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Hecke Operators and the Fundamental Domain for SL(3, Z)

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Dedicated to Daniel Shanks

Abstract. We report on a detailed study of the fundamental domain for the special linear group $\mathrm{SL}(3, \mathbf{Z})$ of 3×3 integral matrices with determinant one. Graphs of points coming from the action of Hecke operators are considered.

1. $\mathrm{SL}(2, \mathbf{Z})$. The *modular group* $\Gamma_2 = \mathrm{SL}(2, \mathbf{Z})$ consists of all 2×2 integer matrices of determinant one. An element $\gamma \in \Gamma_2$ acts, as does any element of $\mathrm{SL}(2, \mathbf{R})$, on the *Poincaré upper half-plane*

$$H = \{ z = x + iy \mid x, y \in \mathbf{R},\ y > 0 \}$$

via fractional linear transformation: $z \mapsto \gamma z = (az + b)/(cz + d)$, for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A *fundamental domain* D for Γ_2 is a connected closed subset $D \subset H$ behaving like the quotient space $\Gamma_2 \backslash H$, at least up to boundary identifications. Thus, for every $z \in H$, there is a $\gamma \in \Gamma_2$ such that $\gamma z \in D$. Moreover, if z and w lie in the interior of D and $z = \gamma w$ for $\gamma \in \Gamma_2$, then $\gamma = \pm I$, where I is the identity matrix.

It is easily seen (cf. Terras [17]) that the region

$$F_2 = \left\{ z \in H \mid -\tfrac{1}{2} \leqslant \operatorname{Re} z \leqslant \tfrac{1}{2},\ |z| \geqslant 1 \right\} \tag{1.1}$$

is a fundamental domain for $\mathrm{SL}(2, \mathbf{Z})$. The usual method of moving $z \in H$ to F_2 is called the "*highest-point method*", i.e., you choose $\gamma \in \Gamma_2$ to maximize $\operatorname{Im}(\gamma z)$. The process of moving z to $\gamma z \in D$ is called a *reduction algorithm*. It can be done by a sequence of *flips* by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and *translations*

$$T = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \qquad n \in \mathbf{Z}.$$

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This gives a continued fraction type algorithm, and the maps

$$z \mapsto -1/z \quad \text{and} \quad z \mapsto z + 1$$

generate the projective linear group $\mathrm{PSL}(2, \mathbf{Z}) = \mathrm{SL}(2, \mathbf{Z})/\pm I$.

In Figure 1, we picture F_2 using coordinates $v = 1/y, x = x$. These coordinates are chosen, since we will use analogous ones for $\mathrm{SL}(3, \mathbf{Z})$. They have the advantage of giving us a bounded region to graph. On the other hand, the coordinates are still Euclidean. One would have to take $\log y$ to obtain noneuclidean coordinates. For the details of the results mentioned above, as well as background for the rest of this paper, see Terras [17]. Our goal here is to come to an understanding of the fundamental domain for $\mathrm{SL}(3, \mathbf{Z})$ which is as good as that for $\mathrm{SL}(2, \mathbf{Z})$.

There are many applications of the study of $\mathrm{SL}(2, \mathbf{Z}) \backslash H$. For example, one can use the fundamental domain to give an easy algorithm for the computation of class numbers of imaginary quadratic number fields, and Sarnak has used Selberg's trace formula for $\mathrm{SL}(2, \mathbf{Z})$ (which is a noneuclidean analogue of the Poisson sum formula) to obtain asymptotic results on units in real quadratic fields. There are also applications in physics (cf. Gutzwiller [5]). Discussions of the fundamental domain for $\mathrm{SL}(2, \mathbf{Z})$ and some of its applications can be found in Terras [17, Sections 3.3–3.7]. Similar applications are envisioned for $\Gamma_3 = \mathrm{SL}(3, \mathbf{Z})$.

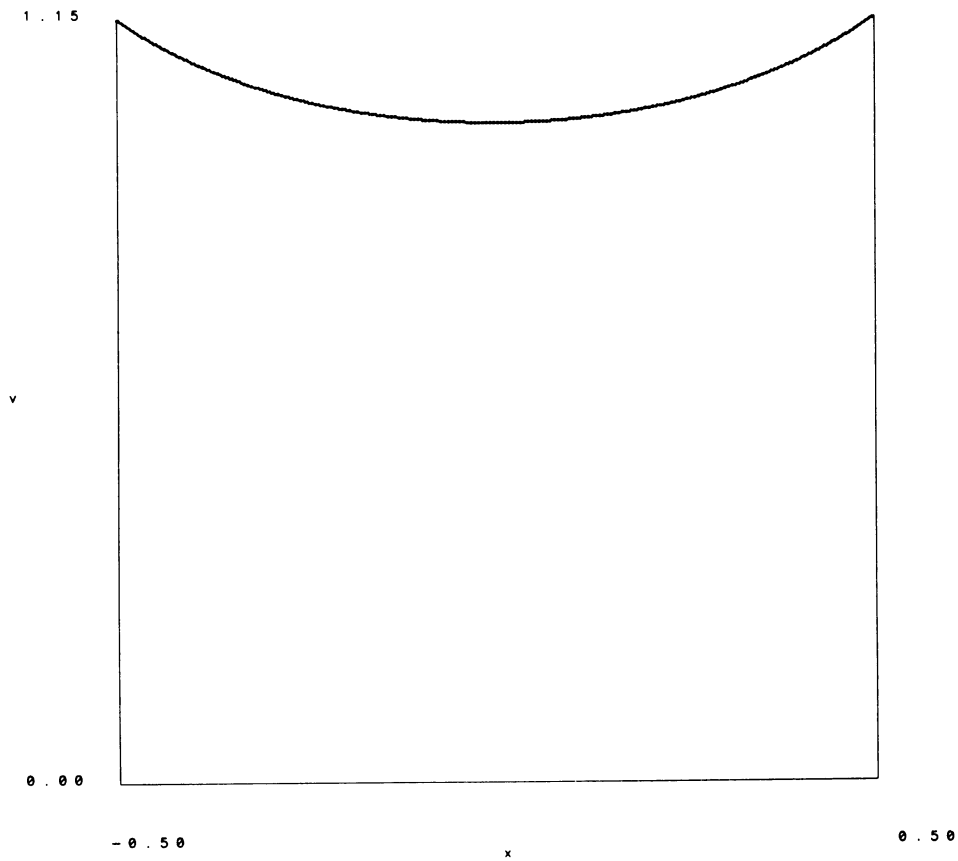

FIGURE 1

The standard fundamental domain F_2 for $\mathrm{SL}(2, \mathbf{Z})$ transformed by $v = 1/y$

Hecke operators T_n have been investigated since the times of Hecke, Hurwitz, Mordell, and Ramanujan. They are important for the derivation of Euler products of L-functions corresponding to automorphic forms. Here we consider the *Hecke operator* T_n acting on functions f: $\mathrm{SL}(2, \mathbf{Z}) \backslash H \to \mathbf{C}$ via

$$T_n f(z) = \sum_{\sigma \in \Gamma_2 \backslash M_2(n)} f(\sigma z),$$

where

$$M_2(n) = \{\sigma \in \mathbf{Z}^{2\times 2} \mid \det \sigma = n\}.$$

It is not hard to find representatives for the quotient $\Gamma_2 \backslash M_2(n)$ and show that

$$T_n f(z) = \sum_{\substack{ad=n,\ d>0,\\ b \bmod d}} f((az+b)/d). \tag{1.2}$$

We have left out the normalizing factors which are usually introduced here. They are not of interest for our discussion, and we are thinking of forms f of "weight 0" in the usual parlance.

Hecke operators are very useful in the study of automorphic forms (i.e., Γ-invariant functions on H satisfying certain partial differential equations). Stark [16] uses these operators, for example, to obtain an algorithm for the computation of Fourier coefficients of Maass wave forms; i.e., eigenfunctions f of the *noneuclidean Laplacian*

$$\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$$

such that f is Γ-invariant on H (and satisfies a certain growth condition).

Here we are interested in images in F_2 of *Hecke points* for T_p, p a prime:

$$S_p(z_0) = \{(z_0 + j)/p \mid 0 \leqslant j \leqslant p-1\}, \tag{1.3}$$

for fixed $z_0 \in H$. One looks at various examples (see Figure 2) and quickly becomes convinced that images of the points in $S_p(z_0)$ become dense in H as the prime p approaches infinity.

The density of the Hecke points in $\Gamma_2 \backslash H$ is connected with the Ramanujan conjecture on the size of the Fourier coefficients of automorphic forms—a conjecture which remains unproved in the Maass wave form case (see Stark [16] and Sarnak [14]).

One obvious consequence is that these Hecke points can be used for numerical integration and differentiation. The Hecke points are particularly nice for the solution of $\Delta f = \lambda f$, where Δ is the noneuclidean Laplacian defined above, for Γ-invariant functions f on H, since eigenfunctions of Δ can also be assumed to be eigenfunctions of the T_n (because $\Delta T_n = T_n \Delta$). And the Fourier coefficients of such $f(z)$, considered as a periodic function of $\operatorname{Re} z$, are basically the eigenvalues of the T_n.

The multiplicative properties of the T_n are given by

$$T_n T_m = T_{mn} \quad \text{if g.c.d.}(n, m) = 1,$$

$$\sum_{r \geqslant 0} T_{p^r} X^r = \left(1 - p^{1/2} T_p X + p X^2\right)^{-1}.$$

These show that the L-functions corresponding to automorphic forms f with $T_n f = \lambda_n f$ and defined by

$$L_f(s) = \sum_{n \geqslant 1} \lambda_n n^{-s} \quad \text{for Re } s \text{ sufficiently large,}$$

have Euler products.

A *horocycle* in H is a horizontal line

$$C_y = \{ z = x + iy \mid x \in \mathbf{R} \}, \tag{1.4}$$

or an image of some C_y under an element g in $\mathrm{SL}(2, \mathbf{R})$. Note that the Hecke points $S_p(z_0)$ are in the horocycle C_y, $y = y_0/p$. The Hecke points $(z_0 + j)/p$, $0 \leqslant j \leqslant p - 1$, give an equally-spaced set of points on the segment of the horocycle with $x \in [0, 1]$. We are seeing in Figure 2 that the images of these points in the standard fundamental domain become dense as p approaches infinity. Stark has made an interesting movie using an Apple 2e computer, showing what happens to the images in the standard fundamental domain $F_2(1.1)$ for $\mathrm{SL}(2, \mathbf{Z})$ of points $z_j(y) = iy + j/N$, $j = 1, 2, \ldots, N$, holding N fixed and letting y approach 0 from above. At first, you see points on a horizontal line segment of length 1 and height 2. Then, as y approaches 0, the line reflects from the boundaries of the fundamental domain when y passes through 1. Then more reflections occur and the picture begins to look very chaotic. The maximum amount of chaos appears to occur near $y = 1/N$. After that, the picture begins to become less random. Ultimately, the points move on vertical line segments (one for each divisor of N) as the points go to the cusp.

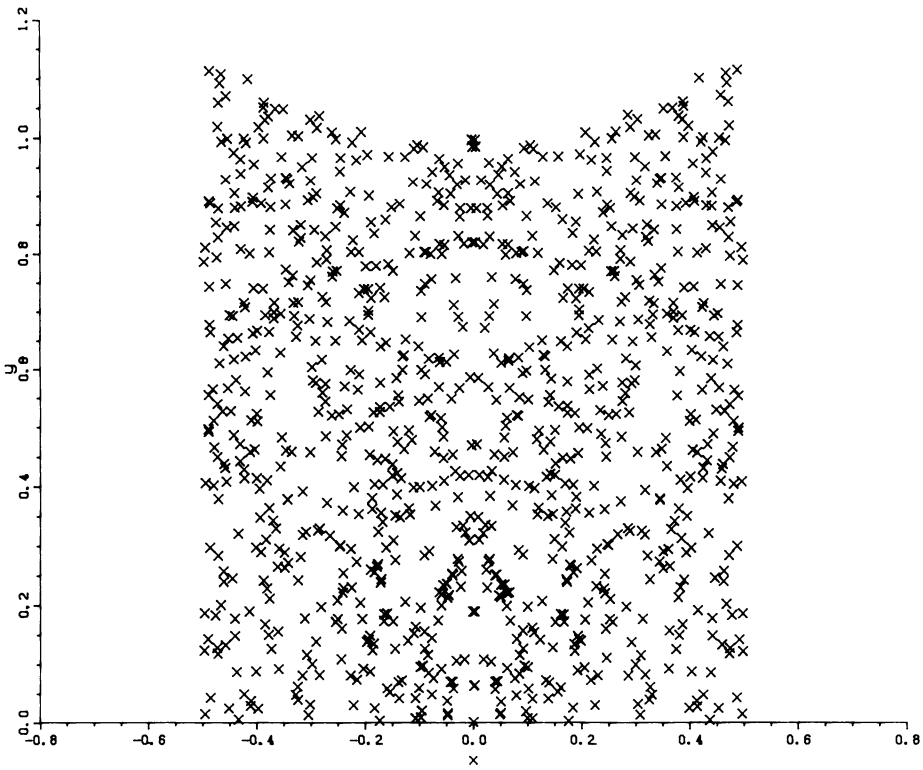

FIGURE 2

Images of Hecke points $S_p(z_0)$ from (1.3) *in the fundamental domain for* $\mathrm{SL}(2, \mathbf{Z})$

$z_0 = 1.4i$, $p = 997$

It is also of interest to consider the images in $\Gamma_2 \backslash H$ of geodesics in H. These are curves minimizing the SL(2, **R**)-*invariant arc length* on H given by

$$ds^2 = y^{-2}(dx^2 + dy^2).$$

It is easily verified that such geodesics are straight lines and circles in H which are orthogonal to the x-axis. This gives a geometry violating Euclid's 5th postulate.

Artin [1] showed in 1924 that almost all geodesics in H induce densely wound lines in the fundamental domain $\Gamma_2 \backslash H$. This is a nice noneuclidean version of the familiar result that lines in $\mathbf{R}^2$ with irrational slope induce densely wound lines in the torus $\mathbf{R}^2/\mathbf{Z}^2$.

Very general results from ergodic theory include the special cases we are considering here (cf. Zimmer [20]).

2. SL(3, **Z**). Next we consider a higher-dimensional analogue of the topics in Section 1. To find the analogue of the Poincaré upper half-plane, note that the action of the special linear group $G = \mathrm{SL}(2, \mathbf{R})$ is transitive on H and that $K = \mathrm{SO}(2)$, the orthogonal group of 2×2 rotation matrices, is the subgroup fixing the point i. Thus we can identify H with G/K via $gK \mapsto gi \in H$. It follows that a natural analogue of H is the *symmetric space* $\mathrm{SL}(3, \mathbf{R})/\mathrm{SO}(3)$. We will often identify this symmetric space with the space of *positive determinant-one quadratic forms*:

$$SP_3 = \{Y \in \mathbf{R}^{3\times 3} \mid Y = {}^tY,\ Y > 0,\ \det Y = 1\}.$$

Here $Y > 0$ means that $Y[x] = {}^txYx > 0$ for every column vector $x \in \mathbf{R}^3 - 0$. We use the notation ${}^tx$ = transpose of x. Then we have the identification

$$\mathrm{SO}(3) \backslash \mathrm{SL}(3, \mathbf{R}) \to SP_3, \qquad Kg \mapsto I[g] = {}^tgg.$$

The right action of $g \in \mathrm{SL}(3, \mathbf{R})$ on $Y \in SP_3$ is via $Y[g] = {}^tgYg$. Recall that we had a left action of $\mathrm{SL}(2, \mathbf{R})$ on H. We prefer a right action of $\mathrm{SL}(3, \mathbf{R})$ on SP_3 following Maass [8] and Siegel. Hopefully, this will not cause too much confusion.

To make SP_3 look like an upper half-space, recall that we can map $z = x + iy \in H$ to a 2×2 determinant-one positive matrix via

$$\begin{pmatrix} 1/y & 0 \\ 0 & y \end{pmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}.$$

So to make SP_3 into an upper half-space we use *Iwasawa coordinates*:

$$Y = \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{pmatrix} \begin{bmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.1}$$

where $y_j > 0$, $x_j \in \mathbf{R}$, $\prod_{j=1}^3 y_j = 1$. We will set

$$v(Y) = v = y_1, \qquad w(Y) = w = v^{1/2}y_2.$$

So we end up with two "y-coordinates": v, w and three "x-coordinates": x_1, x_2, x_3. We write $Y = Y(v, w, x)$ if Y has the decomposition (2.1).

A *horocycle* C_Y in SP_3 has the form

$$C_Y = \{Y[n(x)] \mid x \in \mathbf{R}^3\}, \quad \text{where } n(x) = \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

Here $Y \in SP_3$ is fixed. We also call any image $C_Y[g]$ for $g \in \mathrm{SL}(3, \mathbf{R})$ a horocycle.

A *geodesic* (or distance-minimizing curve) G_a in SP_3 has the form

$$G_a = \{\mathrm{diag}(e^{a_1 t}, e^{a_2 t}, e^{a_3 t}) \mid t \in \mathbf{R}\}, \tag{2.3}$$

where $\mathrm{diag}(u)$ means the diagonal matrix with diagonal entries $u \in \mathbf{R}^3$. And here a is a fixed vector in $\mathbf{R}^3$. And any image of G_a under $g \in \mathrm{SL}(3, \mathbf{R})$ is also a geodesic. See Maass [8] or Terras [17, Vol. II] for the details of the proof that the arclength $ds^2 = \mathrm{Tr}((Y^{-1}dY)^2)$ is minimized by G_a.

Fundamental domains for SP_3/Γ_3, where $\Gamma_3 = \mathrm{SL}(3, \mathbf{Z})$, have been studied for over 100 years. Minkowski [10] obtained fundamental domains for $\mathrm{GL}(n, \mathbf{Z})$ for all values of n and gave the defining inequalities very explicitly for $n \leqslant 6$. Tammella did the case $n = 7$ more recently. In particular, *Minkowski's fundamental domain* for $\Gamma_3 = \mathrm{SL}(3, \mathbf{Z})$ is

$$SM_3 = \left\{ \begin{matrix} Y \in SP_3 \\ Y = (y_{ij}) \end{matrix} \;\middle|\; \begin{matrix} y_{11} \leqslant y_{22} \leqslant y_{33}, 0 \leqslant y_{12} \leqslant y_{11}/2, \\ 0 \leqslant y_{23} \leqslant y_{22}/2, |y_{13}| \leqslant y_{11}/2, \\ Y[e] \geqslant y_{33}, {}^t e = (\pm 1, \pm 1, \pm 1) \end{matrix} \right\}. \tag{2.4}$$

Using the Iwasawa coordinates (2.1), we find that

$$0 \leqslant x_1 \leqslant \tfrac{1}{2}, \quad |x_2| \leqslant \tfrac{1}{2}, \quad 0 \leqslant y_1 x_1 x_2 + y_2 x_3 \leqslant \tfrac{1}{2}(y_1 x_1^2 + y_2).$$

It follows that

$$-\tfrac{1}{4} y_1/y_2 \leqslant x_3 \leqslant \tfrac{1}{2} + (3/8) y_1/y_2.$$

This means that Minkowski's fundamental domain has only an *approximate box shape* at infinity. We prefer an *exact box shape*, that is, we prefer to see the inequality $0 \leqslant x_3 \leqslant \frac{1}{2}$, particularly when computing the integrals of Eisenstein series over truncated fundamental domains which arise when one seeks to generalize Selberg's trace formula to $\Gamma_3 = \mathrm{SL}(3, \mathbf{Z})$.

Minkowski's version of the fundamental domain for $\mathrm{SL}(3, \mathbf{Z})$ does not make use of Iwasawa coordinates—except in the proofs. Grenier [4] describes a fundamental domain for $\mathrm{GL}(n, \mathbf{Z})$, the general linear group of $n \times n$ integral matrices with determinant ± 1, and Grenier's domain makes essential use of the Iwasawa coordinates. Moreover, Grenier's domain has an exact box shape at the boundary. And Grenier develops a reduction algorithm to move $Y \in SP_3$ into this fundamental domain via a "highest-point method". Here we consider only the case $n = 3$. The results are analogous to those of Siegel for $\mathrm{Sp}(n, \mathbf{Z})$ (see Maass [8] and Gottschling [3]). Other fundamental domains and reduction algorithms are considered in [6], [11],

[12], [13], and [19]. Note that $\mathrm{GL}(3,\mathbf{Z})/\mathrm{SL}(3,\mathbf{Z})$ has order 2 and the nontrivial coset is represented by $-I$ which does nothing to an element $Y \in SP_3$. So the fundamental domain for $\mathrm{SL}(3,\mathbf{Z})$ is the same as that for $\mathrm{GL}(3,\mathbf{Z})$.

Grenier's fundamental domain for $\mathrm{SL}(3,\mathbf{Z})$ is

$$F_3 = \left\{ Y \in SP_3 \;\middle|\; \begin{array}{l} 0 \leqslant x_1 \leqslant \frac{1}{2},\ |x_2| \leqslant \frac{1}{2},\ 0 \leqslant x_3 \leqslant \frac{1}{2}, \\ 1 \leqslant w^{-2} + x_3^2, \\ v \leqslant v(a + {}^t xc)^2 + v^{-1/2} W[c] \\ \quad \text{for } a = 0,\ {}^t c = (1,0), (0,1), (1,-1) \\ \qquad a = 1,\ {}^t c = (-1,1) \end{array} \right\}. \tag{2.5}$$

Here,

$$\begin{aligned} W &= \begin{pmatrix} w & 0 \\ 0 & 1/w \end{pmatrix} \begin{bmatrix} 1 & x_3 \\ 0 & 1 \end{bmatrix}, \\ Y = Y(v,w,x) &= \begin{pmatrix} v & 0 \\ 0 & v^{-1/2}W \end{pmatrix} \begin{bmatrix} 1 & (x_1, x_2) \\ 0 & I_2 \end{bmatrix}, \end{aligned} \tag{2.6}$$

where Y has the Iwasawa coordinates (2.1), meaning that

$$Y = \begin{pmatrix} v & vx_1 & vx_2 \\ vx_1 & vx_1^2 + v^{-1/2}w & vx_1x_2 + v^{-1/2}wx_3 \\ vx_2 & vx_1x_2 + v^{-1/2}wx_3 & vx_2^2 + v^{-1/2}wx_3^2 + v^{-1/2}w^{-1} \end{pmatrix}.$$

If we list our inequalities for F_3 more explicitly, we obtain

$$\begin{array}{ll} \text{(i)} & v^{3/2} \leqslant v^{3/2}(1 - x_1 + x_2)^2 + w(1 - x_3)^2 + w^{-1}, \\ \text{(ii)} & v^{3/2} \leqslant v^{3/2}(x_1 - x_2)^2 + w(1 - x_3)^2 + w^{-1}, \\ \text{(iii)} & v^{3/2} \leqslant v^{3/2}x_1^2 + w, \\ \text{(iv)} & v^{3/2} \leqslant v^{3/2}x_2^2 + wx_3^2 + w^{-1}, \\ \text{(v)} & 1 \leqslant w^{-2} + x_3^2, \\ \text{(vi)} & 0 \leqslant x_1 \leqslant \frac{1}{2}, \\ \text{(vii)} & 0 \leqslant x_3 \leqslant \frac{1}{2}, \\ \text{(viii)} & -\frac{1}{2} \leqslant x_2 \leqslant \frac{1}{2}. \end{array} \tag{2.7}$$

Inequalities (v) and (vii) say that the 2×2 matrix W is in our standard fundamental domain for $\mathrm{GL}(2,\mathbf{Z})$, the general linear group of all 2×2 integral matrices whose inverses are also integral. This is the group generated by $\mathrm{SL}(2,\mathbf{Z})$ and the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So a fundamental domain is half of that for $\mathrm{SL}(2,\mathbf{Z})$; e.g., $z = x + iy$ with $0 \leqslant x \leqslant \frac{1}{2}$ and $x^2 + y^2 \geqslant 1$. Here we take $x = x_3$ and $y = w^{-1}$.

Boundary identifications for the fundamental domain F_3 come from the matrices:

$$
\begin{aligned}
&T1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \\
&S1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad S2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad S3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \\
&S4 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad S5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \\
&U1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\end{aligned}
\tag{2.8}
$$

Note: This gives more than enough generators for $\mathrm{SL}(3,\mathbf{Z})/\pm I$, but we do not appear to be able to get rid of any of the inequalities in (2.7).

Grenier's reduction algorithm from [4] is a "highest-point method" where the height of Y is $1/v$, for $v =$ the entry y_{11}, which is the coordinate v in (2.1). Grenier's algorithm goes as follows:

Step I. Set $S0 = I =$ the 3×3 identity matrix. Pick i to minimize $v(Y[Si])$, for $i = 0, 1, 2, 3, 4$ and replace Y by $Y[Si]$.

Step II. Let $W(Y)$ denote the element of SP_2 defined by (2.6). Put $W(Y)$ in our standard fundamental domain for $\mathrm{GL}(2,\mathbf{Z})$ using $\delta \in \mathrm{GL}(2,\mathbf{Z})$. Replace Y by $Y[\gamma]$ for

$$\gamma = \begin{pmatrix} \pm 1 & 0 \\ 0 & \delta \end{pmatrix} \in SL(3,\mathbf{Z}).$$

Here $\gamma = S5$, $U1$ or $(T3)^n$, for some $n \in \mathbf{Z}$.

Step III. Translate the x_1, x_2-coordinates of Y in (2.1) by $\gamma = (Ti)^{p_i}$, $i = 1, 2$, $p_i = [\frac{1}{2} - x_i]$. Here $[x]$ denotes the greatest integer $\leqslant x$. Replace Y by $Y[\gamma]$.

Step IV. Make $x_1 \geqslant 0$ by replacing Y by $Y[U2]$, if necessary.

Keep doing these steps until the process converges.

Jeff Stopple suggested that we use the last test (i.e., see whether the process has repeated) to stop the program. This idea was useful since it allows us not to test all the inequalities at each step, as some might be tempted to do. On the other hand, one might worry that the program would get into an infinite loop. This does not happen if one is careful in writing the code.

Grenier proves in [4] that F_3 is a fundamental domain up to boundary identifications and that Steps I–IV above constitute a reduction algorithm. Let us just sketch the arguments here.

First, note that one obtains a fundamental domain by considering the set of all Y in SP_3 such that Y satisfies the inequalities:

$$
\left\{
\begin{aligned}
&v \leqslant v(a + {}^t xc)^2 + v^{-1/2} W[c], \text{ for all matrices} \\
&\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in } \Gamma = \mathrm{SL}(3,\mathbf{Z}),\ a \in \mathbf{Z},\ c \in \mathbf{Z}^2, \\
&\quad W = \begin{pmatrix} w & 0 \\ 0 & 1/w \end{pmatrix} \begin{bmatrix} 1 & x_3 \\ 0 & 1 \end{bmatrix} = \begin{pmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{pmatrix}, \\
&\quad x_3 + iw^{-1} \text{ in the standard fundamental domain for } \mathrm{GL}(2,\mathbf{Z}) \setminus H, \\
&\qquad\qquad \text{i.e., } 0 \leqslant x_3 \leqslant \tfrac{1}{2},\ x_3^2 + w^{-2} \geqslant 1; \\
&\quad 0 \leqslant x_1 \leqslant \tfrac{1}{2},\ |x_2| \leqslant \tfrac{1}{2}.
\end{aligned}
\right.
\tag{2.9}
$$

We now prove that it suffices for Y to satisfy (2.9) for $a \in \mathbf{Z}$, $c \in \mathbf{Z}^2$ such that $|a| \leqslant 1$ and $|c_i| \leqslant 1$, $i = 1, 2$. These inequalities include the special cases $a = 0$, ${}^t c = (1, 0)$ or $(0, 1)$:

$$v^{3/2} \leqslant v^{3/2} x_1^2 + w_{11}, \qquad v^{3/2} \leqslant v^{3/2} x_2^2 + w_{22}. \tag{2.10}$$

These inequalities imply that $1 \leqslant x_1^2 + v^{-3/2} w_{11}$ and thus, since $0 \leqslant x_1 \leqslant \frac{1}{2}$, we have

$$\tfrac{3}{4} \leqslant v^{-3/2} w_{11}. \tag{2.11}$$

By a standard argument in this business, if an inequality (2.9) is really necessary to define the boundary of the fundamental domain, it must occur with *equality*. So then we have

$$v = v(a + {}^t xc)^2 + v^{-1/2} W[c] \quad \text{for some } Y \text{ in } F_3. \tag{2.12}$$

It follows that since $x_3 + iw^{-1} \in F_2$ (and, in fact half of this fundamental domain for SL(2, Z)), we have

$$1 \geqslant v^{-3/2} W[c] \geqslant v^{-3/2} w_{11} \left(|c_1|^2 - |c_1||c_2| + |c_2|^2 \right).$$

Then (2.11) implies

$$\tfrac{4}{3} \geqslant \left(|c_1|^2 - |c_2| \right)^2 + |c_1 c_2|^2.$$

It follows that since the c_i are integers, $|c_i|^2 \leqslant 1$, $i = 1, 2$. To obtain the bound on $|a|$, use (2.12) again to see that

$$1 \geqslant |a + {}^t xc| \geqslant a + {}^t xc \geqslant |a| - |{}^t xc|.$$

Therefore,

$$|a| \leqslant 1 + 2^{-1/2} \quad \text{and} \quad a \in \mathbf{Z} \text{ implies } |a| \leqslant 1.$$

Thus we have proved that only the inequalities (2.9) with $|c_i| \leqslant 1$ and $|a| \leqslant 1$ are necessary.

Next we want to prove that we can leave out the inequalities with

(a) $a = 0$, $c = \pm(1, 1)$,
(b) $a = 1$, ${}^t c = (1, -1), (1, 1)$,
(c) $a = -1$, ${}^t c = (-1, 1), (-1, -1)$,
(d) $a = 1$, ${}^t c = \pm(1, 0), \pm(0, 1), (0, 0)$,
(e) $a = -1$, ${}^t c = \pm(1, 0), \pm(0, 1), (0, 0)$,
(f) $a = 1$, ${}^t c = (-1, -1)$,
(g) $a = -1$, ${}^t c = (1, 1)$.

If we count the inequalities here, plus those in (2.5), plus that for ${}^t(ac) = 0$, we get the required 27 inequalities.

To prove that we can omit the inequality corresponding to (a), note that this inequality is

$$v \leqslant v(x_1 + x_2)^2 + v^{-1/2}(w_{11} + 2w_{12} + w_{22}). \tag{2.13}$$

Using (2.10), we see that this inequality follows from

$$0 \leqslant v + 2vx_1x_2 + 2v^{-1/2} w_{12} = v(1 + 2x_1x_2) + 2w_{12} v^{-1/2}.$$

But $w_{12} \geqslant 0$ and $1 + 2x_1x_2 \geqslant \frac{1}{2}$ follows from the other inequalities in (2.9).

Now we want to drop the inequalities (b) with $a = 1$, ${}^t c = (1, -1)$ or $(1, 1)$. These inequalities look like

(A) $v \leqslant v(1 + x_1 + x_2)^2 + v^{-1/2}(w_{11} + 2w_{12} + w_{22})$,

(B) $v \leqslant v(1 + x_1 - x_2)^2 + v^{-1/2}(w_{11} - 2w_{12} + w_{22})$.

We can use the inequality (2.7), part (ii), to show that for (B) to hold we need only know the inequality

$$v(1 + 2(x_1 - x_2)) \geqslant v(1 - 2x_2) \geqslant 0.$$

Now similarly (2.7), inequality (ii), implies (2.13) which gives inequality (A) in the same way. Clearly, we can also drop the inequalities (c).

Now we want to show that we can omit the inequalities (d). These inequalities are

$$\begin{aligned} v^{3/2} &\leqslant v^{3/2}(1 \pm x_1)^2 + w, \\ v^{3/2} &\leqslant v^{3/2}(1 \pm x_2)^2 + wx_3^2 + w^{-1}, \\ v^{3/2} &\leqslant v^{3/2}. \end{aligned}$$

The last inequality is clearly true. The first follows from (2.7), parts (iii) and (vi). The second follows from (2.7), parts (iv) and (viii). One proves similarly that the inequalities (e) can be eliminated.

The argument to leave out the inequality (f) goes as follows. We need to show that

$$v^{3/2} \leqslant v^{3/2}(1 - x_1 - x_2)^2 + W\begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Note that $W\left[\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}\right] = w_{11} + 2w_{12} + w_{22} \geqslant 2w_{11} \geqslant (3/2)v^{3/2}$ by (2.11). Similarly, one proves that the inequality (g) can be omitted.

We have thus completed our sketch of a proof of the following theorem.

THEOREM [4]. *The set F_3 defined by* (2.5) *is a fundamental domain for $SP_3/\mathrm{SL}(3, \mathbf{Z})$, up to boundary identifications. And a reduction algorithm is given by Steps* I–IV *listed following Eqs.* (2.8).

Note that we need to complete the matrices

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

to a 3×3 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(3, \mathbf{Z})$$

to obtain the matrices Si, $i = 1, 2, 3, 4$ in (2.8). This can be done in a number of ways—each differing by matrices of the form

$$\begin{pmatrix} 1 & {}^t q \\ 0 & R \end{pmatrix}$$

with $q \in \mathbf{Z}^2$ and $R \in \mathbf{Z}^{2\times 2}$. The choice of Si will affect the later steps of the reduction algorithm, but not the final result that the point lands in the fundamental domain.

3. The Figures. We want to use Hecke points to help us visualize the fundamental domain F_3 for $\mathrm{SL}(3, \mathbf{Z})$ which was considered in Section 2. Since F_3 is 5-dimensional, we will take the easy way out and look at graphs of 2 coordinates from

$(v, w, x1, x2, x3)$. So there are 10 possible graphs. The most interesting is that of (w, v) showing the shape of the *cuspidal region*, where v or w approach 0.

We quickly see that (2.7), formulae (v) and (vii), imply that

$$w \leqslant 2/\sqrt{3} \cong 1.154701. \tag{3.1}$$

And (2.11) implies that

$$v \leqslant 4/3 \cong 1.333333. \tag{3.2}$$

Hecke operators for $\Gamma_3 = \mathrm{SL}(3, \mathbf{Z})$ are described in many places (see, for example, Bump [2], Shimura [15], Terras [17, Vol. II] and [18]). For f: $SP_3/\Gamma_3 \to \mathbf{C}$ and $m \in \mathbf{Z}^+$, define *the Hecke operator* T_m by

$$T_m f(Y) = \sum_{A \in M_m/\Gamma_3} f\left(Y[A]^0\right), \tag{3.3}$$

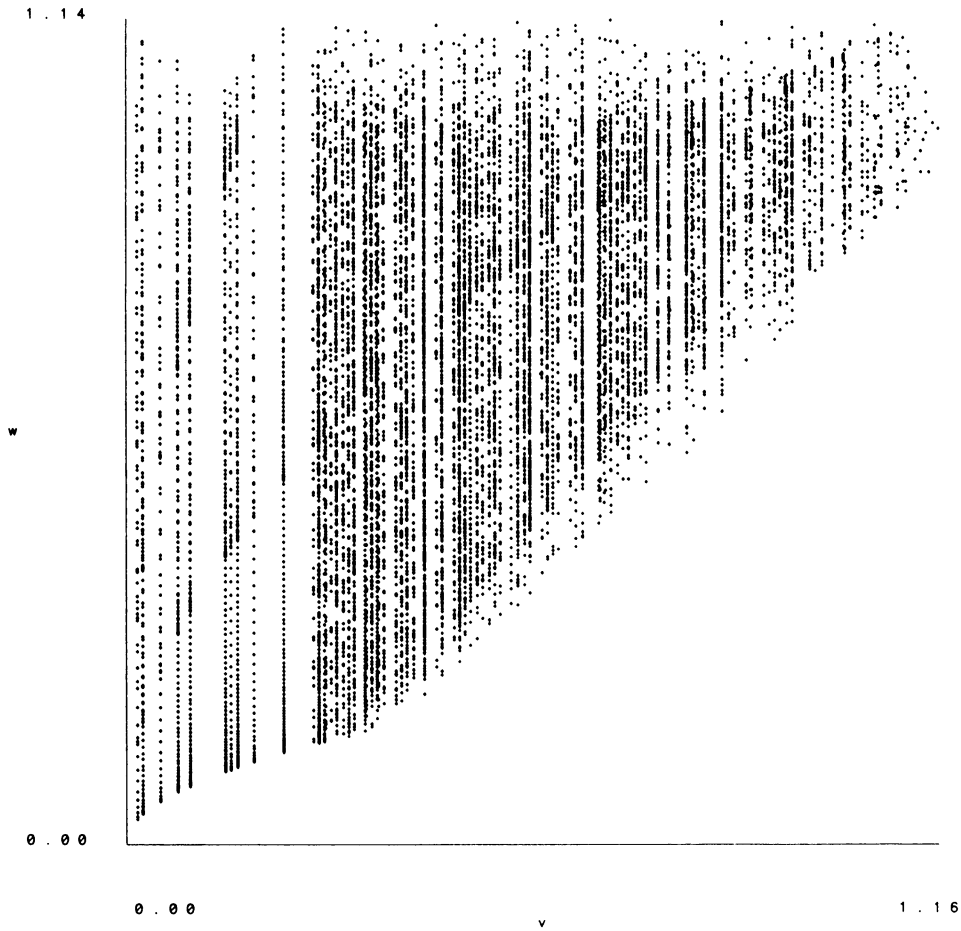

FIGURE 3

(v, w) coordinates of images of the fixed point Y_0 under transformation by Hecke matrices $M(p; a, b)$ in the fundamental domain F_3 for $\mathrm{SL}(3, \mathbf{Z})$ *defined in* (2.5) *and* (3.6) *with Y_0 as in* (3.7).

$$p = 163$$

where

$$M_m = \{ A \in \mathbf{Z}^{3\times 3} \mid \det A = m \} \quad \text{and} \quad Y^0 = (\det Y)^{-1/3} Y \in SP_3.$$

It is easily seen that one can take representatives of $M_3(m)/\Gamma_3$ of the form

$$\begin{pmatrix} d_1 & d_{12} & d_{13} \\ 0 & d_2 & d_{23} \\ 0 & 0 & d_3 \end{pmatrix}, \qquad d_i > 0, \prod_{i=1}^{3} d_i = m, 0 \leqslant d_{ij} < d_i. \tag{3.4}$$

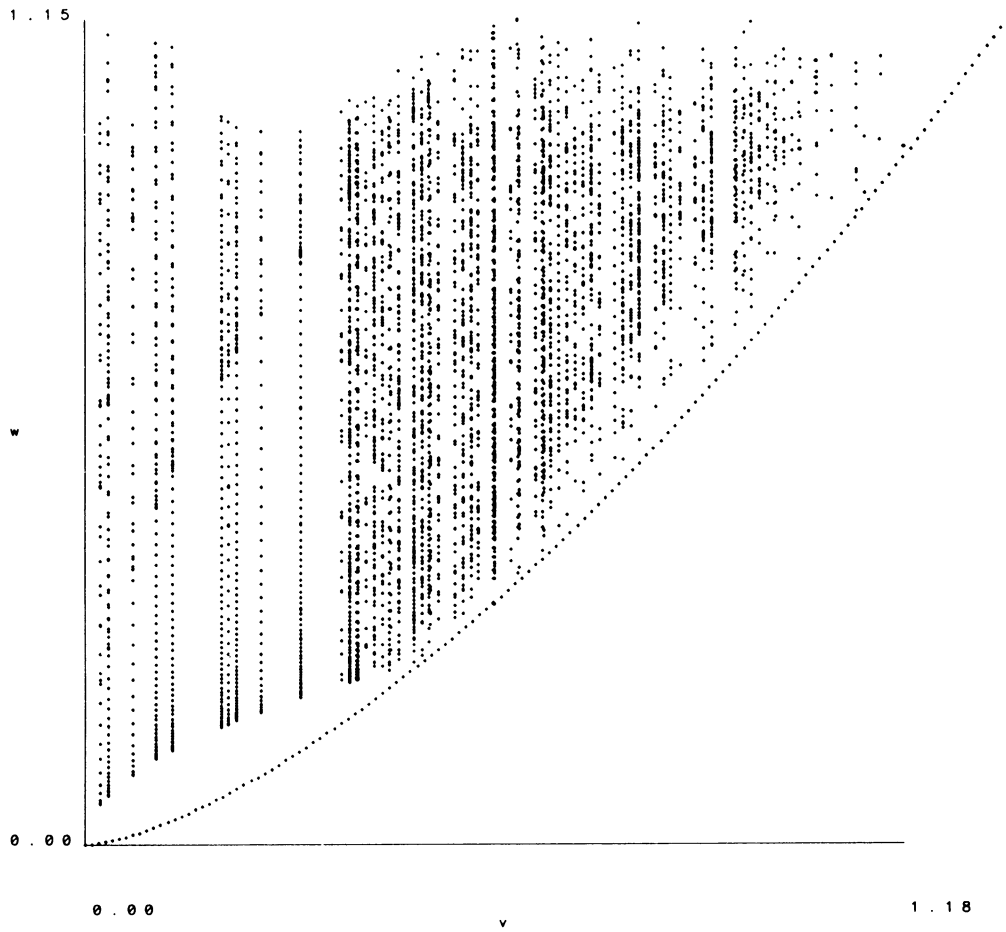

FIGURE 4

(v, w) coordinates of images of the fixed point Y_0 under transformation by Hecke matrices $M(p; a, b)$ in the fundamental domain F_3 for $\mathrm{SL}(3, \mathbf{Z})$ *defined in* (2.5) *and* (3.6) *with Y_0 as in* (3.7).

$$p = 101$$

The lower curve is the graph of $\frac{3}{4}v^{3/2} = w$.

Maass [9] studied Hecke operators for the Siegel modular group $\mathrm{Sp}(n, \mathbf{Z})$ in 1951. We are imitating his version of the theory. It is a theory which is basic to the study of automorphic forms on higher-rank symmetric spaces G/K and it connects with many questions in representation theory, p-adic group theory, combinatorics and number theory. Applications of Hecke operators to numerical integration on spheres are given in [7].

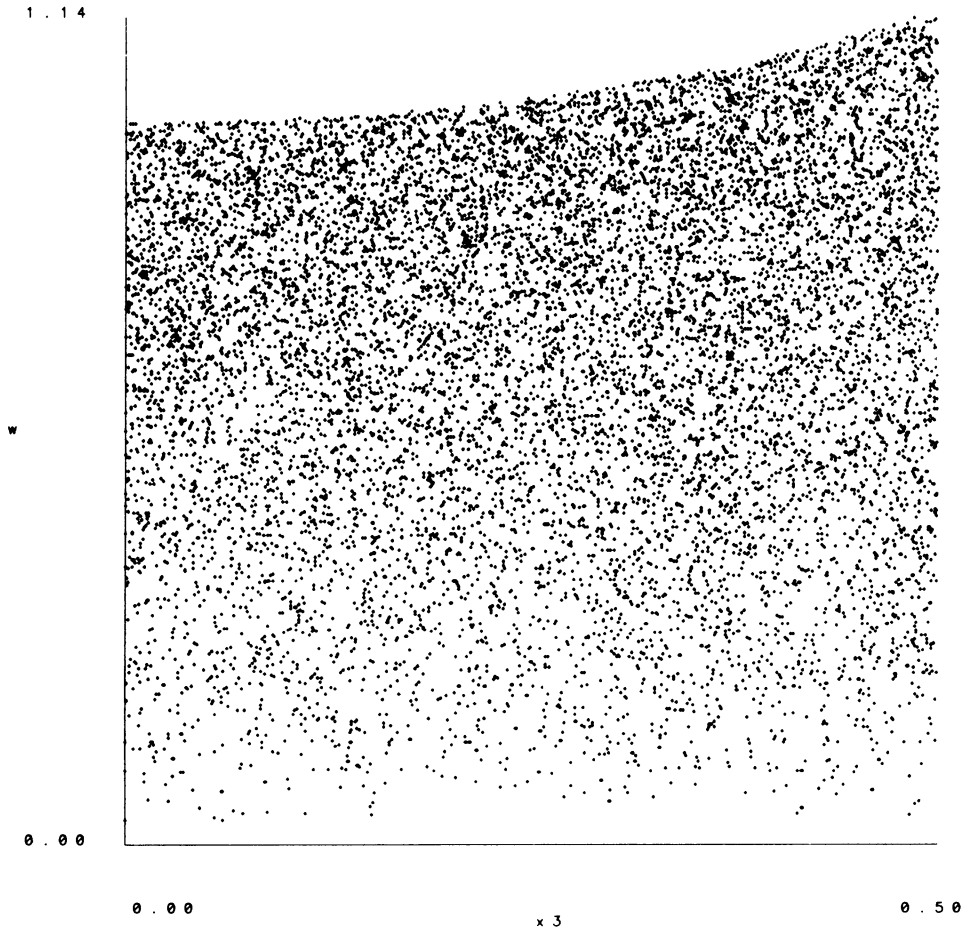

FIGURE 5

$(x3, w)$ coordinates of images of the fixed point Y_0 under transformation by Hecke matrices $M(p; a, b)$ in the fundamental domain F_3 for $\mathrm{SL}(3, \mathbf{Z})$ *defined in* (2.5) *and* (3.6) *with Y_0 as in* (3.7).

$p = 163$

It is not hard to see that the Hecke operators for $\mathrm{SL}(3, \mathbf{Z})$ have the following properties:

$$
(3.5)\qquad
\begin{aligned}
&\text{(i)} \quad T_n T_m = T_{mn} \quad \text{if g.c.d.}(m, n) = 1;\\
&\text{(ii)} \quad \text{for } p = \text{prime}\\
&\qquad \sum_{r \geqslant 0} T_{p^r} X^r = \left(I - T_p X + \left[(T_p)^2 - T_{p^2}\right] X^2 - p^3 X^3\right)^{-1}.
\end{aligned}
$$

It follows that L-functions associated with eigenforms f of the Hecke operators must have Euler products.

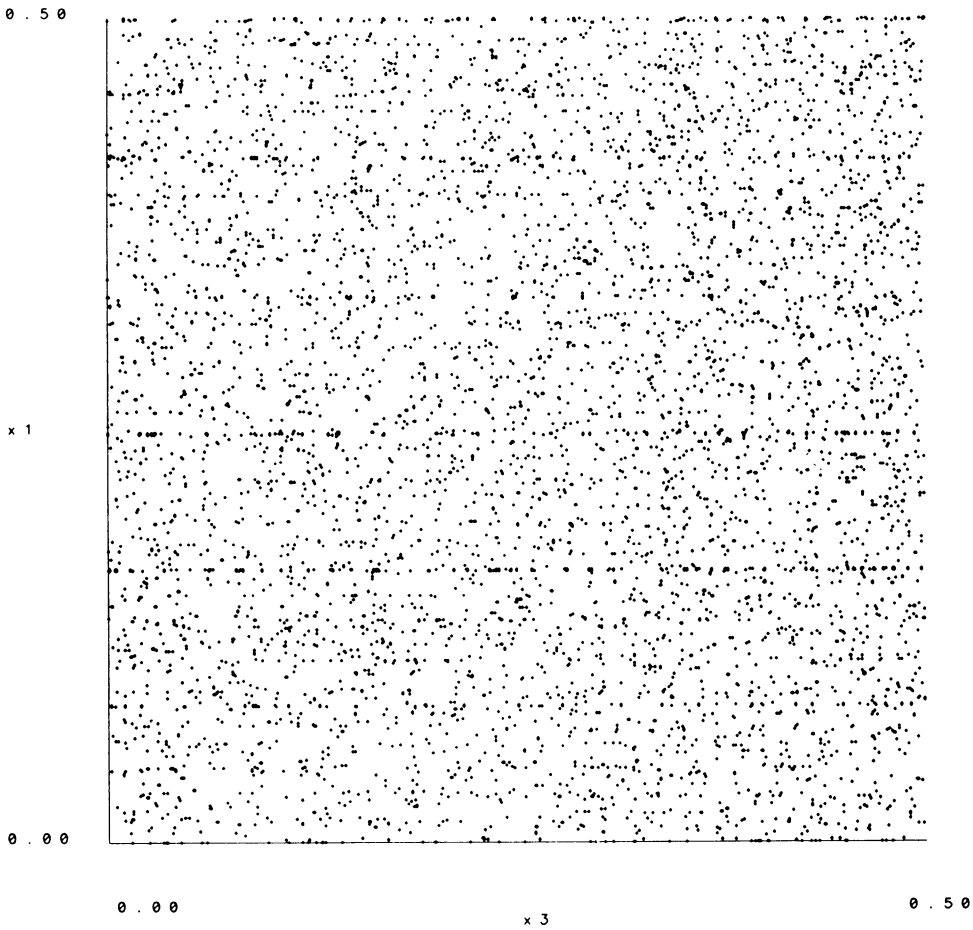

FIGURE 6

$(x3, x1)$ coordinates of images of the fixed point Y_0 under transformation by Hecke matrices $M(p; a, b)$ in the fundamental domain F_3 for $\mathrm{SL}(3, \mathbf{Z})$ *defined in* (2.5) *and* (3.6) *with Y_0 as in* (3.7).

$p = 101$

Here we intend to graph points from the operator T_p, p = prime. We use only the matrices

$$M(p; a, b) = p^{-1/3}\begin{pmatrix} p & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad 0 \leqslant a,\ b \leqslant p-1. \tag{3.6}$$

The other matrices in T_p do not appear to be necessary.

Figures 3–9 show plots of pairs of coordinates of Γ_3-images in F_3 of points

$$Y_0[M(p; a, b)], \qquad 0 \leqslant a, b \leqslant p-1,$$

for $M(p; a, b)$ as in (3.6) and fixed Y_0 equal to

$$Y_0 = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 8 \end{pmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.7}$$

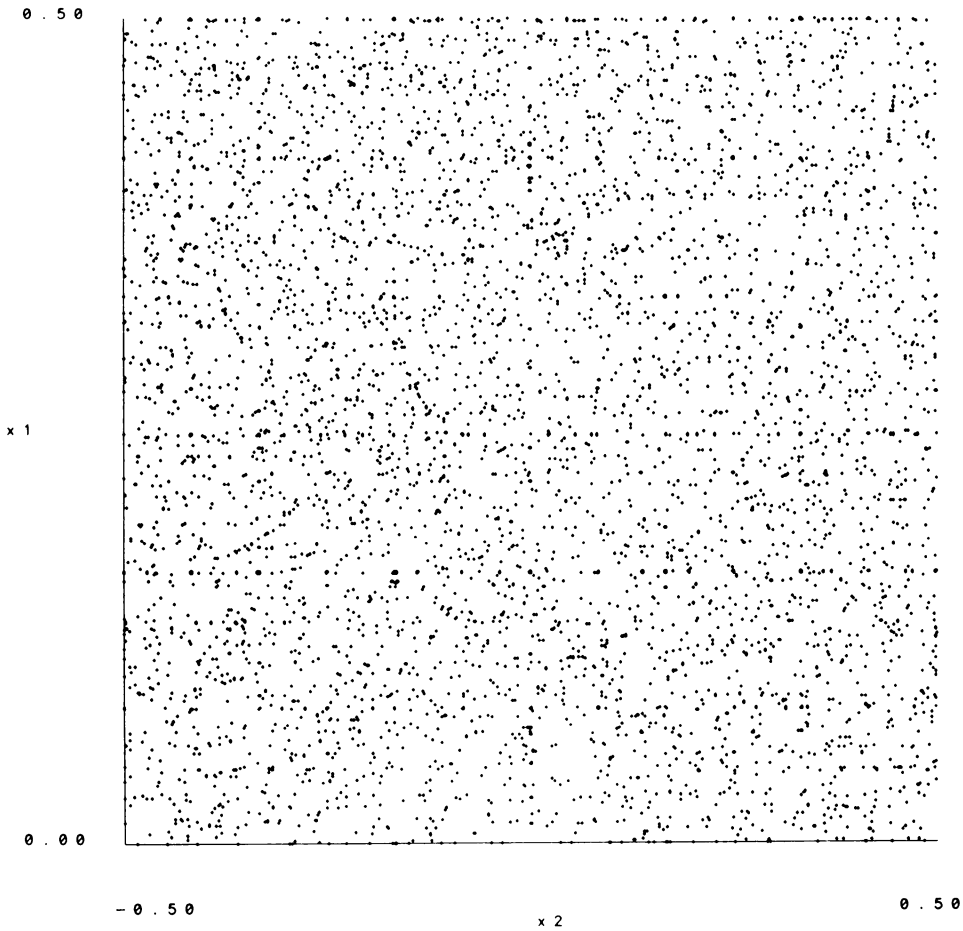

FIGURE 7

$(x2, x1)$ coordinates of images of the fixed point Y_0 under transformation by Hecke matrices $M(p; a, b)$ in the fundamental domain F_3 for SL(3, **Z**) *defined in* (2.5) *and* (3.6) *with Y_0 as in* (3.7).

$p = 101$

In Figure 3, $p = 163$ and the graph shows v versus w. The lower curve fits well with the curve $3/4v^{3/2} = w$, as might be expected from Eq. (2.11). Figure 4 shows v versus w for the prime $p = 101$ as well as the curve $3/4v^{3/2} = w$. Note that the values of w get close to the bound 1.154701 of (3.1) while the values of v do not come so close to 1.333333 (the bound of (3.2)). There are points with v closer to 1.333333 than the value 1.16 seen in Figure 3. For example, take $v = 2^{1/3} \cong 1.25992$, $w = 3/(2\sqrt{2})$, $x_1 = x_2 = \frac{1}{2}$, $x_3 = \frac{1}{3}$. It is likely that taking larger p will fill out the region more. The problem then becomes one of storing and plotting huge numbers of points. Here we needed more storage than the standard quota and the laser printer was not happy having to plot 26569 points. This is the reason that we have stopped at $p = 163$. But this is only a temporary phenomenon. More experiments are in order.

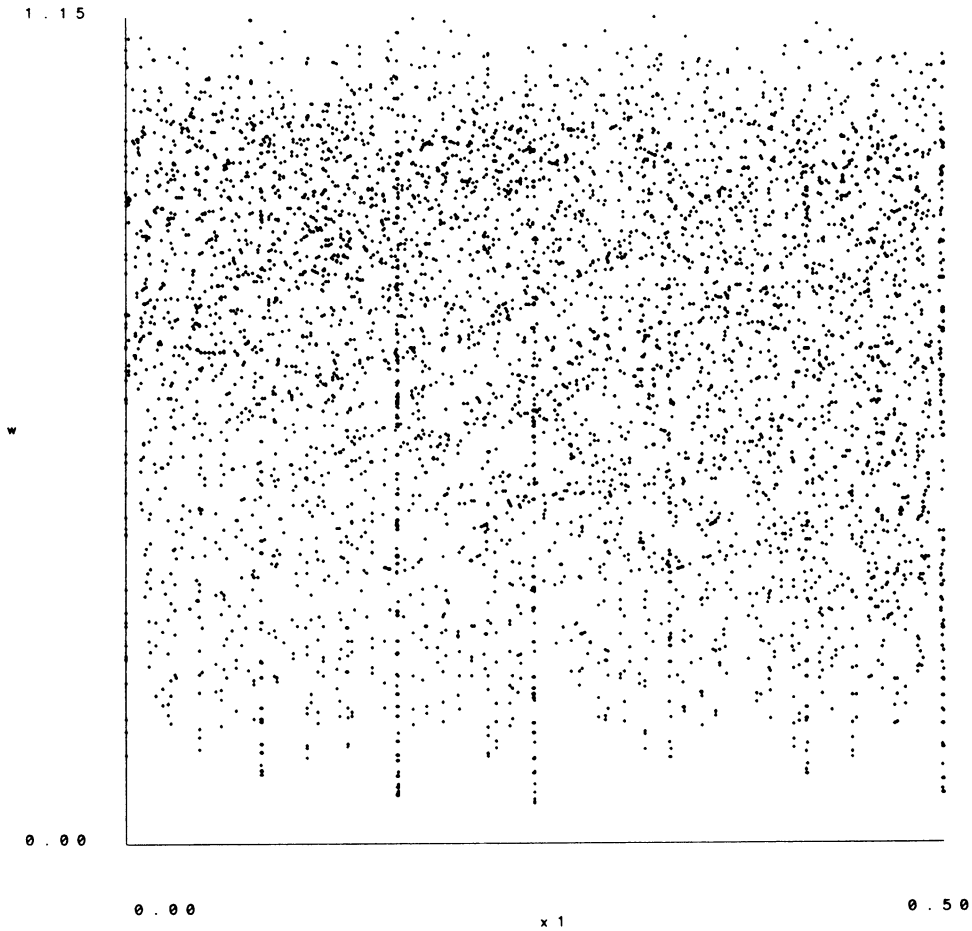

FIGURE 8

$(x1, w)$ coordinates of images of the fixed point Y_0 under transformation by Hecke matrices $M(p; a, b)$ in the fundamental domain F_3 for $\mathrm{SL}(3, \mathbf{Z})$ *defined in* (2.5) *and* (3.6) *with Y_0 as in* (3.7).

$p = 101$

Figure 5 shows $x3$ versus w for the prime $p = 163$. These are the variables in the copy of the Poincaré upper half-plane in our matrices Y. The figure shows a good approximation to half of Figures 1 and 2, as expected.

Figures 6 and 7 give plots of $(x3, x1)$ and $(x2, x1)$, respectively. Here the prime $p = 101$. The plots look like randomly placed points in $[0, \frac{1}{2}]^2$ and $[0, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, respectively.

Figures 8 and 9 give graphs of $(x1, w)$ and $(x2, w)$ for $p = 101$. The result should be compared with Figure 5. If we do so, we see that now the top curve of Figures 8 and 9 cannot be that of Figure 5. The variables in Figures 8 and 9 are less closely related.

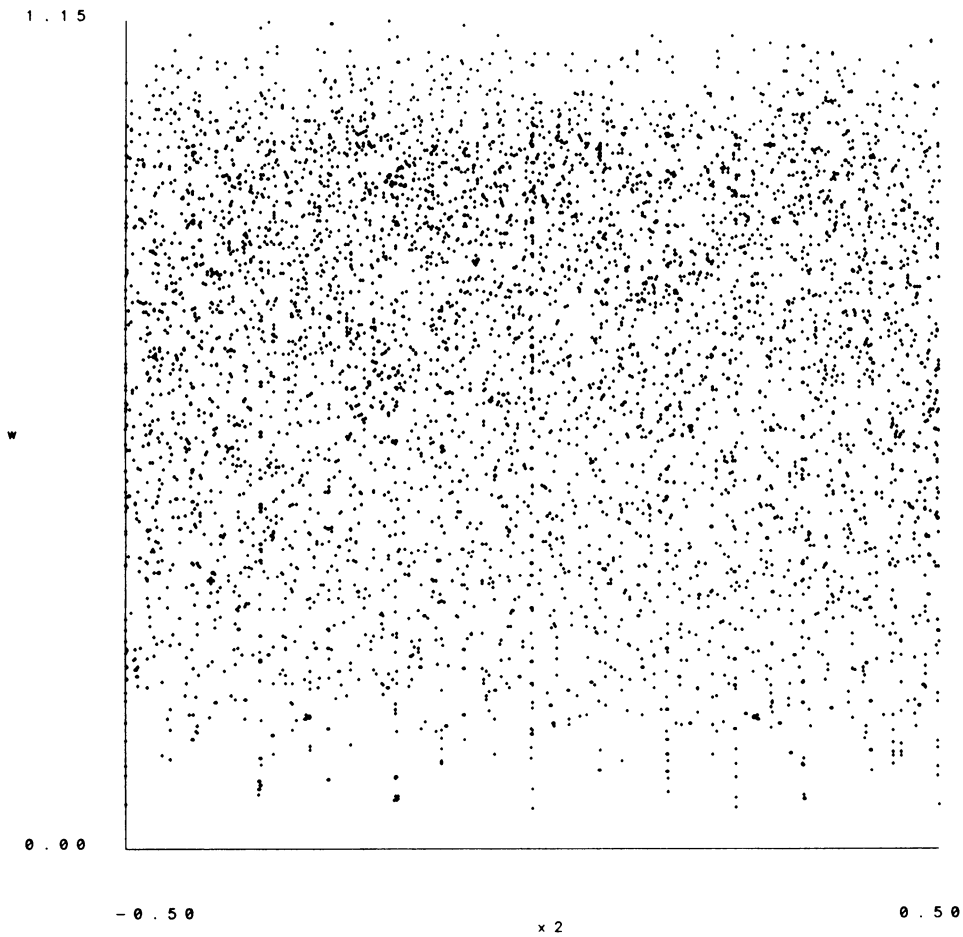

FIGURE 9

$(x2, w)$ coordinates of images of the fixed point Y_0 under transformation by Hecke matrices $M(p; a, b)$ in the fundamental domain F_3 for SL(3, Z) *defined in* (2.5) *and* (3.6) *with Y_0 as in* (3.7).

$p = 101$

Figure 10 shows the points that result from $M(997; a, 0)$ for $a = 0, \ldots, 996$. Many of the plots of points from $M(a, 0)$ tended to be uninteresting since v was essentially constant. It is interesting to compare Figures 10 and 8.

One might complain that our graphs still do not give a real 5-dimensional feeling for the fundamental domain. We hope to make "F_3 THE MOVIE" some day, making use of motion and color. This would be a noneuclidean analogue of Banchoff's movie of a rotating 4-dimensional cube. For you may view our region F_3 as a 5-dimensional noneuclidean crystal. It would also be nice to produce a figure representing the tessellation of the 5-dimensional space SP_3 corresponding to $\mathrm{SL}(3, \mathbf{Z})$ images of F_3. These would be 5-dimensional analogues of pictures that inspired the artist M. C. Escher.

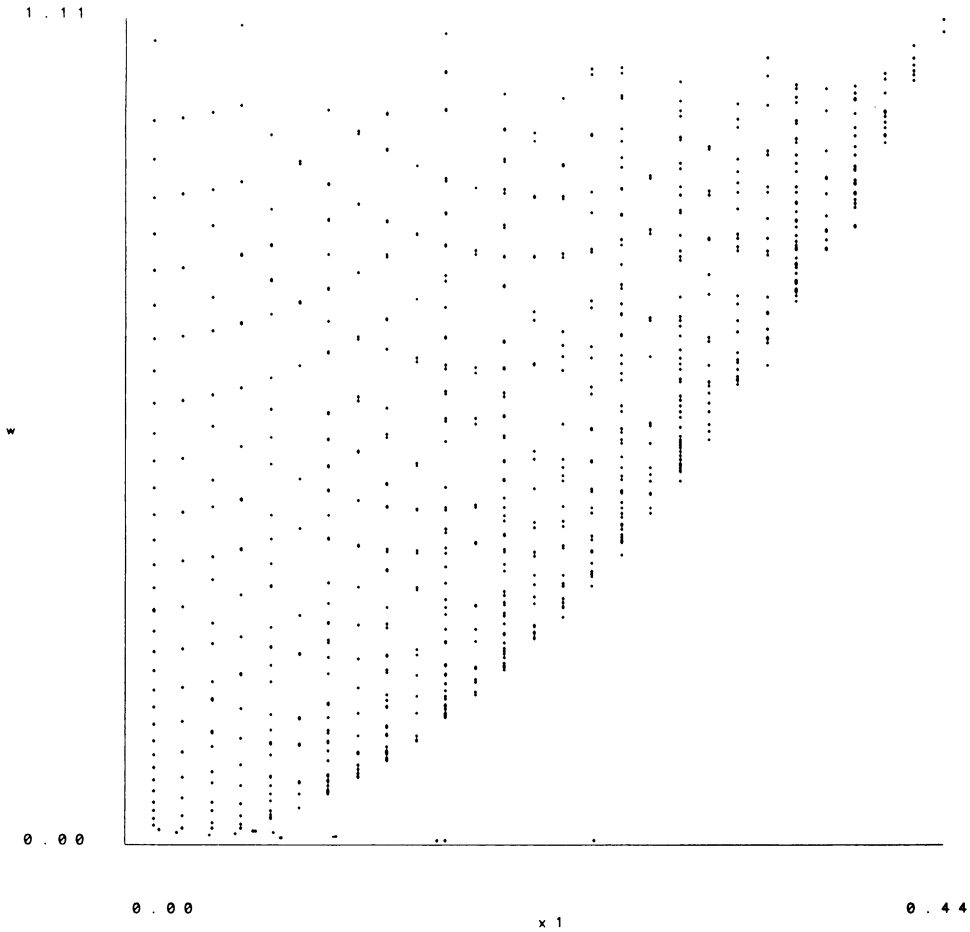

FIGURE 10

$(x1, w)$ coordinates of images of the fixed point Y_0 under transformation by Hecke matrices $M(p; a, b)$ for $b = 0$ in the fundamental domain F_3 of $\mathrm{SL}(3, \mathbf{Z})$ *defined in* (2.5) *and* (3.6) *with Y_0 as in* (3.7).

$p = 997$

The figures (except Figure 2) were produced by Dan Gordon using one of the U.C.S.D. VAX computer (sdcc6) and a laser printer. A. Terras did Figure 2 using a U.C.S.D. VAX and plotter.

There are various ways of understanding why the Hecke points should be dense in F_3. One could imitate an argument of Zagier using Eisenstein series (cf. Terras [17, p. 248]) for the SL(2, **Z**)-version of the argument) to show that the image of a horocycle C_Y in (2.2) becomes dense as F_3 in Y approaches the boundary of SP_3.

This result is also related to standard results in ergodic theory for connected noncompact simple Lie groups G with finite center (e.g., $G = \mathrm{SL}(3, \mathbf{R})$), saying that if H is a closed noncompact subgroup of G and Γ is an irreducible lattice (e.g., $\Gamma = \mathrm{SL}(3, \mathbf{Z})$) then H acts ergodically on G/Γ. Here we are closest to looking at an equally-spaced finite set of points in

$$H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y \in \mathbf{R} \right\}.$$

For we are looking at points from T_p acting on a fixed $Y_0 \in SP_3$ via

$$Y_0 \left[\begin{pmatrix} p^{2/3} & 0 & 0 \\ 0 & p^{-1/3} & 0 \\ 0 & 0 & p^{-1/3} \end{pmatrix} \begin{pmatrix} 1 & a/p & b/p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right], \qquad 0 \leqslant a, b \leqslant p - 1.$$

For the ergodic theory result, see Zimmer [20, p. 19 ff].

Ultimately, one would hope to be able to use the points $M(p; a, b)$ to generalize the results of Stark [16]. This will require programs for the computation of matrix argument K-Bessel or Whittaker functions.

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1. E. Artin, *Collected Papers*, Addison-Wesley, Reading, Mass., 1965.

2. D. Bump, *Automorphic Forms on* GL(3, **R**), Lecture Notes in Math., vol. 1083, Springer-Verlag, Berlin and New York, 1984.

3. E. Gottschling, "Explizite Bestimmung der Randflächen des Fundamentalbereiches der Modulgruppe zweiten Grades," *Math. Ann.*, v. 138, 1959, pp. 103–124.

4. D. Grenier, *Fundamental Domains for* $P_n/\mathrm{GL}(n, \mathbf{Z})$, Ph.D. Thesis, Univ. of California at San Diego, June, 1986.

5. M. Gutzwiller, "Physics and Selberg's trace formula," in *The Selberg Trace Formula and Related Topics* (D. Hejhal, P. Sarnak, and A. Terras, eds.), Contemp. Math., vol. 53, Amer. Math. Soc., Providence, R.I., 1986, pp. 215–251.

6. A. Korkine & G. Zolotareff, "Sur les formes quadratiques," *Math. Ann.*, v. 6, 1873, pp. 366–389.

7. A. Lubotzky, R. Phillips & P. Sarnak, "Hecke operators and distributing points on the sphere, I." (Preprint.)

8. H. Maass, "*Siegel's Modular Forms and Dirichlet Series*, Lecture Notes in Math., vol. 216, Springer-Verlag, Berlin and New York, 1971.

9. H. Maass, "Die Primzahlen in der Theorie der Siegelschen Modulfunktionen," *Math. Ann.*, v. 124, 1951, pp. 87–122.

10. H. Minkowski, *Gesammelte Abhandlungen*, Chelsea, New York, 1911 (reprinted, 1967).

11. N. V. Novikova, "Korkin-Zolotarev reduction domains of positive quadratic forms in $n \leqslant 8$ variables and a reduction algorithm for these domains," *Soviet Math. Dokl.*, v. 27, 1983, pp. 557–560.

12. S. S. Ryškov, "The theory of Hermite-Minkowski reduction of positive quadratic forms," *J. Soviet Math.*, v. 6, 1976, pp. 651–676.

13. S. S. RYŠKOV & E. P. BARANOVSKIĬ, "Classical methods in the theory of lattice packings," *Russian Math. Surveys*, v. 34, no. 4, 1979, pp. 1–68.

14. P. SARNAK, "Statistical properties of eigenvalues of Hecke operators." (Preprint.)

15. G. SHIMURA, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton Univ. Press, Princeton, N. J., 1971.

16. H. STARK, "Fourier coefficients of Maass wave forms," in *Modular Forms* (R. A. Rankin, ed.), Horwood, Chichester (distributed by Wiley), 1984, pp. 263–269.

17. A. TERRAS, *Harmonic Analysis on Symmetric Spaces and Applications*, Vol. I, Springer-Verlag, Berlin and New York, 1985; Vol. II, to appear.

18. A. TERRAS, "On automorphic forms for the general linear group," *Rocky Mountain J. Math.*, v. 12, 1982, pp. 123–143.

19. B. A. VENKOV, "Über die Reduction positiver quadratischer Formen," *Izv. Akad. Nauk SSSR*, v. 4, 1940, pp. 37–52. (Russian)

20. R. J. ZIMMER, *Ergodic Theory and Semisimple Groups*, Birkhäuser, Boston, 1984.