$.

The present authors have computed the $E$-extremal formulas in $T(n, d_n)$ for $n \leq 25$ by a different method with the help of Theorem 2 resp. Theorem 3 [6]. The $E$-optimal formulas for $n = 18, \ldots, 25$ are given at the end of this section. These formulas are all definite. Theorem 5 implies that every $Q_n \in T(n, d_n)$ is also definite for $n \leq 25$, $n \neq 12$; for $n = 8, 10, 11, 13$ see Förster [4]. In case of definiteness, the comparison of the coefficients of $f^{(d_n+1)}(\xi)$ in (4.1) between the $E$-minimal and the $E$-maximal formula gives information as to how useful the choice of the $E$-optimal formula is in $T(n, d_n)$. These coefficients are listed in Table 1. In every case, the $E$-minimal formula is $E$-optimal. The numerical results correspond to the interval of integration $[-1, 1]$.

A conclusion of the above theorems is that the results of Gautschi and Monegato [8] and Förster [4] for $n = 8, 10, 11, 13$ remain valid for all $n \leq 25$, $n \neq 12$. 

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Corollary 3. Let \( n \leq 25 \) and \( w = 1 \). Let \( Q_n^{\text{opt}} \) be the \( E \)-optimal formula and \( Q_n \in T(n, d_n) \).

(a) If in (1.1) \( b = -a \), then for every \( m \in \mathbb{N} \),

\[
0 \leq R_n^{\text{opt}}[p_m] \leq R_n[p_m].
\]

(b) If \( n \neq 12 \) and \( f \in C_{d_n+1}^{d_n+1}, f(d_n+1) \geq 0 \), then

\[
0 \leq R_n^{\text{opt}}[f] \leq R_n[f].
\]
Therefore, these $E$-optimal formulas satisfy also every optimality criterion of the form

$$\min \left\{ \sum_{i=d_n+1}^{\infty} a_i(R_n[p_i])^2 \mid Q_n \in T(n, d_n) \right\}$$

with any $a_i \geq 0$ [7, p. 113]. They are, in particular for $n \neq 12$, also optimal in the sense of Sard [7, p. 112], [4].

The $E$-Optimal Formulas for $w = 1$ \hspace{1em} $18 \leq n \leq 25$

$n = 18$

$-x_1 = 0.95611589370931681977$ \hspace{1em} $= x_{18}$
$-x_2 = -x_3 = 0.7833953833119703042$ \hspace{1em} $= x_{16} = x_{17}$
$-x_4 = 0.58679047283945639018$ \hspace{1em} $= x_{15}$
$-x_5 = -x_6 = 0.45756408008040941541$ \hspace{1em} $= x_{14}$
$-x_7 = 0.25373493728377540704$ \hspace{1em} $= x_{12}$
$-x_8 = -x_9 = 0.1206841192781514185$ \hspace{1em} $= x_{10} = x_{11}$

$n = 19$

$-x_1 = 0.95841522638659246454$ \hspace{1em} $= x_{19}$
$-x_2 = -x_3 = 0.79485226355878236323$ \hspace{1em} $= x_{17} = x_{18}$
$-x_4 = 0.60772484959475892451$ \hspace{1em} $= x_{16}$
$-x_5 = -x_6 = 0.4868851013054279206$ \hspace{1em} $= x_{14} = x_{15}$
$-x_7 = 0.29638895564058655907$ \hspace{1em} $= x_{13}$
$-x_8 = -x_9 = 0.1631508328419371742$ \hspace{1em} $= x_{11} = x_{12}$
$-x_{10} = 0.0$ \hspace{1em} $= x_{10}$

$n = 20$

$-x_1 = 0.96051482286129288228$ \hspace{1em} $= x_{20}$
$-x_2 = -x_3 = 0.80496515092537905967$ \hspace{1em} $= x_{18} = x_{19}$
$-x_4 = 0.63049631592920524269$ \hspace{1em} $= x_{17}$
$-x_5 = -x_6 = 0.50749481899047359478$ \hspace{1em} $= x_{15} = x_{16}$
$-x_7 = 0.35906562874648327105$ \hspace{1em} $= x_{14}$
$-x_8 = -x_9 = -x_{10}$ \hspace{1em} $= x_{11} = x_{12} = x_{13}$

$n = 21$

$-x_1 = 0.96243015157286074846$ \hspace{1em} $= x_{21}$
$-x_2 = -x_3 = 0.81403490074542027161$ \hspace{1em} $= x_{19} = x_{20}$
$-x_4 = 0.65167313907372323093$ \hspace{1em} $= x_{18}$
$-x_5 = -x_6 = 0.5255764207596964732$ \hspace{1em} $= x_{16} = x_{17}$
$-x_7 = 0.40559995128245393129$ \hspace{1em} $= x_{15}$
$-x_8 = -x_9 = -x_{10} = 0.18868995126113857640$ \hspace{1em} $= x_{12} = x_{13} = x_{14}$
$-x_{11} = 0.0$ \hspace{1em} $= x_{11}$

(continues)
5. Examples. Table 1 shows that in case of \( w = 1 \) the sets \( T(n, d_n) \) for \( n < 25 \) and \( d_n < n \) contain an infinite number of elements (see Theorem 3). The same is true for the examples computed by Anderson and Gautschi [2] in case of other weight functions. The following example shows with the help of Theorem 2 the possibility that for \( d_n < n \) the set \( T(n, d_n) \) contains only one element.

Let the weight function \( w \) be given by \( w(x) := \sqrt{1 - x^2} \). The corresponding Gauss-formula \( G_5 \) with 5 nodes and therefore degree 9 is given by (see Szegö [12, p. 344])

\[
G_5[f] = \frac{\pi}{24} \left( f\left(-\frac{3}{2}\right) + 3f\left(-\frac{1}{2}\right) + 4f(0) + 3f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right) \right).
\]

Because of \( m_0 = \pi/2 \) the formula \( G_5 \) is a Chebyshev-type quadrature formula (1.2)
with the twelve nodes

$$x_1 = -\frac{\sqrt{3}}{2} = -x_{12},$$

$$x_2 = x_3 = x_4 = -\frac{1}{2} = -x_{11} = -x_{10} = -x_9,$$

$$x_5 = x_6 = 0 = -x_8 = -x_7.$$  

So $G_5$ is an element of $T(12, 9)$. By (2.4) the sequence $S(G_5)$ is given by

$$S(G_5) = (0, 1, -1, 0, -1, 1, -1, 0, -1, 1, 0)$$

and has four changes of sign; see (2.5). Theorem 2(i) shows that $T(12, 9)$ contains only the element $G_5$. Furthermore, $G_5$ is also the only element of $T(12, 8)$, and $G_5$ is the $E$-maximal formula $Q_{12}^{\text{max}}$ in $T(12, 7)$.

In the case of $w = 1$ and $n \leq 25$ the nodes of the $E$-minimal formula $Q_{n,d_n}^{\text{min}}$ are contained in the interval $(-1, 1)$. Therefore, in these cases, using Theorem 4, the nodes of every formula $Q_n \in T(n, d_n)$ are also contained in $(-1, 1)$. But this is not so, in general, for every weight function $w$ and every $n \in \mathbb{N}$. The following example shows that there exist even Chebyshev quadrature formulas in the strict sense, i.e., $d_n > n$, with nodes not all contained in $[-1, 1]$.

Let $w$ be a weight function on $(-1, 1)$ with $w(x) := (1 - x^2)^{-4/5}$. A simple calculation with the help of Newton's identities (see [7, p. 104]) shows that for $n = 3, 4, 6, 7$ the Chebyshev quadrature formulas in the strict sense exist and that their first and last nodes are not contained in $[-1, 1]$.

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2. L. A. ANDERSON & W. GAUTSCHI, "Optimal weighted Chebyshev-type quadrature formulas."


4. K.-J. FÖRSTER, "Bemerkungen zur optimalen Tschebyscheff-Typ Quadratur,"


11. P. RABINOWITZ & N. RICHTER, "Chebyshev-type integration rules of minimum norm,"

Supplement to
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6. PROOFS OF THE THEOREMS

Fundamental for our considerations are Newton's well-known identities [7, p. 104]:

Let $t_1, \ldots, t_n$ be a solution of

\begin{equation}
\sum_{j=1}^{n} t_j^j = u_j, \quad j = 1, 2, \ldots, n,
\end{equation}

for given values $u_j \in \mathbb{R}$. Let

\begin{equation}
g(t) := \sum_{i=1}^{n} (t-t_i) := t^n + \sum_{i=1}^{n} c_i t^{n-i}.
\end{equation}

Then

\begin{equation}
\begin{align*}
c_1 &= -u_1, \\
c_1 u_1 + 2 c_2 &= -u_2, \\
c_1 u_2 + c_2 u_1 + 3 c_3 &= -u_3, \\
&\vdots \\
c_1 u_{n-1} + c_2 u_{n-2} + \ldots + c_{n-1} u_1 + n c_n &= -u_n.
\end{align*}
\end{equation}

If, conversely, the coefficients $c_1$ are a solution of the system (6.3) then the roots $t_i$ of the corresponding function $g(t)$ in (6.2) satisfy (6.1).

With respect to (1.3) the following holds for all $Q_n \in \mathcal{T}(n,d)$ and $j = 1, \ldots, d$

\begin{equation}
\sum_{i=1}^{n} x_i^j = \frac{n m_j}{\tilde{p}_0}.
\end{equation}

Let $P_n(Q_n)$ be the polynomial of degree $n$ whose roots are the nodes of $Q_n$. 

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S21
(6.5) \[ P_n(Q_n)(x) := \sum_{i=1}^{n} (x-x_i) \]
\[ = x^n + \sum_{i=1}^{d} a_i x^{n-i} + \sum_{i=d+1}^{n} b_i x^{n-i} . \]

It follows from (6.3) and (6.4) that the coefficients \( a_i \) in (6.5) of all \( P_n(Q_n) \) with \( Q_n \in \mathcal{T}(n,d) \) are identical. Conversely, every \( P_n \) of type (6.5) with these coefficients \( a_i \) and only real roots represents a formula \( Q_n \in \mathcal{T}(n,d) \).

Let \( R_n \) be that polynomial of type (6.2) whose coefficients (6.3)

are uniquely given by (6.4) with \( j=1, \ldots, n \):

(6.6) \[ R_n(x) := x^n + \sum_{i=1}^{n} e_i x^{n-i} . \]

It follows from (6.3) for \( Q_n \in \mathcal{T}(n,d) \) and the corresponding \( P_n(Q_n) \)

\[ R_n [P_{d+1}] = \int_a^b w(x) x^{d+1} dx - \frac{m_d}{n} \sum_{i=1}^{n} x_i^{d+1} \]

\[ = \frac{m_d}{n} (d+1) b_{d+1} \]

\[ + \sum_{i=1}^{d+1} a_i \sum_{j=1}^{n} m_{d+1-j} \]

\[ = \frac{m_d}{n} (d+1) (b_{d+1} - e_{d+1}) . \]

\( Q_n \in \mathcal{T}(n,d) \) is therefore E-minimal (E-maximal) in \( \mathcal{T}(n,d) \) if and only if the coefficient \( b_{d+1} \) of \( P_n(Q_n) \) is minimal (maximal) under the restriction of only real roots of \( P_n(Q_n) \) [11].

Corresponding to the sequence \( S(Q_n) \) (see (2.4)) we define a sequence \( V(Q_n) \) as follows. Let \( k \) be the number of sign changes of \( S(Q_n) \). If there is a sign change between \( s_i(Q_n) \) and \( s_{i+1}(Q_n) \), \( \Omega_{i-1} \in Z \cap Z \), the pair \((u,v)\) is given by

\[ (u,v) := (x_{n+1-i}, x_{n-i}) . \]

The sequence \( V(Q_n) := \{(u_i,v_i)\}_{i=1}^{k} \) consists of all these pairs ordered by \( u_i \leq u_{i+1} \). By this definition, it is possible that \( u_1 = v_1 = u_1+1 = v_1+1 \) and for \( k = 0 \) that \( V(Q_n) \) is empty. We define

\[ K(Q_n) \in (0,1) \] by

(6.8) \[ K(Q_n) := \begin{cases} 0, & \text{if all elements of } S(Q_n) \text{ are zero} \\ 1, & \text{if an element of } S(Q_n) \text{ is different from zero.} \end{cases} \]

Lemma 1: Let \( Q_n \in \mathcal{T}(n,d) \). Let \( k \) be the number of sign changes of \( S(Q_n) \). If

(6.9) \[ k + K(Q_n) < n-d, \]

then \( Q_n \) is not E-extremal in \( \mathcal{T}(n,d) \).

Proof: We have to show the existence of \( \bar{Q}_n = \hat{Q}_n \in \mathcal{T}(n,d) \) with

(6.10) \[ \bar{Q}_n [P_{d+1}] < R_n [P_{d+1}] < \hat{Q}_n [P_{d+1}] . \]

One has \( k = 0 \) for \( d = n-1 \) and \( k \leq n-d-2 \) for \( d > n-1 \). If \( d = n-1 \) we choose two real numbers \( \gamma \geq x_n \) and \( y_1 \leq x_1 \). If \( d < n-1 \) we choose \( (n-d) \) real numbers \( \gamma \geq x_n \) and \( y_1 \leq x_1 \) by means of the sequence \( V(Q_n) = \{(u_i,v_i)\}_{i=1}^{k} \) as follows

\[ y_1 \leq (u_i,v_i) \] and \( y_j \leq x_j \) \( \forall j=1, \ldots, n \), for \( u_1 = v_1 \)

\[ y_1 = u_1 \]

\[ \gamma \geq x_n \]

\[ y_{n-d-1} \leq \cdots \leq y_{k+1} \leq x_1 . \]

Let \( a \) be the first nonzero element of \( S(Q_n) \). If all elements of \( S(Q_n) \) are zero we define \( a := -1 \). We consider polynomials \( \bar{h}, \hat{h} \) of degree \( n-d-1 \) given by
\[ h(x) := \frac{n-d-1}{d} \sum_{i=1}^{d-1} (x-y_i). \]

For \( d = n-1 \) we obtain \( \tilde{h} = -a \) and \( \hat{h} = a \). First, we consider the case \( a = -1 \). In every interval \((y_{2i+1}, y_{2i})\), \( i = 0, 1, \ldots, \) by definition of \( S(Q_n) \) and \( V(Q_n) \) all relative minima of \( P_n(Q_n) \) are negative. Let \( M_1 \) be the (negative) maximum of all these minima. A similar argument yields the positivity of all relative maxima in every interval \((y_{2i}, y_{2i-1})\). We denote by \( M_2 \) the (positive) minimum of all these maxima. Therefore the following numbers \( \bar{m}, \bar{n} \) are positive,

\[ \bar{m} := \frac{\min\{M_1, M_2\}}{\max\{|h(x)|\}}. \]

Each \( Y_p \in (y_{r+1}, y_{r-1}) \) is a root of \( P_n(Q_n) \) and \( Y_{r+1} \) as well as \( Y_{r-1} \) are not roots of \( P_n(Q_n) \). Especially \( Y_0 \) and \( Y_{n-d-1} \) are not roots of \( P_n(Q_n) \) by (6.11).

Let \( \ell \) be the number of distinct \( Y_p \in (y_{r+1}, y_{r-1}) \), i.e., the number of elements of the set \( \{y_{r+1}, y_{r+1-1}, \ldots, y_{r-2, r-1}\} \). We denote these elements by \( z_0, z_1, z_2, \ldots, z_\ell \). Let \( z_p \) be the multiplicity of the roots of \( \tilde{h} \) at \( z_p \) and let \( w_p \) be the number of roots of \( P_n(Q_n) \) in the interval \((z_{p+1}, z_p)\) for \( p = 1, \ldots, \ell-1 \). We remark that \( P_n(Q_n) \) has for \( 1 < p < \ell \) in \( z_p \) a root of multiplicity \( z_p > 2 \). \( P_n(Q_n) \) has in \((y_{r+1}, y_{r-1})\) the same number of roots as \( P_n(Q_n) \) if \( P_n(Q_n) \) - \( \tilde{h} \) has in \((z_{p+1}, z_p)\) for \( p + 1, p + \ell-1 \) at least \( w_p+2 \) roots and for \( p = 1 \) and \( p = \ell-1 \) at least \( w_p+1 \) roots.

This can be shown by similar argument as in case 1), considering the fact, that on the one hand for \( 1 < p < \ell-1 \) the multiplicity of the root \( z_p \) of \( P_n(Q_n) \) is equal to the multiplicity of the root of \( \tilde{h} \) in \( z_p \) plus two and that on the other hand there exists a \( c > 0 \) so that \( \tilde{h} \) and \( P_n(Q_n) \) are of same constant sign in \( z_p-\ell \) as well as in \( z_p+\ell \).

By these two cases all roots of \( P_n(Q_n) \) have been taken into account.
once. So $P_n(Q_n)$ and $P_n(Q_n) - \hat{m} \hat{n}$ have the same number of real roots. $\hat{n}$ is a polynomial of degree $n-d-1$. Therefore the formula $\hat{\Omega}_n$ resulting from $\hat{\Omega}_n = P_n(Q_n) - \hat{m} \hat{n}$ is in $T(n,d)$. The leading coefficient $\hat{m} \hat{n}$ is positive by virtue of (6.12). Therefore (6.7) implies the left part of (6.10). For $q=1$ the argumentation is similar. $\hat{n}$ and $\hat{n}$ have to be changed.

Remark 1: The proof remains valid if in (6.13) the value $\hat{m} (m)$ is replaced by $\hat{m} = c \hat{m}$ $(m := c \hat{m})$ with $c \in (0,1)$. Therefore (6.10) implies the fact that under the assumptions of Lemma 1 there is a positive $s \in \mathbb{R}$ so that for any $r$ with

$$r \in \{ R_{n}(P_{d+1}) - s, R_{n}(P_{d+1}) + s \}$$

a formula $\hat{\Omega}_n$ exists in $T(n,d)$ with $\hat{\Omega}_n = r$.

Remark 2: If $P_n(Q_n)$ has roots of multiplicity higher than one, then the maximal multiplicity of the roots of $P_n(Q_n) - \hat{m} \hat{n}$ or $P_n(Q_n) - \hat{m} \hat{n}$ with any $\hat{m}$ chosen as in Remark 1) is lower than the maximal multiplicity before. For these formulas $\hat{\Omega}_n$ and $\hat{\Omega}_{n+1}$, Eq. (6.9) is also valid. So if a formula $Q_n \in T(n,d)$, then there exists an infinite number of formulas in $T(n,d)$ with pairwise distinct nodes. Let $\hat{\Omega}_n$ be an $E$-extremal formula in $T(n,d)$. Lemma 1 implies that $S(\hat{\Omega}_n)$ has for $d = n-1$ at least one nonzero element and for $d < n-1$ at least $n-d-1$ changes of sign. With definition (2.4), this yields the proof for the first part of Theorem 2.

The next lemma follows from Theorem 1 of [5] together with Remark 1, there.

Lemma 2: Let $Q_n$ and $\hat{\Omega}_n$ be in $T(n,d)$. Let $L := Q_n - \hat{\Omega}_n$ with

$$L[f] := \sum_{i=1}^{\hat{m}} A_i f(B_i), \quad i=1, \ldots, m, \quad B_1 < B_2 < \ldots < B_\hat{m}.$$ 

Let $q$ be defined by $B_q := \max \{ B_i | A_i \not\in \emptyset, i=1, \ldots, m \}$ and let $A$ be the following sequence

$$A := \left( \sum_{k=1}^{m} \sum_{i=1}^{n_d} A_{i \hat{k}} \right).$$

Then

1) $A$ has at least $d$ changes of sign.
2) If $A$ has at most $d$ changes of sign, then for every $f \in \mathbb{C}^{d+1}$, $\mathbb{R}[B_1, B_\hat{m}], f(d+1) > 0$, $\lim_{x \to \infty} f(x) = \infty$, and $\lim_{x \to -\infty} f(x) = -\infty$,

$$\text{sign}(A_q) \geq \text{sign}(A_{q+1}) \varrho_{n}[f] \geq \text{sign}(A_{q+1}) \varrho_{n}[f].$$

In the following we first consider the case $d < n-1$. Let $\bar{X}_i$ be the nodes of an $E$-extremal formula $\hat{\Omega}_n$ in $T(n,d)$. Lemma 1 and (2.4) imply that there exist $n-d$ nodes $\bar{X}_i$ with $g(i) \neq g(i+1)$ for $i=1, \ldots, n-d-1$ so that

$$\bar{X}_i = \bar{X}_i, \quad i=1, \ldots, n-d-1.$$ 

(6.15)

$$g(i) + g(i+1) \text{ is odd for } i=1, \ldots, n-d-1.$$ 

Let $\bar{X}_i = \bar{X}_i + \bar{X}_i$. Then the number of nodes between $\bar{X}_i$ and $\bar{X}_i$ is equal to $g(i+1) - g(i+1) - 2$. Eq. (6.15) shows that this number is odd. Let this number be $2 \delta(i)-1$. In case of $\bar{X}_i = \bar{X}_i$ for avoiding multiple counting of the same node - we define $2 \delta(i)-1 := -1$. Let $\ell(0) := g(i)-1$ and $\ell(n-d) := g(n-d)-1$. Then follows

$$n = \ell(0) + \ell(n-d) + 2(n-d) + \sum_{i=1}^{n-d-1} (2 \delta(i)-1),$$

(6.16)

$$d = 1 + \ell(0) + \ell(n-d) + 2 \sum_{i=1}^{n-d-1} \delta(i).$$

In Lemma 2 let for $Q_n \in T(n,d)$
For the nodes $B_1$ we require
\[ B_1 < B_2 < \ldots < B_m \cdot \]
\[
\left( B_1, B_2, \ldots, B_m \right) = \left( x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n} \right) .
\]

We see that some coefficients $\alpha_i$ may be zero. Let $A_{r,s}$ be the sequence
\[
A_{r,s} := \left\{ \frac{k}{\gamma} \right\} \mathbf{A}_r \quad 1 \leq r \leq s \leq m
\]
and by (6.17)
\[
\tilde{B}_i = x_i .
\]

We require that $\tilde{x}_i(i) = \tilde{x}_{i+1}(i)$, implies $\tilde{g}(i) = \tilde{g}(i+1)$ and therefore implies that
$A_2(i), \tilde{g}(i+1)$ has no sign change, i.e., by the above definition of $\ell(i)$ this sequence has $2\ell(i)$ sign changes. In the case
\[
\tilde{g}(i) = \tilde{g}(i+1)
\]
the number of nodes in the interval $B_{\tilde{g}(i)}, B_{\tilde{g}(i+1)}$ is at least $2\ell(i)-1$. For an estimation of the number of sign changes of $A_{\tilde{g}(i)}, \tilde{g}(i+1)$ we remark, that every element of the sequence $A_r$ is of the type $c_1 \epsilon$ with $c_1 \epsilon x$ and $c_1$ defined in (1.4). The case $A_r$ and $A_{r+1}$, $r \geq 0$ negative implies therefore that in the interval $B_r, B_{r+1}$ there are at least two nodes of $\tilde{g}_n$. This means that the sequence $A_2(i), \tilde{g}(i+1)$ has at most $2\ell(i)$ changes of sign. This number is reduced to $2\ell(i)-1$ if
$A_2(i), \tilde{g}(i)$ or $A_2(i+1), \tilde{g}(i+1)$ is not negative. In the following this last remark is of importance in the case $\ell(n-d) = 0$ and
$x_n \geq x_n$, because $A_{\tilde{g}(n-d)}, \tilde{g}(n-d) = A_{\tilde{g},m} = 0$. So we have shown that the number of sign changes of the sequence $A_{\tilde{g}(1)}, \tilde{g}(n-d)$ is at most
\[
2 \ell(i) - 1
\]
\[
\text{or even}
\]
\[
2 \ell(i) - 1
\]
\[
\text{if } \ell(n-d) = 0 \text{ and } x_n \geq x_n
\]
\[
\text{and } \tilde{g}(i) \neq \tilde{g}(n-d).
\]

We now consider the number of sign changes of the sequences $A_1, \tilde{g}(1)$ and $A_{\tilde{g}(n-d),m}$. We distinguish between the following cases.

I) $\ell(0)$ even or $\ell(0) = 0$
II) $\ell(0)$ even or $\ell(0) = 0$

(6.21) Ia) $\ell(0)$ odd
Ib) $\ell(0)$ odd

1) Ia) $x_n \geq x_n$
2) 2a) $x_n \geq x_n$

(6.22) Ia) $x_n \geq x_n$

1b) $x_n \geq x_n$
2b) $x_n \geq x_n$

1c) $x_n \geq x_n$
2c) $x_n \geq x_n$

By the same argumentation as above we get the following results.

The number of sign changes of $A_1, \tilde{g}(1)$ is at most
\[
\ell(0) + 1 \quad \text{for Ia) or Ib)}
\]
\[
\text{if } A_{\tilde{g}(1)}, \tilde{g}(1) < 0
\]
\[
\text{In case of every other combination of I and I1) the number of sign changes of } A_1, \tilde{g}(1) \text{ is at most } \ell(0).
\]

Because of $A_{n,m} = 0$ the number of sign changes of $A_{\tilde{g}(n-d),m}$ is at most
\[
\ell(n-d) \quad \text{for Ia) or IIa2a)
\]
\[
\text{II) } \ell(n-d) = 0 .
\]
For any other combination of II) and 2) there are at most \(\ell(n-d)-1\) sign changes of this sequence.

By virtue of (6.16) and (6.19), (6.20), (6.2), (6.24) the sequence \(A\) has at most \(d\) changes of sign. This number can only be achieved in the following cases

A) \(x_n < x_{n+1}, x_1 > x_1, \ell(n-d)\) odd, \(\ell(0)\) odd,

B) \(x_n < x_{n+1}, x_1 < x_1, \ell(n-d)\) odd, \(\ell(0)\) even or \(\ell(0)=0\),

C) \(x_n > x_{n+1}, x_1 > x_1, \ell(n-d)\) even or \(\ell(n-d)=0, \ell(0)\) odd,

D) \(x_n > x_{n+1}, x_1 < x_1, \ell(n-d)\) even or \(\ell(n-d)=0, \ell(0)\) even or \(\ell(0)=0\).

So we have shown for every \(Q_n\) in \(T(n,d)\) and for every \(f (c^{d+1})\), \(f(d+1) \geq 0\), with the help of Lemma 2, the following result

(6.25)

\[ R_n[f] \geq \hat{R}_n[f] \quad \text{for } \ell(n-d) \text{ odd,} \]

\[ R_n[f] \leq \hat{R}_n[f] \quad \text{for } \ell(n-d) \text{ even or } \ell(n-d)=0. \]

The number \(\ell(n-d)\) resp. \(\ell(0)\) is independent of the choice of \(Q_n\) in \(T(n,d)\). Therefore, \(Q_n\) is E-minimal for odd \(n-d\) and E-maximal for even \(n-d\) or \(\ell(n-d)=0\). This proves Theorem 5.

(6.24) resp. (6.25) follows only from the fact that \(S(Q_n)\) has at least \(n-d-1\) changes of sign. So by means of Lemma 1, we have proven the first part of Theorem 2a. The second part of Theorem 2a follows from the definition of \(\ell(n-d)\) and (2.4).

In the cases A and D in (6.24), \(d\) is odd; in the other two cases \(d\) is even. This proves Theorem 4 and therefore also Theorem 1.

In the case \(d = n-1\), these theorems can be proven in the same way or with the help of the methods given in [4]. We remark that by Lemma 1 an E-extremal formula has at least one multiple node \(x_g(t)\).

Theorem 3 follows immediately from Theorem 2 and Remarks 1 and 2. To prove Theorem 3 we consider the corresponding formula \(Q_n\) in Theorem 3 and ask for "E-extremal" formulas under the additional assumption \(R_n[\hat{P}_{d^*}]=r\). By the methods given above, it follows that these "E-extremal" formulas have at most \(d+1\) distinct nodes (see Theorem 2).

Let \(n \geq 2\) and therefore \(d_n \geq 1\). Let \(Q_{n,d}^{\min}\) and \(Q_{n,d}^{\max}\) be the E-minimal and the E-maximal formula in \(T(n,d)\). It follows from the definition of \(d_n\) and from Theorem 3 that

(6.26) \(0 \in [Q_{n,d}^{\min}[P_{d_n^*}], Q_{n,d}^{\max}[P_{d_n^*}]]\).

By definition, the E-optimal formula \(Q_n^{\opt}\) must be either E-minimal or E-maximal. This proves Theorem 6 for \(n \geq 2\).

For \(n = 1\), \(T(1,1)\) has only one element, so the conclusion is valid. For \(n = 2\), there remains the case \(d_2 = 1\). For every \(Q_2\) in \(T(2,1)\) \(P(Q_2)\) is a parabola with positive leading coefficient. So there exists a unique E-maximal formula in \(T(2,1)\). This is E-optimal by the same arguments as above. An E-minimal formula doesn't exist.

For the proof of Theorem 3 we first consider the case that \(Q_n^{\opt}\) is the E-minimal formula in \(T(n,d_n)\). It follows from (6.26) and Theorem 5 that

(6.27) \(R_n^{\opt}[P_{d_n^*}] > 0\),

(6.28) \(R_n^{\opt}[f] \leq \bar{R}_n[f]\)

for every \(Q_n\) in \(T(n,d)\) and every \(f \in T(n,d)\). (6.27) and (6.28) imply the assertion. E-maximal \(Q_n^{\opt}\) the inequality signs in (6.27) and (6.28) have to be reversed.

It remains to prove Corollary 3a for \(n = 12\). One has \(d_{12} = 9\) and \(R_{12}^{\opt}[P_{10}] > 0\). By Theorem 5 and the symmetry of \(Q_n^{\opt}\) we have to show...
(6.29) \[ r_{12}^{opt} \{P_{21+10}\} \geq 0 \]

for every \( i \in \mathbb{N} \). With Peano's representation of the remainder term - cf. [3,p.39] - there follows

\[ (6.30) \quad \frac{1}{2} \frac{\Delta (2i)}{i(2i+10)} r_{12}^{opt} \{P_{21+10}\} = \int_{a}^{b} x^{2i} k_{10}^{opt}(x) \, dx. \]

\( k_{10}^{opt} \) denotes the Peano kernel of highest degree with respect to \( \Delta_{12} \cdot k_{10}^{opt} \) has in \((0,b)\) only one change of sign and is negative at the origin - see [1,p.65]. The positivity of \( r_{12}^{opt} \) therefore implies

\[ (6.31) \quad \int_{0}^{b} k_{10}^{opt}(x) \, dx > 0. \]

From (6.30) and (6.31) the inequality (6.29) follows in view of the monotonicity of \( P_{21} \) in \((0,b)\).