A Lower Bound for the Class Number of Certain Cubic Number Fields

By Günter Lettl

Abstract. Let \( K \) be a cyclic number field with generating polynomial
\[
\chi^3 - \frac{a - 3}{2} \chi^2 - \frac{a + 3}{2} \chi - 1
\]
and conductor \( m \). We will derive a lower bound for the class number of these fields and list all such fields with prime conductor \( m = \frac{a^2 + 27}{4} \) or \( m = \frac{1 + 27b^2}{4} \) and small class number.

1. Introduction. Let \( h_m \) denote the class number of the cyclotomic field \( \mathbb{Q}(\zeta_m) \), and \( h_m^+ \), the class number of its maximal real subfield \( \mathbb{Q}(\cos(2\pi/m)) \). It is a well-known conjecture of Vandiver that \( p \mid h_p^+ \) holds for all primes \( p \in \mathbb{P} \). This is a customary assumption for proving the second case of Fermat’s Last Theorem (for more details see Washington [16]). Since \( h_p^+ \) grows slowly (\( h_p^+ = 1 \) for \( p < 163 \) with the use of the Generalized Riemann Hypothesis (GRH), van der Linden [10]), for no \( p \) with \( h_p^+ > 1 \) the exact value of \( h_p^+ \) is known without using GRH. Masley suggested that perhaps \( h_p^+ < p \) always holds, but a counterexample was found in [3], [12]. The class number of each real subfield of \( \mathbb{Q}(\zeta_p) \) divides \( h_p^+ \), and in this way one can find primes with \( h_p^+ > 1 \). Using the quadratic subfield, Ankeny, Chowla and Hasse [1] showed that \( h_p^+ > 1 \) if \( p \) belongs to certain quadratic sequences in \( \mathbb{N} \), and S.-D. Lang [9] and Takeuchi [15] found more such sequences. Similar results were obtained for \( h_{4p}^+ \) by Yokoi [17]. Using the cubic subfield of \( \mathbb{Q}(\zeta_p) \), which has been thoroughly investigated (e.g., [2], [5], [8]), the theorem of the present paper yields the following results:

If \( a \) is an odd integer, \( a > 23 \), and \( p = \frac{a^2 + 27}{4} \), a prime, then \( h_p^+ > 1 \).

If \( b \) is an odd integer, \( b > 1 \), and \( p = \frac{1 + 27b^2}{4} \), a prime, then \( h_p^+ > 1 \).

A conjecture about primes in quadratic sequences (Hardy and Wright [7, I.2.8]) implies that there exist infinitely many primes \( p \) of each of these two forms, because one can write
\[
\frac{a^2 + 27}{4} = \left( \frac{a - 3}{2} \right)^2 + 3 \left( \frac{a - 3}{2} \right) + 9
\]
and
\[
\frac{1 + 27b^2}{4} = 3 \left( \frac{3b - 1}{2} \right)^2 + 3 \left( \frac{3b - 1}{2} \right) + 1.
\]
2. Class Number Bounds and Main Results. Let $K$ be a cyclic cubic number field with conductor $m$ and class number $h$. It is well known that $m$ is the product of distinct primes, which are congruent to 1 mod $(6)$, and of 9, if 3 ramifies in $K$. The class number $h$ is congruent to $1$ mod $(3)$, if $m$ is a prime or $m = 9$, and $h$ is divisible by $3$ otherwise. Set $f(s) = L(s, \chi) \cdot L(s, \bar{\chi})$ for $s \in \mathbb{C}$, where $\chi$ and $\bar{\chi}$ are the nontrivial cubic Dirichlet characters modulo $(m)$ belonging to $K$. Since the discriminant of $K$ equals $m^2$, the analytic class number formula yields

$$h = \frac{m \cdot f(1)}{4 \cdot R},$$

where $R$ is the regulator of $K$. Moser [11] showed that for prime conductors, $h < m/3$ holds, so cubic fields will never lead to a contradiction to Vandiver's conjecture. Our aim is to establish a lower bound for the class number of a special family of cubic fields and to list all fields of some special types with prime conductor and small class number. From a result of Stark [14] one can deduce $f(1) > c/\log m$, where $c$ is effectively computable, but this bound is not suited for our purposes. From the results of the next section we will obtain:

$$f(1) > 0.023 \cdot m^{-0.054}.$$  

The harder problem is to find an upper bound for the regulator, which is only achieved for the following family of cyclic cubic fields. The polynomial

$$f_a(X) = X^3 - \frac{a - 3}{2} X^2 - \frac{a + 3}{2} X - 1, \quad a \in \mathbb{N} \text{ odd},$$

is irreducible over $\mathbb{Q}$, has discriminant $D(f_a) = ((a^2 + 27)/4)^2$, and if $\epsilon$ is a zero of $f_a$, the other zeros are $\epsilon' = -1/(\epsilon + 1)$ and $\epsilon'' = -(\epsilon + 1)/\epsilon$. Therefore, $f_a$ is a generating polynomial of a cyclic cubic field $K$ with conductor $m$, and we define $k \in \mathbb{N}$ by $D(f_a) = (a^2 + 27)/4 = km$.

We call the field $K$ of type A, if $k = 1$, and of type B, if $k = 27$ and $a = 27b$ with $b \in \mathbb{N}$ odd, $b \neq 1$ (in this case we have $m = (1 + 27b^2)/4$). It is well known that fields of type A or B have relatively large class numbers (see, for example, the tables of Gras [5]). Shanks [13] states that for cubic fields of type A with prime conductor "a rough mean value for $h$ is given by $h \approx 12m/35(\log m)^2$".

**Lemma 1.** Let $K$ be a cyclic cubic field with generating polynomial $f_a$, conductor $m$ and regulator $R$. Then,

$$4R < \left(\frac{1}{2} \log D(f_a)\right)^2 = (\log(km))^2.$$  

**Proof of Lemma 1.** Since the zeros $\epsilon, \epsilon', \epsilon''$ of $f_a$ are units of $K$, we can estimate the regulator of $K$ by $R \leqslant \text{Reg}(\{\epsilon, \epsilon', \epsilon''\}) =: R'$, if $R' \neq 0$ (see Lemma 4.15 in [16]). Choosing

$$\epsilon = \frac{a - 3 + 4\sqrt{km} \cdot \cos(1/3 \cdot \arctan(\sqrt{27}/a))}{6} \sim \sqrt{km}$$
with the principal value of arctan, we obtain

\[
R' = \left| \det \begin{pmatrix} \log|e| & \log|e'| \\ \log|e'| & \log|e''| \end{pmatrix} \right|
\]

\[
= (\log|e + 1|)^2 - \log|e + 1| \log|e| + (\log|e|)^2.
\]

Series expansions yield

\[
R' = \frac{1}{4} (\log km)^2 - \frac{3 \log(km)}{2km} + \frac{3}{4km} + O\left(\frac{\log km}{(km)^2}\right)
\]

and elementary calculus explicitly gives (4).

With (2) and Lemma 1 we immediately obtain from (1):

Let \( K \) be a cyclic cubic field with conductor \( m > 10^5 \) and with generating polynomial \( f_a \). Then,

\[
h > 0.023 \frac{m^{0.946}}{(\log km)^2}.
\]

**Theorem.** (a) Let \( K \) be a cyclic cubic field of type A with prime conductor \( m \). Then \( h < 16 \) holds only for the following values of \( m \):

<table>
<thead>
<tr>
<th>( h )</th>
<th>( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7, 13, 19, 37, 79, 97, 139</td>
</tr>
<tr>
<td>4</td>
<td>163, 349, 607, 709, 937</td>
</tr>
<tr>
<td>7</td>
<td>313, 877, 1129, 1567, 1987, 2557</td>
</tr>
<tr>
<td>13</td>
<td>1063</td>
</tr>
</tbody>
</table>

(b) Let \( K \) be a cyclic cubic field of type B with prime conductor \( m \). Then \( h < 43 \) holds only for the following values of \( m \):

<table>
<thead>
<tr>
<th>( h )</th>
<th>( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>61, 331</td>
</tr>
<tr>
<td>4</td>
<td>547, 1951</td>
</tr>
<tr>
<td>7</td>
<td>2437, 3571</td>
</tr>
<tr>
<td>13</td>
<td>9241</td>
</tr>
<tr>
<td>28</td>
<td>4219, 25117</td>
</tr>
<tr>
<td>31</td>
<td>23497</td>
</tr>
<tr>
<td>37</td>
<td>8269</td>
</tr>
</tbody>
</table>

**Proof of the Theorem.** From (5) we obtain \( h > 14 \) for fields of type A with \( m \geq 169339 \), and \( h > 37.2 \) for fields of type B with \( m > 10^6 \). It is well known (see, e.g., Gras [4]) that primes \( q \equiv -1 \pmod{3} \) divide the class number of a cyclic cubic field only with an even exponent. The table of class numbers of Shanks [13], and Table 1 below, complete the proof of the theorem.
Table 1
Class numbers of cyclic cubic fields of type B
with prime conductor \( m < 10^6 \)

<table>
<thead>
<tr>
<th>( b )</th>
<th>( m = \frac{1 + 27b^2}{4} )</th>
<th>( h )</th>
<th>( b )</th>
<th>( m = \frac{1 + 27b^2}{4} )</th>
<th>( h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>61</td>
<td>1</td>
<td>173</td>
<td>202021</td>
<td>316 = 2^2 \cdot 79</td>
</tr>
<tr>
<td>7</td>
<td>331</td>
<td>1</td>
<td>185</td>
<td>231019</td>
<td>343 = 7^3</td>
</tr>
<tr>
<td>9</td>
<td>547</td>
<td>4 = 2^2</td>
<td>189</td>
<td>241117</td>
<td>1216 = 2^6 \cdot 19</td>
</tr>
<tr>
<td>17</td>
<td>1951</td>
<td>4 = 2^2</td>
<td>191</td>
<td>246247</td>
<td>175 = 5^3 \cdot 7</td>
</tr>
<tr>
<td>19</td>
<td>2437</td>
<td>7</td>
<td>193</td>
<td>251431</td>
<td>247 = 13 \cdot 19</td>
</tr>
<tr>
<td>23</td>
<td>3571</td>
<td>7</td>
<td>199</td>
<td>267307</td>
<td>196 = 2^2 \cdot 7^2</td>
</tr>
<tr>
<td>25</td>
<td>4219</td>
<td>28 = 2^2 \cdot 7</td>
<td>205</td>
<td>283669</td>
<td>541</td>
</tr>
<tr>
<td>33</td>
<td>7351</td>
<td>49 = 7^2</td>
<td>221</td>
<td>329677</td>
<td>316 = 2^2 \cdot 79</td>
</tr>
<tr>
<td>35</td>
<td>8269</td>
<td>37</td>
<td>227</td>
<td>347821</td>
<td>331</td>
</tr>
<tr>
<td>37</td>
<td>9241</td>
<td>13</td>
<td>231</td>
<td>360187</td>
<td>1732 = 2^3 \cdot 433</td>
</tr>
<tr>
<td>39</td>
<td>10267</td>
<td>49 = 7^2</td>
<td>235</td>
<td>372769</td>
<td>553 = 7 \cdot 79</td>
</tr>
<tr>
<td>45</td>
<td>13669</td>
<td>109</td>
<td>243</td>
<td>398581</td>
<td>1075 = 5^2 \cdot 43</td>
</tr>
<tr>
<td>59</td>
<td>23497</td>
<td>31</td>
<td>259</td>
<td>452797</td>
<td>769</td>
</tr>
<tr>
<td>61</td>
<td>25117</td>
<td>28 = 2^2 \cdot 7</td>
<td>261</td>
<td>459817</td>
<td>2257 = 37 \cdot 61</td>
</tr>
<tr>
<td>91</td>
<td>55897</td>
<td>133 = 7 \cdot 19</td>
<td>297</td>
<td>595411</td>
<td>2299 = 11^2 \cdot 19</td>
</tr>
<tr>
<td>95</td>
<td>60919</td>
<td>193</td>
<td>299</td>
<td>603457</td>
<td>739</td>
</tr>
<tr>
<td>105</td>
<td>74419</td>
<td>688 = 2^4 \cdot 43</td>
<td>301</td>
<td>611557</td>
<td>889 = 7 \cdot 127</td>
</tr>
<tr>
<td>115</td>
<td>89269</td>
<td>211</td>
<td>305</td>
<td>627919</td>
<td>1552 = 2^4 \cdot 97</td>
</tr>
<tr>
<td>117</td>
<td>92401</td>
<td>532 = 2^2 \cdot 7 \cdot 19</td>
<td>341</td>
<td>784897</td>
<td>688 = 2^4 \cdot 43</td>
</tr>
<tr>
<td>123</td>
<td>102121</td>
<td>307</td>
<td>347</td>
<td>812761</td>
<td>769</td>
</tr>
<tr>
<td>129</td>
<td>112327</td>
<td>604 = 2^2 \cdot 151</td>
<td>361</td>
<td>879667</td>
<td>688 = 2^4 \cdot 43</td>
</tr>
<tr>
<td>131</td>
<td>115837</td>
<td>148 = 2^2 \cdot 37</td>
<td>367</td>
<td>909151</td>
<td>787</td>
</tr>
<tr>
<td>137</td>
<td>126691</td>
<td>97</td>
<td>371</td>
<td>929077</td>
<td>1588 = 2^3 \cdot 397</td>
</tr>
<tr>
<td>147</td>
<td>145861</td>
<td>652 = 2^2 \cdot 163</td>
<td>373</td>
<td>939121</td>
<td>661</td>
</tr>
<tr>
<td>159</td>
<td>170647</td>
<td>628 = 2^2 \cdot 157</td>
<td>383</td>
<td>990151</td>
<td>532 = 2^2 \cdot 7 \cdot 19</td>
</tr>
</tbody>
</table>

The class numbers of Table 1 were calculated with a “Sirius 1 Personal Computer”, using the analytic class number formula (1). We also used that for fields of type B the roots of \( f_a \) are already fundamental units, and therefore \( R = R' \) can be calculated with the explicit formula for \( e \), given in the proof of Lemma 1. In the following way it can be proved that \( e \) is a fundamental unit:

Let \( K \) be a field of type B with generating polynomial \( f_a, \ a = 27b \) and \( m = (1 + 27b^2)/4 \). Hasse [8] investigated the arithmetic of cyclic cubic fields, using the Gauss sums of the corresponding Dirichlet characters. With Hasse’s notation, every integer \( \alpha \in K \) can be written as \( \alpha = \{x, y\} \) with \( x \in \mathbb{Z}, \ y \in \mathbb{Z}[\rho] \), where \( \rho^2 + \rho + 1 = 0 \), and \( x \equiv y \mod (1 - \rho) \). If \( \alpha \) is a unit of \( K \), \( N(\alpha) = 1 \) implies \( x^3 \equiv 27 \mod (m) \) and \( |x| \leq 2\sqrt{m|y|} \) (Satz 8, [8]). For the roots of \( f_{27b} \) we have \( e = [(27b - 3)/2, 3i\sqrt{3}] \) and its conjugates. Since Godwin’s conjecture about fundamental units holds for cyclic cubic fields with \( m > 9 \) (see Gras [6]), we have to show:

There exists no unit \( \alpha = \{x, y\} \in K, \ \alpha \neq \pm 1, \ \text{with} \ |y| < 2\sqrt{27m} \). Suppose the contrary. Then \( x^3 \equiv 27 \mod (m) \) and \( |x| < 2\sqrt{27m} \) imply \( x \in \{3, (27b - 3)/2, -(27b + 3)/2\} \) for \( b \geq 7 \). Considering \( 0 \equiv x \equiv y \mod (1 - \rho) \) and \( xy < 27 \) yields only a few possibilities for \( y \in \mathbb{Z}[\rho] \), and one can check that for each
of these \( y \), \( N(\alpha) = 1 \) has no solution \( \alpha \neq 1 \). For small values of \( b \), one can consult the table in [5].

In the same way, but with much less computation, one can prove that for \( k = 1 \) (type A) and \( k = 3 \) the roots of \( f_a \) are also fundamental units. In these cases one has \( \varepsilon = [(a - 3)/2, \pm 1] \) with \((a - 3)/2 \equiv \pm 1 \pmod{3}\), and \( \varepsilon = [(9b - 3)/2, i\sqrt{3}] \), respectively.

3. A Lower Bound for \( L(1, \chi) \cdot L(1, \bar{\chi}) \). Let \( m \) be the conductor of a cyclic cubic field \( K \), \( \chi \) and \( \bar{\chi} \) the nontrivial cubic Dirichlet characters modulo \( m \) associated with \( K \), and \( f(s) = L(s, \chi)L(s, \bar{\chi}) \). To find a lower bound for \( f(1) \), we first need an upper bound for \( |f(s)| \) in a disk in \( \mathbb{C} \) containing 1. Consider \( C = C(\mu, \rho) = \{ s \in \mathbb{C} | |s - \mu| < \rho \} \) with \( 1 < \mu \) and \( \mu - 1 < \rho < \mu \), and set \( \sigma_0 = \mu - \rho \). Let \( s = \sigma + it \in \mathbb{C} \). For \( \sigma > 0 \) we have the representation

\[
L(s, \chi) = \sum_{n=1}^{m} \frac{\chi(n)}{n^s} + s \int_{m}^{\infty} \frac{S(x, \chi)}{x^{s+1}} \, dx \quad \text{with} \quad S(x, \chi) = \sum_{1 \leq n < x} \chi(n)
\]

(see [16, p. 211]). The inequality of Pólya-Vinogradov [16, Lemma 11.8] states that \( |S(x, \chi)| < \sqrt{m} \cdot \log m \). For \( s \in C(\mu, \rho) \), the function \( |s|/\sigma = 1/\cos(\arg s) \) attains its maximum \( \mu/\sqrt{\mu^2 - \rho^2} \) if \( s \) is the point of contact of a tangent of \( C \) through 0.

Combining these results, we obtain for every \( s \in C(\mu, \rho) \):

\[
|L(s, \chi)| < 1 + \int_{1}^{m} \frac{1}{\sigma_0} \, dx + |s|\sqrt{m} \cdot \log m \int_{m}^{\infty} \frac{1}{x^{\sigma+1}} \, dx
\]

\[
< \frac{1}{1 - \sigma_0} m^{1-\sigma_0} + \frac{\mu}{\sqrt{\mu^2 - \rho^2}} \log m \cdot m^{0.5 - \sigma_0}.
\]

Since \( \log x/\sqrt{x} \) is monotone decreasing for \( x \geq e^2 \), we conclude that for \( m \geq m_0 \geq e^2 \),

\[
|f(s)| < c_1 \cdot m^{2-2\sigma_0}
\]

holds for all \( s \in C(\mu, \rho) \), with

\[
c_1 = \left( \frac{1}{1 - \sigma_0} + \frac{\mu}{\sqrt{\mu^2 - \rho^2}} \cdot \frac{\log m_0}{m_0} \right)^2.
\]

Lemma 2. If \( K \) is a cyclic cubic number field with conductor \( m \), then \( f(1) > c_6 \cdot m^{-c_7} \), with \( c_6, c_7 > 0 \) as given in the course of the proof. Furthermore, \( c_7 \) can be made arbitrarily small.

Proof of Lemma 2. The proof follows mainly Washington [16, pp. 212–214]. Let \( \xi(s) \) be the Riemann zeta function and \( \xi_K(s) = \xi(s)/f(s) \) the zeta function of the cyclic cubic field \( K \) with conductor \( m \). If \( s = \sigma + it \in \mathbb{C} \), we have

\[
\xi_K(s) = 1 + \sum_{n=2}^{\infty} a_n/n^s \quad \text{for} \quad \sigma > 1,
\]

with \( a_n \geq 0 \), and \( a_n \geq 1 \) if \( n \) is a cube. Developing \( \xi_K \) in a power series around \( \mu > 1 \) gives

\[
\xi_K(s) = \sum_{j=0}^{\infty} b_j (\mu - s)^j,
\]
with
\( b_0 = \xi_K(\mu) > \xi(3\mu) > 1 \) and \( b_j = \frac{1}{j!} \sum_{n=2}^{\infty} (\log n)^j \cdot \frac{a_n}{n^\mu} > 0 \) for \( j \geq 1 \).

The integral representation of \( \xi(s) \) for \( \sigma > 0 \) yields
\[
|\xi(s)| \leq \left| \frac{s}{s-1} \right| + |s| \int_1^\infty \frac{1}{u^{\sigma+1}} \, du = \left| \frac{s}{s-1} \right| + \frac{|s|}{\sigma}
\]
and
\[
|\xi(s)| \leq \left| \frac{s}{s-1} \right| + |s| \cdot \sum_{n=1}^{\infty} \frac{1}{n^{\sigma+1}} \cdot \int_1^{n+1} (u-[u]) \, du
\]
\[
< \left| \frac{s}{s-1} \right| + \frac{|s|}{2} \left( 1 + \frac{1}{\sigma} \right).
\]

Let \( C = C(\mu, \rho) \), with \( \mu - 1 < \rho < \mu \), be the disk with center \( \mu \) and radius \( \rho \), and denote its boundary by \( \partial C \). Using (6), we get for all \( s \in \partial C \):
\[
|\xi_K(s) - f(1)| \leq |\xi(s)| \cdot |f(s)| + \frac{1}{|s-1|} \cdot |f(1)| < c_2 \cdot m^{2-2\sigma_0},
\]
with
\[
c_2 = c_1 \cdot \max_{s \in \partial C} \left( \left| \frac{s}{s-1} \right| + |s| \cdot \min \left( \frac{1}{\sigma}, \frac{1}{2} \left( 1 + \frac{1}{\sigma} \right) \right) \right).
\]

Since \( \xi_K(s) - f(1)/(s - 1) \) is holomorphic in the whole complex plane, (8) holds for all \( s \in C(\mu, \rho) \). Computing the coefficients of
\[
\xi_K(s) - \frac{f(1)}{s-1} = \sum_{j=0}^{\infty} \left( b_j - \frac{f(1)}{(\mu - 1)^{j+1}} \right) \cdot (\mu - s)^j
\]
with a Cauchy integral gives
\[
\left| b_j - \frac{f(1)}{(\mu - 1)^{j+1}} \right| = \left| \frac{1}{2\pi i} \int_{\partial C} \left( \xi_K(s) - \frac{f(1)}{s-1} \right) \frac{ds}{(s-\mu)^{j+1}} \right| < \frac{c_2}{\rho^j} \cdot m^{2-2\sigma_0}.
\]

For \( 0 < \sigma < 1 \), the integral representation of \( \xi(s) \), and \( f(\sigma) = |L(\sigma, \chi)|^2 \), show that \( \xi_K(\sigma) \leq 0 \). So for any \( \alpha \) with \( \sigma_0 < \alpha < 1 \), and any \( \nu \in \mathbb{R}^+ \) with \( 1 < \nu \), we have
\[
-\frac{f(1)}{\alpha-1} \geq \xi_K(\alpha) - \frac{f(1)}{\alpha-1} > \sum_{j=0}^{[\nu]-1} \left( b_j - \frac{f(1)}{(\mu - 1)^{j+1}} \right) \cdot (\mu - \alpha)^j
\]
\[
\geq c_3 - \frac{f(1)}{\alpha-1} - \frac{f(1)}{1-\alpha} \left( \frac{\mu - \alpha}{\mu - 1} \right)^\nu - c_2 \cdot m^{2-2\sigma_0} \left( \frac{\mu - \alpha}{\rho} \right)^{\nu-1} \cdot \frac{\rho}{\alpha - \sigma_0},
\]
where \( \sum_{j=0}^{[\nu]-1} b_j (\mu - \alpha)^j \geq c_3 \geq 1 \).

From this inequality, we obtain
\[
f(1) > c_3 (1-\alpha) \left( \frac{\mu - 1}{\mu - \alpha} \right)^\nu - c_2 \cdot m^{2-2\sigma_0} \cdot \frac{\rho^2 (1-\alpha)}{(\mu - \alpha)(\alpha - \sigma_0)} \left( \frac{\mu - 1}{\rho} \right)^{\nu}.
\]
Choosing \( \nu = c_4 \cdot \log m + c_5 \), with
\[
c_4 = \frac{2 - 2\alpha_0}{\log \frac{\rho}{\mu - \alpha}}
\]
and
\[
c_5 = \frac{\log \frac{c_2 \cdot \rho^2}{(\mu - \alpha)(\alpha - \sigma_0)} + \log \log \frac{\rho}{\mu - 1} - \log \log \frac{\mu - \alpha}{\mu - 1}}{\log \frac{\rho}{\mu - \alpha}},
\]
gives \( f(1) > c_6 \cdot m^{-c_7} \), with
\[
c_6 = c_3 (1 - \alpha) \left( \frac{\mu - 1}{\mu - \alpha} \right)^{c_5} - c_2 \cdot \frac{\rho^2 (1 - \alpha)}{(\mu - \alpha)(\alpha - \sigma_0)} \cdot \left( \frac{\mu - 1}{\rho} \right)^{c_5}
\]
and
\[
c_7 = (2 - 2\alpha_0) \cdot \log \frac{\mu - \alpha}{\mu - 1} / \log \frac{\rho}{\mu - \alpha}.
\]
Since \( c_7 \to 0 \) for \( \alpha \to 1 \), the proof of Lemma 2 is completed.

Numerical computations show that for \( m_0 = 10^5 \) good results can be obtained by choosing \( m = 10, \rho = 9.9 \) and \( \alpha = 0.975 \). With these values we obtain \( c_2 = 10.8685 \) and \( \nu = 315 \).

Using (7), and \( a_n \geq 1 \) for \( n \) a cube, we obtain the following estimations for \( c_3 \):
\[
\sum_{j=0}^{[\nu] - 1} b_j (\mu - \alpha)^j \geq \xi K(\mu) + \sum_{j=1}^{300} \frac{1}{j!} \sum_{n=2}^{\infty} \frac{a_n}{n^\mu} ((\mu - \alpha) \log n)^j
\]
\[
> \xi (3\mu) + \sum_{k=2}^{N_0} \frac{1}{k^{3\mu}} \sum_{j=1}^{300} \frac{1}{j!} ((\mu - \alpha) 3 \cdot \log k)^j
\]
\[
> 1 + \sum_{k=2}^{N_0} \frac{1}{k^{3\mu}} \left( k^{3(\mu - \alpha)} - \frac{((\mu - \alpha) 3 \cdot \log k)^{301}}{301!} \cdot \frac{302}{302 - (\mu - \alpha) 3 \cdot \log k} \right),
\]
where \( N_0 < e^{302/(3\mu - \alpha)} \). With the special values for \( \mu, \rho \) and \( \alpha \), and \( N_0 = 100 \), we obtain
\[
\sum_{j=0}^{[\nu] - 1} b_j (\mu - \alpha)^j > \sum_{k=1}^{100} k^{-2.925} - 10^{-40} > 1.2175 = c_3.
\]
These values yield \( c_6 > 0.023 \) and \( c_7 < 0.054 \), and thus (2) is proved.

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