An Estimate of Goodness of Cubatures for the Unit Circle in $\mathbb{R}^2$

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Abstract. The Sarma-Eberlein estimate $s_E$ is an estimate of goodness of cubature formulae for $n$-cubes defined as the integral of the square of the formula truncation error, over a function space provided with a measure. In this paper, cubature formulae for the unit circle in $\mathbb{R}^2$ are considered and an estimate of the above type is constructed with the desirable property of being compatible with the symmetry group of the circle.

1. Isometries and Two-dimensional Cubature Formulae. Let

$$(1.1) \quad S_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

be the unit circle in the two-dimensional Euclidean space $\mathbb{R}^2$ and let $\mathcal{U}(S_2)$ denote the symmetry group of $S_2$. This group consists of all linear bijective maps $u: \mathbb{R}^2 \to \mathbb{R}^2$ which preserve the Euclidean distance (that is, isometries of $\mathbb{R}^2$ leaving the origin invariant). Each element of $\mathcal{U}(S_2)$ can be identified with a $2 \times 2$ real orthogonal matrix and therefore

$$(1.2) \quad \mathcal{U}(S_2) = \{ u_\alpha, u_\alpha \circ v; \alpha \in [0, 2\pi)\},$$

where $u_\alpha$ denotes the rotation of $\alpha$ radians around the origin and $v$ is the reflection about any fixed straight line passing through the origin; thus

$$(1.3) \quad u_\alpha(x, y) = (x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha), \quad v(x, y) = (x, -y).$$

Let $w(x, y)$ be a normalized weight function compatible with $\mathcal{U}(S_2)$, that is, a real positive continuous function in the interior of $S_2$ such that

$$(1.4) \quad \iint_{S_2} w(x, y) \, dx \, dy = 1 \quad \text{and} \quad w \circ u = w \quad \text{for all} \ u \in \mathcal{U}(S_2).$$

A cubature formula for the $w$-weighted circle $S_2$ has the form

$$(1.5) \quad I(f) = Q_N(f) + E(f),$$
where

\[ I(f) = \iint_{S^2} w(x, y)f(x, y) \, dx \, dy, \]

(1.6)

\[ Q_N(f) = \sum_{i=1}^{N} A_i f(x_i, y_i), \quad (x_i, y_i) \in S^2, \]

and the constants \( A_i \) are independent of \( f \).

Let us consider a symmetry \( u \in \mathcal{U}(S^2) \) acting on (1.5). Since \( I(f \circ u) = I(f) \), it leads to another cubature formula

\[ I(f) = Q'_N(f) + E'(f), \]

(1.7)

where

\[ Q'_N(f) = Q_N(f \circ u) = \sum_{i=1}^{N} A_i f(u(x_i, y_i)), \]

(1.8)

\[ E'(f) = E(f \circ u). \]

Definition 1. For every \( u \in \mathcal{U}(S^2) \), the cubature formulae (1.5) and (1.7) are said to be \( \mathcal{U}(S^2) \)-equivalent or equivalent with respect to the symmetry group of \( S^2 \).

The integration of a function on the \( w \)-weighted circle \( S^2 \) is independent of the pair of orthogonal axis \( OX, OY \) whose origin \( O \) lies in the center of the circle. Therefore, all \( \mathcal{U}(S^2) \)-equivalent formulae have identical characteristics when they are considered as approximations of the operator \( I \).

Therefore, any estimate of goodness for cubature formulae (1.5) should be compatible with the \( \mathcal{U}(S^2) \)-equivalence relation, that is, all \( \mathcal{U}(S^2) \)-equivalent formulae should have the same estimate of goodness. For instance, the degree of precision of a cubature formula (1.5) is an estimate compatible with \( \mathcal{U}(S^2) \), because the space of polynomials of degree at most \( n \) is invariant under all the symmetries in (1.2).

The aim of this paper is to construct an \( \mathcal{U}(S^2) \)-compatible estimate of goodness of cubature formulae for \( S^2 \) similar to that defined by V. L. N. Sarma in [3] for cubatures for the square.

The next section is devoted to recalling briefly some characteristics of the Sarma-Eberlein estimate that are useful for our purpose. A detailed exposition of the construction of this estimate can be found in [3], [4] and [5] and an excellent summary of these results in [6, pp. 188–192].

2. The Sarma-Eberlein Estimate of Goodness \( s_E \). Let us consider the square

\[ C^2 = \{(x, y) \in \mathbb{R}^2: |x| \leq 1, |y| \leq 1\} \]

and cubature formulae

\[ I(f) = Q_N(f) + E(f), \]

(2.1)

where

\[ I(f) = \frac{1}{4} \iint_{C^2} f(x, y) \, dx \, dy, \]

(2.2)

\[ Q_N(f) = \sum_{i=1}^{N} A_i f(x_i, y_i), \quad (x_i, y_i) \in C^2. \]
Sarma in [3], [4] defines the estimate of goodness of the cubature formula (2.1) as

\[ s_E^2 = \int_{F_{S_\infty}} E(f)^2 \, df, \]

where the integral is defined over the unit sphere of a normed space of functions provided with a measure defined as follows:

Let \( l_1 \) be the space of real sequences

\[ f = \{ f_{nk}; \; n = 0, 1, \ldots; \; k = 0, 1, \ldots, n \} \]

such that

\[ \| f \|_1 = \sum_{n,k} |f_{nk}| < \infty; \quad n = 0, 1, \ldots; \; k = 0, 1, \ldots, n. \]

The unit sphere \( S_\infty = \{ f \in l_1; \| f \|_1 \leq 1 \} \) is compact in the weak*-topology of \( l_1 \), and an elementary integral defined for the weak*-continuous real functions on \( S_\infty \) can be extended by the Daniell process inducing a countably additive measure on \( S_\infty \).

Among the properties of this measure, let us recall that

\[ \int_{S_\infty} f_{nk} f_{ml} \, df = 0 \quad \text{if} \quad (n, k) \neq (m, l), \]

\[ \int_{S_\infty} f_{nk}^2 \, df = \frac{2^{n+2}}{(n+2)!(n+3)!} = q_n^2. \]

Real two-dimensional power series

\[ f(x, y) = \sum_{n,k} f_{nk} x^{n-k} y^k; \quad n = 0, 1, \ldots; \; k = 0, 1, \ldots, n, \]

whose coefficients satisfy the condition

\[ \| f \|_1 = \sum_{n,k} |f_{nk}| < \infty \]

converge uniformly and absolutely for all points \((x, y) \in C_2\).

The space \( Fl_1 \) of all functions defined by (2.8) and (2.9) can be identified with the sequence space \( l_1 \) and is dense in the space \( \mathcal{C}(C_2) \) of all real continuous functions on \( C_2 \) with the uniform norm. This identification allows us to consider the above integral as an integral over the unit sphere \( F_{S_\infty} \) of the function space \( Fl_1 \).

The truncation error \( E(f) \) of the cubature formula (2.1) is a continuous linear form over \( \mathcal{C}(C_2) \) with the uniform norm and therefore also over \( Fl_1 \) with the \( \| \cdot \|_1 \)-norm. Using (2.6) and (2.7), it follows that the estimate \( s_E \) defined by (2.3) can be written as

\[ s_E^2 = \sum_{n=0}^\infty q_n^2 e_n^2, \]

where \( q_n \) is defined in (2.7) and

\[ e_n^2 = \sum_{k=0}^n E(x^{n-k} y^k)^2. \]

It should be noted that the identification of \( l_1 \) and \( Fl_1 \) is made through the monomials \( x^{n-k} y^k \) and the use of these particular functions makes \( s_E \) compatible with \( \mathcal{C}(C_2) \), the symmetry group of \( C_2 \), in the sense described in the previous
section. In effect, \( \mathcal{U}(C_2) \) consists of the eight symmetries
\[
(x, y) \to (\pm x, \pm y); \quad (x, y) \to (\pm y, \pm x)
\]
and the equalities
\[
e_n^2 = \sum_{k=0}^{n} E(x^{n-k}y^k)^2 = \sum_{k=0}^{n} E((\pm x)^{n-k}(\pm y)^k)^2
\]
\[
e_n^2 = \sum_{k=0}^{n} E((\pm y)^{n-k}(\pm x)^k)^2
\]
imply that \( \mathcal{U}(C_2) \)-equivalent cubature formulae have the same estimate of goodness \( s_E \). Unfortunately, this estimate of goodness is not useful for cubature formulae (1.5), (1.6) for the unit circle \( S_2 \), because it is not compatible with \( \mathcal{U}(S_2) \), as can be computationally checked. For instance, taking \( w(x, y) = 1/\pi \), the cubature formula (degree 3, 4 points) given by
\[
Q_4(f) = \frac{1}{4} \left[ f(\sqrt{2}/2, 0) + f(-\sqrt{2}/2, 0) + f(0, \sqrt{2}/2) + f(0, -\sqrt{2}/2) \right]
\]
has an estimate of goodness \( s_E = (-4)^{1.75032} \), whereas the \( \mathcal{U}(S_2) \)-equivalent formula (use a rotation of \( \pi/4 \) radians) given by
\[
Q_4(f) = \frac{1}{4} \left[ f(1/2, 1/2) + f(-1/2, 1/2) \right.
\]
\[
\left. + f(1/2, -1/2) + f(-1/2, -1/2) \right]
\]
has an estimate of goodness \( s_E = (-4)^{3.81547} \).

3. An Estimate of Goodness of Cubatures for the Unit Circle. In the previous section, the sequence space \( l_1 \) was identified with the space of functions \( F_{l_1} \) by using the family of monomials \( \{ x^{n-k}y^k; n = 0, 1, \ldots; k = 0, 1, \ldots, n \} \), but we can also identify \( l_1 \) with other subspaces of \( \mathcal{U}(C_2) \) or \( \mathcal{U}(S_2) \) by using other families of polynomials. For each \( n \), let us denote
\[
M_n = \{ a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n; a_i \in \mathbb{R} \}
\]
and let
\[
\Phi_n = \{ \varphi_{n0}, \ldots, \varphi_{nn} \} \subset M_n
\]
be a basis of \( M_n \), i.e., \( M_n = \text{span} \Phi_n \).

If the polynomials \( \varphi_{nk} \) satisfy
\[
\| \varphi_{nk} \|_{\infty} = \max_{(x, y) \in S_2} |\varphi_{nk}(x, y)| \leq c; \quad n = 0, 1, \ldots; k = 0, 1, \ldots, n,
\]
then the series
\[
f(x, y) = \sum_{n,k} f_{nk} \varphi_{nk}(x, y)
\]
whose coefficients satisfy (2.9) converge uniformly and absolutely for all points \( (x, y) \in S_2 \). If we denote \( \Phi = \{ \Phi_1, \Phi_2, \ldots \} \), the space \( F_{l_1}(\Phi) \) of all functions defined by (3.4) and (2.9) can be identified with the sequence space \( l_1 \). Let us note that \( F_{l_1}(\Phi) \) contains all real polynomials in two variables and therefore is dense in \( \mathcal{U}(S_2) \) with the uniform norm.

This identification allows us to define, in a natural way, an estimate of goodness for cubatures (1.5) by
\[
s_E^2(\Phi) = \int_{F_{l_1}(\Phi)} E(f)^2 \, df,
\]
where
where
\[(3.6)\] \(FS_\infty(\Phi) = \left\{ f \in Fl_1(\Phi) : \sum_{n,k} |f_{nk}| \leq 1 \right\}.\]

It is straightforward to deduce that this estimate can be expressed by
\[(3.7)\] \(s_E^2(\Phi) = \sum_{n=0}^{\infty} q_n^2 e_n^2(\Phi_n),\]
where \(q_n^2\) is given in (2.7) and
\[(3.8)\] \(e_n^2(\Phi_n) = \sum_{k=0}^{n} E(\varphi_{nk})^2.\)

Our problem at this stage is to choose suitable families \(\Phi_n\) satisfying (3.3), such that the estimate \(s_E^2(\Phi)\) is compatible with the symmetry group \(\mathcal{U}(S_2)\) in the sense described in Section 1.

As the matrix
\[(3.9)\] \[
\begin{pmatrix}
\cos \alpha, & -\sin \alpha \\
\sin \alpha, & \cos \alpha
\end{pmatrix}
\]
associated with the rotation \(u_\alpha \in \mathcal{U}(S_2)\) has eigenvalues \(e^{i\alpha}, e^{-i\alpha}\) and eigenvectors \((1, i)^T, (1, -i)^T\), the use of complex arithmetic will simplify the calculations. Let us denote
\[(3.10)\] \(M_n^* = \left\{ a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n ; a_i \in \mathbb{C} \right\},\)
and let
\[(3.11)\] \(\Phi_n^* = \left\{ \varphi_{n0}^*, \ldots, \varphi_{nn}^* \right\}\)
be a basis of \(M_n^*\), i.e., \(M_n^* = \text{span}^* (\Phi_n^*)\).

Considering the natural complexification of linear operators
\[(3.12)\] \(E(f + ig) = E(f) + iE(g)\)
with the standard complex notation
\[(3.13)\] \[|E(f + ig)|^2 = E(f + ig) E(f + ig) = E(f)^2 + E(g)^2,\]
we can define
\[(3.14)\] \[e_n^2(\Phi_n^*) = \sum_{k=0}^{n} |E(\varphi_{nk}^*)|^2.\]

**Theorem 1.** For every \(n\), let \(\Phi_n^* = \{ \varphi_{n0}^*, \ldots, \varphi_{nn}^* \}\) and \(\Phi_n = \{ \varphi_{n0}, \ldots, \varphi_{nn} \}\) be bases of \(M_n^*\) and \(M_n\), respectively, satisfying
(i) \((\varphi_{n0}, \ldots, \varphi_{nn})^T = A_n (\varphi_{n0}^*, \ldots, \varphi_{nn}^*)^T\)
where \(A_n\) is an \(n \times n\) complex unitary matrix, i.e., \(A^\dagger = A^{-1}\);
(ii) \(\sum_{k=0}^{n} |E(\varphi_{nk}^*)|^2 = \sum_{k=0}^{n} |E(\varphi_{nk} \circ u_\alpha)|^2 = \sum_{k=0}^{n} |E(\varphi_{nk} \circ u_\alpha \circ v)|^2\) for all \(\alpha \in [0, 2\pi)\);
(iii) there exists a \(c \in \mathbb{R}\) such that \(\|\varphi_{nk}\|_\infty \leq c\) for all \(n, k\).

Then, the estimate \(s_E(\Phi)\) associated with the family \(\Phi = \{ \Phi_0, \Phi_1, \ldots \}\) is compatible with the symmetry group \(\mathcal{U}(S_2)\).
Proof. Let us remark that the operators
\[ f^* \in M_n^* \rightarrow E(f^*) \in \mathbb{C}, \]
\[ f^* \in M_n^* \rightarrow f^* \circ u_a \in M_n^*, \]
\[ f^* \in M_n^* \rightarrow f^* \circ u_a \circ v \in M_n^* \]
are linear and therefore "pass through the matrix \( A_n^* \)."
Moreover, \( E(\varphi_{nk}) \) and \( E(\varphi_{nk} \circ u) \) are real and therefore
\[
\sum_{k=0}^{n} E(\varphi_{nk} \circ u_a)^2 \\
= (E(\varphi_{n0} \circ u_a), \ldots, E(\varphi_{nn} \circ u_a))(E(\varphi_{n0} \circ u_a), \ldots, E(\varphi_{nn} \circ u_a))^T \\
= (\sum_{k=0}^{n} |E(\varphi_{nk} \circ u_a)|^2)^2 \\
= (E(\varphi_{n0}^*), \ldots, E(\varphi_{nn}^*))E(\varphi_{n0}), \ldots, E(\varphi_{nn}))^T \\
= (E(\varphi_{n0}), \ldots, E(\varphi_{nn})) A_n A_n^H E(\varphi_{n0}), \ldots, E(\varphi_{nn}))^T \\
= \sum_{k=0}^{n} E(\varphi_{nk}^2),
\]
given that \( A_n \) is unitary. Similarly, it can be shown that
\[
\sum_{k=0}^{n} E(\varphi_{nk} \circ u_a \circ v)^2 = \sum_{k=0}^{n} E(\varphi_{nk})^2,
\]
and therefore it follows in a straightforward way that \( s_E(\Phi) \) is compatible with \( \mathcal{U}(S_2) \).

Now let us consider the complex polynomials
\[
(3.15) \quad \varphi_{nk}^* = (x + iy)^{n-k}(x - iy)^k \in M_n^*
\]
on obtained from the monomials \( x^{n-k}y^k \) by a linear transformation with Jacobian
\[
J = \begin{vmatrix} 1 & i \\ 1 & -i \end{vmatrix} = -2i,
\]
so that \( \varphi_{n0}^*, \ldots, \varphi_{nn}^* \) are linearly independent in \( M_n^* \).

Also,
\[
(\varphi_{nk}^* \circ u_a)(x, y) = \varphi_{nk}^*(x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha) \\
= e^{i(n-k)\alpha}(x + iy)^{n-k}e^{-ik\alpha}(x - iy)^k = e^{i(n-2k)\alpha} \varphi_{nk}^*(x, y),
\]
thus
\[
(3.16) \quad \sum_{k=0}^{n} |E(\varphi_{nk}^* \circ u_a)|^2 = \sum_{k=0}^{n} |E(\varphi_{nk}^*)|^2.
\]
Similarly,
\[
(\varphi_{nk}^* \circ u_a \circ v)(x, y) = (\varphi_{nk}^* \circ u_a)(x, -y) = e^{i(n-2k)\alpha}(x - iy)^{n-k}(x + iy)^k \\
= e^{i(n-2k)\alpha} \varphi_{n,n-k}^*(x, y),
\]
and then

\[ \sum_{k=0}^{n} \left| E \left( \varphi_{nk}^{*} \circ u_{n} \circ v \right) \right|^2 = \sum_{k=0}^{n} \left| E \left( \varphi_{nk}^{*} \right) \right|^2. \]  

(3.17)

Therefore, for each \( n \), the family \( \Phi_{n}^{*} = \{ \varphi_{00}^{*}, \ldots, \varphi_{nn}^{*} \} \) is a basis of \( M_{n}^{*} \) which satisfies the hypothesis (ii) of Theorem 1.

For \( k < n/2 \) let us define

\[ \varphi_{nk} = \frac{1}{\sqrt{2}} \left( \varphi_{nk}^{*} + \varphi_{n,n-k}^{*} \right) \]

\[ = \frac{1}{\sqrt{2}} \left( (x^2 + y^2)^k \left( (x + iy)^{n-2k} + (x - iy)^{n-2k} \right) \right), \]

(3.18)

\[ \varphi_{n,n-k} = \frac{1}{\sqrt{2}i} \left( \varphi_{nk}^{*} - \varphi_{n,n-k}^{*} \right) \]

\[ = \frac{1}{\sqrt{2}i} \left( (x^2 + y^2)^k \left( (x + iy)^{n-2k} - (x - iy)^{n-2k} \right) \right), \]

(3.19)

and if \( n \) is even,

\[ \varphi_{n,n/2} = \varphi_{n,n/2}^{*} = (x^2 + y^2)^{n/2}. \]

(3.20)

Then, \( \Phi_{n} = \{ \varphi_{00}, \ldots, \varphi_{nn} \} \) is formed by polynomials with real coefficients and is a basis of \( M_{n} \). Also the matrix \( A_{n} \) of Theorem 1 that relates the elements of \( \Phi_{n} \) and \( \Phi_{n}^{*} \) is a unitary matrix, because the matrices

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}}, & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}i}, & -\frac{1}{\sqrt{2}i}
\end{pmatrix}
\]

that relate the pairs \((\varphi_{nk}, \varphi_{n,n-k})\) and \((\varphi_{nk}^{*}, \varphi_{n,n-k}^{*})\) are unitary. Moreover, it can easily be shown that

\[ \| \varphi_{nk} \|_{\infty} = \| \varphi_{n,n-k} \|_{\infty} = \frac{\sqrt{2}}{k < n/2}, \]

and \( \| \varphi_{n,n/2} \|_{\infty} = 1 \) for \( n \) even.

Using the results above, and applying Theorem 1, we deduce the following

**Theorem 2.** Let \( \Phi = \{ \Phi_{0}, \Phi_{1}, \ldots \} \) where, for each \( n \), \( \Phi_{n} = \{ \varphi_{00}, \ldots, \varphi_{nn} \} \) is the basis of \( M_{n} \) defined by (3.18), (3.19) and (3.20). Then the estimate \( s_{E}(\Phi) \) defined by (3.5) is an estimate of goodness of cubature formulae for the unit circle that is compatible with the symmetry group \( \mathcal{U}(S_{2}) \).

Following the proof of Theorem 1, we can also deduce that

\[ s_{E}^{2}(\Phi) = \sum_{n=0}^{\infty} q_{n}^{2} \sum_{k=0}^{n} E(\varphi_{nk})^2 = \sum_{n=0}^{\infty} q_{n}^{2} \sum_{k=0}^{n} \left| E(\varphi_{nk}) \right|^2, \]

(3.21)

and therefore the estimate \( s_{E}(\Phi) \) can be calculated using any of these two expressions.
Table 1 shows the values of $s_E(\Phi)$ for some cubature formulae (1.5) for the unit circle with $w(x, y) = 1/\pi$. The nomenclature of these formulae corresponds to the one in [6, pp. 277–289]. N stands for the number of nodes and D for the degree of precision.

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