Computation of the Néron-Tate Height on Elliptic Curves

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For Daniel Shanks on the occasion of his 70th birthday

Abstract. Using Néron's reduction theory and a method of Tate, we develop a procedure for calculating the local and global Néron-Tate height on an elliptic curve over the rationals. The procedure is illustrated by means of two examples of Silverman and is then applied to calculate the global Néron-Tate height of a series of rank-one curves of Bremner-Cassels and of a series of rank-two curves of Selmer. In the latter case, the regulator is also computed, and a conjecture of S. Lang is investigated numerically.

In dealing with the arithmetic of elliptic curves $E$ over a global field $K$, the task arises of computing the Néron-Tate height on the group $E(K)$ of rational points of $E$ over $K$. Solving this task in an efficient manner is important, for instance, in view of calculations concerning the Birch and Swinnerton-Dyer conjecture (see [2]) or of the conjectures of Serge Lang [6]. The purpose of this note is to suggest a procedure for performing the necessary calculations.

1. Multiplication Formulas. Let the elliptic curve $E$ over any field $K$ be defined by a generalized Weierstrass equation

\[ (E) \quad y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \quad (a_i \in K). \]

As usual, we introduce the quantities (see [10], [11])

\[ b_2 = a_1^2 + 4a_2, \quad b_4 = a_1 a_3 + 2a_4, \quad b_6 = a_3^2 + 4a_6, \]
\[ b_8 = a_1^2 a_6 - a_1 a_3 a_4 + 4a_2 a_6 + a_2 a_3^2 - a_4^2, \]
\[ c_4 = b_2^2 - 24b_4, \quad c_6 = -b_2^3 + 36b_2 b_4 - 216b_6, \]

and the discriminant

\[ \Delta = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6 \neq 0, \]

as well as the absolute invariant

\[ j = c_4^3/\Delta \]

belonging to $E$ over $K$.

The fact that $E$ is nonsingular implies the nonvanishing of the partial derivatives of the polynomial

\[ F(x, y) = y^2 + a_1 xy + a_3 y - x^3 - a_2 x^2 - a_4 x - a_6 \]
at every rational point $P \in E(K)$:

$$\left( \frac{\partial F}{\partial x}(P), \frac{\partial F}{\partial y}(P) \right) \neq (0, 0).$$

The addition law in the additive Abelian group $E(K)$ of rational points on $E$ over $K$ is given by the following formulas:

For $P = (x_P, y_P), \ Q = (x_Q, y_Q) \in E(K)$, denote the sum by $P + Q = (x_{P+Q}, y_{P+Q})$. Then,

$$x_{P+Q} = -(x_P + x_Q) + \frac{y_P - y_Q}{x_P - x_Q} \left( x_P - x_{P+Q} \right) - a_1x_{P+Q} - a_3 - y_P \text{ if } P \neq Q$$

and

$$y_{P+Q} = \frac{y_P - y_Q}{x_P - x_Q} \left( x_P - x_{P+Q} \right) - a_1x_{P+Q} - a_3 - y_P$$

(1)

and

$$x_{2P} = -2x_P + t_P^2 + a_1t_P - a_2, \quad y_{2P} = t_P(x_P - x_{2P}) - a_1x_{2P} - a_3 - y_P$$

for $t_P = \frac{3x_P^2 + 2a_2x_P + a_4 - a_1y_P}{2y_P + a_1x_P + a_3}$ if $P = Q$.

Generalizing classical formulas (see [3], [4], [15]), we obtain

**Proposition 1.** For a rational point $P \in E(K)$ and an $r \in \mathbb{N}$, the $r$-fold rational point has coordinates

$$rP = (x_{rP}, y_{rP}) = \left( \frac{\Phi_r(P)}{\Psi^2_r(P)}, \frac{\Omega_r(P)}{\Psi^3_r(P)} \right),$$

where $\Phi_r, \Psi_r, \text{ and } 2\Omega_r$ are polynomials in $x$ and $y$ with coefficients in $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ given by the following recursion formulas:

$$\Phi_1 = x, \quad \Phi_2 = x^4 - b_4x^2 - 2b_6x - b_8,$$

$$\Omega_1 = y, \quad \Psi_0 = 0, \quad \Psi_1 = 1, \quad \Psi_2 = 2y + a_1x + a_3,$$

$$\Psi_3 = 3x^4 + 2b_2x^3 + 3b_4x^2 + 3b_6x + b_8,$$

$$\Psi_4 = \Psi_2 \left[ 2x^6 + b_2x^5 + 5b_4x^4 + 10b_6x^3 + 10b_8x^2 + (b_2b_8 - b_4b_6)x + b_4b_8 - b_6^2 \right]$$

and for $r \geq 2$,

$$\Phi_r = x\Psi_r^2 - \Psi_{r-1}\Psi_{r+1},$$

$$2\Psi_{2r}\Omega_r = \Psi^2_{r-1}\Psi_{r+2} - \Psi_{r-2}\Psi^2_{r+1} - \Psi_2\Phi_r[a_1\Phi_r + a_3\Psi_r^2],$$

$$\Psi_{2r+1} = \Psi^2_r\Psi_{r+2} - \Psi_{r-1}\Psi^3_{r+1},$$

$$\Psi_{2r+2} = \Psi_r\left[ \Psi^2_{r-1}\Psi_{r+2} - \Psi_{r-2}\Psi^2_{r+1} \right].$$

Moreover, $\Phi_r$, as a polynomial in $x$, has degree $r^2$ and leading coefficient 1, whereas $\Psi_r$ (resp. $\Psi_r^{-1}\Psi_r$), as a polynomial in $x$, has degree $(r^2 - 1)/2$ (resp. $(r^2 - 4)/2$) and leading coefficient $r$ (resp. $r/2$) provided that $r$ is odd (resp. even). If we assign the weight 2, 3 or i to $x$, $y$ or $a_i$, then each term of $\Phi_r$ has weight $2r^2$ and each term of
\(\Psi_r\) (resp. \(\Psi_r^{-1}\Psi_r\)) has weight \(r^2 - 1\) (resp. \(r^2 - 4\)). The coefficients of \(\Phi_r, \Psi_r\) as polynomials in \(x, y\), belong already to \(\mathbb{Z}[b_2, b_4, b_6, b_8]\).

From Proposition 1, one derives the following

**Corollary.** For \(r \in \mathbb{N}\), we put \(\Psi_\epsilon = -\Psi_r\). Then, for \(r, n \in \mathbb{N}\), we have

\[
(4) \quad \Psi_{rn}^2(P) = \Psi_r^{2^r}(P)\Psi_r^2(nP)
\]

and, more generally,

\[
(4') \quad \Psi_{m^r}(P) = \prod_{r=1}^n \Psi_m^{2^{m^r} - 1}(m^r - 1)P).
\]

Furthermore,

\[
(5) \quad x_{rP} - x_{nP} = -\frac{\Psi_{r+n}(P)\Psi_{r-n}(P)}{\Psi_r^2(P)\Psi_n^2(P)},
\]

\[
(6) \quad x_{rP} - x_{rQ} = (-1)^{r+1}\frac{\Psi_r(P + Q)\Psi_r(P - Q)}{\Psi_r^2(P)\Psi_r^2(Q)}(x_P - x_Q)^2
\]

and finally, for \(r \in \mathbb{N}_0\),

\[
(7) \quad \Phi_2(2^rP) = \Phi_{2^r+1}(P)\Psi_2^{-8}(P), \quad \Psi_2^2(2^rP) = \Psi_{2^r+1}(P)\Psi_2^{-8}(P).
\]

These formulas will be needed in the sequel.

**2. Reduction Theory.** Now let the elliptic curve \(E\) be defined by \((E)\) over a complete field \(K\) with respect to a discrete normalized additive valuation \(v\), and suppose that the corresponding residue field \(\bar{K}\) of \(K\) is perfect. We assume the equation defining \(E\) over \(K\) to be minimal with respect to the valuation \(v\) (see [11]). Reducing \(E\) modulo \(v\) yields a cubic curve

\[
(\bar{E}) \quad y^2 + \bar{a}_1\bar{x}\bar{y} + \bar{a}_3\bar{y} = \bar{x}^3 + \bar{a}_2\bar{x}^2 + \bar{a}_4\bar{x} + \bar{a}_6 \quad (\bar{a}_i \in \bar{K})
\]

over \(\bar{K}\) with discriminant \(\bar{\Delta}\). If \(\bar{\Delta} \neq 0\), i.e., \(v(\Delta) = 0\), then \(\bar{E}\) is an elliptic curve over \(\bar{K}\), and \(E\) has **good reduction** at \(v\). If, however, \(\bar{\Delta} = 0\), i.e., \(v(\Delta) > 0\), then \(\bar{E}\) is a rational curve over \(\bar{K}\), and \(E\) has **bad reduction** at \(v\). In the latter case, \(E\) is said to have **multiplicative reduction** or **additive reduction** modulo \(v\), according as \(v(c_4) = 0\) or \(v(c_4) > 0\), respectively.

Denote by \(E_0(K)\) the set of points in \(E(K)\) whose image under the reduction map modulo \(v\),

\[
p: E(K) \to \bar{E}(\bar{K}),
\]

is a nonsingular point on \(\bar{E}\) over \(\bar{K}\). Then, \(E_0(K)\) is a subgroup of finite index in \(E(K)\). Further, the set

\[
E_1(K) = \{P = (x_p, y_p) \in E(K) | v(x_p) \leq -2, v(y_p) \leq -3\}
\]
is a subgroup of \( E_0(K) \), and the restriction \( \rho_0 \) to \( E_0(K) \) of the reduction map \( \rho \) induces an injective homomorphism of the factor group

\[
\tilde{\rho}_0: E_0(K)/E_1(K) \to \tilde{E}_0(\tilde{K})
\]

to the nonsingular part \( \tilde{E}_0(\tilde{K}) \) of \( \tilde{E}(\tilde{K}) \).

We shall use the following result (see [11]).

**Proposition 2.** The above groups satisfy

\[
E(K) = E_0(K) \text{ if } E \text{ has good reduction at } v,
\]

\[
\#(E(K)/E_0(K)) \text{ divides } v(j) \text{ if } E \text{ has multiplicative reduction at } v,
\]

and

\[
\#(E(K)/E_0(K)) \leq 4 \text{ if } E \text{ has additive reduction at } v.
\]

3. **Definition of Height Functions.** Now let \( K \) be a global field, that is, an algebraic number field or a function field of finite transcendence degree over its field of constants \( k \). Then \( K \) possesses a complete set \( M_K \) of nonequivalent additive valuations \( v \) satisfying the sum formula

\[
\sum_{v \in M_K} \lambda_v v(c) = 0 \text{ for } 0 \neq c \in K
\]

with some positive multiplicities \( \lambda_v \in \mathbb{R} \) (cf. [7], [13]).

For an elliptic curve \( E \) over \( K \), given by the Weierstrass equation (E), we introduce the quantities

\[
\mu_v = \min \{ v(b_2), \frac{1}{2} v(b_4), \frac{1}{2} v(b_6) \}
\]

Let \( P = (x_P, y_P) \in E(K) \) be any rational point and \( \mathcal{O} = (\infty, \infty) \) designate the point at infinity. Then we define the local Weil height on \( E(K) \) with respect to \( v \) by setting

\[
d_v(P) = \begin{cases} 
-\frac{1}{2} \min \{ \mu_v, v(x_P) \} & \text{if } P \neq \mathcal{O}, \\
-\frac{1}{2} \mu_v & \text{if } P = \mathcal{O}.
\end{cases}
\]

Then the global Weil height on \( E(K) \) is simply the sum, with multiplicities, over the local Weil heights

\[
d(P) = \sum_{v \in M_K} \lambda_v d_v(P)
\]

(see [13]).

In order to define the global Néron-Tate height on \( E(K) \), we proceed in the same way as with the global Weil height. However, before introducing the local Néron-Tate height on \( E(K) \), we need some estimates.

**Proposition 3.** The local Weil height on \( E(K) \) satisfies the following estimates:

\[
\frac{1}{2}(6\mu_v - v(\Delta)) + 5\alpha_v \leq d_v(P + Q) + d_v(P - Q)
\]

\[
-2d_v(P) - 2d_v(Q) + v(x_P - x_Q)
\]

\[
\leq -2\alpha_v \text{ if } P, Q, P \pm Q \neq \mathcal{O}.
\]
and

\[ \frac{1}{2}(6\mu_v - \nu(\Delta)) + 4\alpha_v \leq d_v(2P) - 4d_v(P) - \frac{1}{2}v\left(\Psi_2^2(P)\right) \leq -\frac{1}{2}\alpha_v \quad \text{if } 2P \neq \emptyset, \]

where the constant \(\alpha_v\) can be chosen to be 0 or \(-\log 2\) according as the valuation \(v\) of \(K\) is discrete or archimedean, respectively (see [13]).

These estimates are obtained as generalizations of those given in [13], [14]. At the same time, they sharpen those cited.

Remark 1. It is interesting to note that the authors of [2] suggested that a sharpening of the estimates in [13], [14] should be possible. Proposition 3 appears to be a step in this direction.

Employing (10), the inequalities (11) can be further generalized.

**Corollary.** For any \(m \in \mathbb{N}\), there are (recursively computable) nonnegative constants \(c_{1,m}, c_{2,m} \in \mathbb{R}\) depending on \(E, K,\) and \(v\) such that, given an arbitrary point \(P \in E(K)\) with \(mP \neq \emptyset\), we have

\[ (11') \quad c_{1,m} \leq d_v(mP) - m^2d_v(P) - \frac{1}{2}v\left(\Psi_m^2(P)\right) \leq c_{2,m}. \]

We are now in a position to define the local Néron-Tate height on \(E(K)\) with respect to \(v\). Let \(m, n \in \mathbb{N}\) and \(m \geq 2\). Then, for a rational point \(P \in E(K)\) such that \(m^nP \neq \emptyset\) for each \(n \in \mathbb{N}\), we define the local Néron-Tate height of \(P\) with respect to \(v\) by the limit formula

\[ (12) \quad \delta_{v,m}(P) = \lim_{n \to \infty} \left( \frac{d_v(m^nP)}{m^{2n}} - \frac{1}{2} \frac{v\left(\Psi_m^2(P)\right)}{m^{2n}} \right) + \frac{1}{12}v(\Delta). \]

**Proposition 4.** For an elliptic curve \(E\) defined by a Weierstrass equation (E) over a global field \(K\) and any valuation \(v\) of \(K\), the function \(\delta_{v,m}\), defined by (12) on the rational point group \(E(K)\), exists, is independent of the choice of \(m \in \mathbb{N}\), so that \(\delta_{v,m} = \delta_v\), and fulfills the relations

\[ (13) \quad \delta_v(P + Q) + \delta_v(P - Q) - 2\delta_v(P) - 2\delta_v(Q) - v(x_P - x_Q) + \frac{1}{2}v(\Delta) = 0 \]

for any two points \(P = (x_P, y_P), Q = (x_Q, y_Q) \in E(K)\) such that \(P, Q, P \pm Q \neq \emptyset\), and

\[ (14) \quad \delta_v(rP) - r^2\delta_v(P) - \frac{1}{2}v\left(\Psi_r^2(P)\right) + \frac{r^2 - 1}{12}v(\Delta) = 0 \]

for any \(P = (x_P, y_P) \in E(K)\) and \(r \in \mathbb{N}\) such that \(rP \neq \emptyset\).

**Proof.** The proof is an adaptation of the corresponding proof of the existence theorem in [14]. Indeed, one exploits (10), (11) from Proposition 3 and (11') from the corollary to establish the existence of \(\delta_{v,m}\). Then formulas (6) and (4) from the corollary to Proposition 1 are utilized to prove that \(\delta_{v,m}\) fulfills the asserted relations.
Corollary 1. The function \( \delta_{v,m} \) on \( E(K) \) is related to the local Weil height on \( E(K) \) through the estimate

\[
(15) \quad \left| \delta_{v,m}(P) - \left( d_v(P) + \frac{1}{12}v(\Delta) \right) \right| \leq c_m,
\]

where

\[
c_m = \frac{1}{m^2 - 1} \cdot \max\{|c_{1,m}|, |c_{2,m}|\}.
\]

In fact, \( \delta_{v,m} = \delta_v \) is uniquely determined by the properties (14) and (15) and hence is independent of the choice of \( m \).

We can now define the global Néron-Tate height on \( E(K) \) as the sum, with multiplicities, over the local Néron-Tate heights as follows (see [14]):

\[
(16) \quad \delta(P) = \sum_{v \in M_K} \lambda_v \delta_v(P) \quad \text{if } P \neq \emptyset,
\]

\[
0 \quad \text{if } P = \emptyset.
\]

By the sum formula (S) we then obtain on the basis of (13) and (14):

Corollary 2. The global Néron-Tate height on \( E(K) \) fulfills the relations

\[
(13') \quad \delta(P + Q) + \delta(P - Q) - 2\delta(P) - 2\delta(Q) = 0
\]

and, for \( r \in \mathbb{N} \),

\[
(14') \quad \delta(rP) - r^2\delta(P) = 0.
\]

Remark 2. Corollary 2 shows that the global Néron-Tate height \( \delta \) is a quadratic form on \( E(K) \), whereas Proposition 4 implies that the local Néron-Tate height \( \delta_v \) is "almost" a quadratic form on \( E(K) \).

4. Computation of the Néron-Tate Height. Again, let the elliptic curve \( E \) be given by (E) over a global field \( K \). Fix a nonarchimedean (discrete) valuation \( v \) of \( K \). Suppose that \( P = (x_P, y_P) \in E(K) \) is a rational point satisfying \( v(x_P) < \mu_v \).

By Proposition 1, on choosing an \( m \in \mathbb{N} \) such that \( m \geq 2 \) and \( v(m) = 0 \), we have

\[
x_{m^*P} = \frac{\Phi_{m^*}(P)}{\Psi_{m^*}(P)}.
\]

Now \( v(x_P) < \mu_v \) together with \( v(a_i) \geq \mu_v \) entails

\[
v(\Phi_{m^*}(P)) = m^{2n}v(x_P), \quad v(\Psi_{m^*}(P)) = (m^{2n} - 1)v(x_P).
\]

Hence

\[
v(x_{m^*P}) = v(x_P).
\]
Thus we obtain from the limit formula (12) and the definition (9) of $d_v$ the asserted relation

$$\delta_v(P) = -\frac{1}{2}v(x_P) + \frac{1}{12}v(\Delta) = d_v(P) + \frac{1}{12}v(\Delta).$$

**Proposition 5.** Suppose that a rational point $P = (x_P, y_P) \in E(K)$ satisfies the inequality $v(x_P) < \mu_v$ for a nonarchimedean (discrete) valuation $v$ of the global field $K$. Then the local Néron-Tate height of $P$ essentially coincides with the local Weil height of $P$ with respect to $v$; more precisely,

$$\delta_v(P) = d_v(P) + \frac{1}{12}v(\Delta).$$

From Proposition 5 we get the following theorem, which is crucial for the calculation of the Néron-Tate height on $E(K)$.

**Theorem 1.** Let $E$ be an elliptic curve defined by a Weierstrass equation $(E)$ over an algebraic number field $K$. Choose a discrete normalized additive valuation $v$ of $K$ and suppose that the equation $(E)$ is minimal with respect to $v$. Then, for each nontorsion point $P \in E_0(K)$, the local Néron-Tate height of $P$ is essentially equal to the local Weil height of $P$ with respect to $v$; more precisely,

$$\delta_v(P) = d_v(P) + \frac{1}{12}v(\Delta).$$

**Proof.** The theorem can be found in [9]. For the convenience of the reader, however, we give a proof.

By Proposition 5, we may confine ourselves to the case in which $v(x_P) > \mu_v$. The subcase in which $v(x_P) > \mu_v > 0$ would lead to a contradiction to the choice of $P \in E_0(K)$. Hence it remains to consider the subcase in which $v(x_P) > \mu_v = 0$.

The reduction map of Section 2,

$$\tilde{\rho}_0: E_0(K)/E_1(K) \rightarrow \tilde{E}_0(\tilde{K}),$$

is an injective homomorphism. Since $K$ is a number field, the residue field $\tilde{K}$ of $K$ with respect to $v$ is finite and hence so is the group $\tilde{E}_0(\tilde{K})$. Therefore, for any $P \in E_0(K)$, there exists a number $r \in \mathbb{N}$ such that $rP \in E_1(K)$. Choose $r \in \mathbb{N}$ minimal with this property. Then we have

$$v(x_{rP}) < \mu_v = 0.$$  

From this, since $v(x_P) \geq \mu_v = 0$ and $v(a_i) \geq \mu_v = 0$, we conclude that

$$v(\Phi_r(P)) \geq 0 \quad \text{and} \quad v(\Psi_r(P)) > 0.$$  

We claim

$$v(x_{rP}) = -v(\Psi_r^2(P)).$$

*The required minimal model of $E$ is found by Tate's algorithm [11].*
By Proposition 5, Formula (14) of Proposition 4, and the definition (9) of $d_v$, this claim yields the asserted identity

$$\delta_v(P) = \frac{1}{r^2} \left( \delta_v(rP) - \frac{1}{2} v(\Psi^2_v(P)) + \frac{r^2 - 1}{12} v(\Delta) \right)$$

$$= d_v(P) + \frac{1}{12} v(\Delta)$$

since $v(x_p) \geq \mu_v = 0$.

To prove (17) it suffices to show that

$$v(\Phi_r(P)) = 0.$$  \hfill (18)

This is accomplished by verifying (18), first for the lower $r \in \mathbb{N}$ and then for general $r \in \mathbb{N}$.

Let $r = 2$.

If $v(3x_p^2 + 2a_2x_p + a_4 - a_1y_p) > 0$ we would get a contradiction to the assumption that $P \in E_0(K)$. Hence it is enough to consider $v(3x_p^2 + 2a_2x_p + a_4 - a_1y_p) = 0$. But then the asserted relation (18) follows directly from the formula (2) for $x_2P$ and Proposition 1.

Let $r = 3$.

By the minimal choice of $r$, we have

$$v(\Psi_2(P)) = 0 \quad \text{and} \quad v(\Psi_3(P)) > 0.$$  \hfill (2.2)

Now the decomposition formula (which can be verified without trouble)

$$\Psi_4(P) = \Psi_2(P) [\Psi_3(P)(6x_p^2 + b_2x_p + b_4) - \Psi_2^2(P)]$$

yields $v(\Psi_4(P)) = 0$, and hence the relation from Proposition 1,

$$\Phi_3(P) = x_p\Psi_3^2(P) - \Psi_2(P)\Psi_4(P),$$

leads to the identity $v(\Phi_3(P)) = 0$, as asserted in (18).

Finally, let $r \geq 4$.

Again, by the choice of $r$, we have

$$v(\Psi_2(P)) = v(\Psi_3(P)) = \cdots = v(\Psi_{r-1}(P)) = 0 \quad \text{and} \quad v(\Psi_r(P)) > 0.$$

Then Formula (5) from the corollary to Proposition 1 yields

$$v(x_2P - x_p) = 0 \quad \text{and} \quad v(x_{(r-1)p} - x_p) > 0,$$

so that another consequence of Formula (5), viz.,

$$\Psi_{r+1}(P) = -[(x_{(r-1)p} - x_p) + (x_p - x_2P)] \frac{\Psi_{r-1}^2(P)\Psi_2^2(P)}{\Psi_{r-3}(P)},$$

leads to $v(\Psi_{r+1}(P)) = 0$. Now the identity from Proposition 1,

$$\Phi_r(P) = x_p\Psi_r^2(P) - \Psi_{r-1}(P)\Psi_{r+1}(P),$$

reveals that $v(\Phi_r(P)) = 0$, as asserted in (18). This proves Theorem 1.
Remark 3. Theorem 1 makes it possible to calculate the local Néron-Tate height \( \delta_v(P) \) with respect to all discrete valuations \( v \) of the number field \( K \) for all nontorsion points \( P \in E(K) \).

This is true because Proposition 2 tells us that a suitable multiple \( rP \) of \( P \) belongs to \( E_0(K) \). Then we apply Theorem 1 to calculate \( \delta_v(rP) \) and use Formula (14) from Proposition 4 to get the desired value of \( \delta_v(P) \) itself.**

Remark 4. Torsion points \( P \in E(K) \) are of no interest in this connection since their global Néron-Tate height is \( \delta(P) = 0 \).

It remains to show how to compute the local Néron-Tate height \( \delta_v \) for archimedean valuations \( v \) of the number field \( K \). From (4') in the corollary to Proposition 1, we get the formula

\[
\frac{1}{2} v\left( \frac{\Psi_m^2(P)}{m^{2n}} \right) = \sum_{v=1}^{m} \frac{1}{2} v\left( \frac{\Psi_m^2(m^{r-1}P)}{m^{2v}} \right),
\]

which proves to be useful in the sequel.

Now, since we are interested here only in the case of \( K = \mathbb{Q} \), the field of rational numbers, we confine ourselves to considering its completion \( K_\infty = \mathbb{Q}_\infty = \mathbb{R} \) with respect to the ordinary absolute value \( v = v_\infty = -\log || \). Then Tate's method is best suited for calculating \( \delta_{v_\infty} \) (see [12]).

**Theorem 2.** Let \( E \) be an elliptic curve defined by a Weierstrass equation (E) over the field \( \mathbb{R} \) of real numbers and denote by \( v_\infty = -\log || \) the ordinary additive archimedean valuation of \( \mathbb{R} \). Take an open subgroup \( \Gamma \) of \( E(\mathbb{R}) \) such that all \( P = (x_P, y_P) \in \Gamma \) satisfy \( x_\neq 0 \).*** For \( P \in \Gamma \) such that \( 2^n P \neq 0 \) for all \( n \in \mathbb{N} \), define the entities \( T_n, W_n, \text{ and } Z_n \) by putting

\[
T_n = \frac{1}{x_P}, \quad T_{n+1} = \frac{W_n}{Z_n} \quad \text{for } n \in \mathbb{N}_0,
\]

where

\[
W_n = 4T_n + b_2T_n^2 + 2b_4T_n^3 + b_6T_n^4, \quad Z_n = 1 - b_4T_n^2 - 2b_6T_n^3 - b_8T_n^4.
\]

Let

\[
\mu(P) = \sum_{n=0}^{\infty} \frac{\log |Z_n|}{2^{2n}}, \quad \lambda(P) = \frac{1}{2} \log |x_P| + \frac{1}{8} \mu(P).
\]

Then the local Néron-Tate height of \( P \) with respect to \( v_\infty \) is

\[
\delta_{v_\infty}(P) = \lambda(P) - \frac{1}{12} \log |\Delta|.
\]

**Proof.** See [12]. However, the assertion of Theorem 2 also follows from

**Proposition 6.** In the situation of Theorem 2 we have for \( n \in \mathbb{N}_0 \),

\[
T_n = \frac{1}{x_{2^n P}}, \quad W_n = \frac{\Psi_2(2^n P)}{x_{4^{2n} P}^4}, \quad Z_n = \frac{\Phi_2(2^n P)}{x_{4^{2n} P}^4}.
\]

*** Added in proof. Joe Silverman, whom we wish to thank for some valuable hints, told us that he has carried out similar height computations (unpublished) avoiding, however, the use of Proposition 2 by employing Tate's local formulas (see [14]).

**** Hence \( \Gamma \) is either \( E(\mathbb{R}) \) or the identity component of \( E(\mathbb{R}) \) according as \( E(\mathbb{R}) \) is connected or disconnected.
Proof. The proof is carried out easily by means of the formulas (7) in the corollary to Proposition 1.

Remark 5. The simplest way of finding a subgroup $\Gamma$ of $E(\mathbb{R})$ of the type desired in Theorem 2 is by applying a birational transformation to $(E)$ to obtain a model $(E')$ such that $b'_0 < 0$. Then $\Gamma = E'(\mathbb{R})$ itself will do.

In the special case of $K = \mathbb{Q}$ we are interested in, the set $M_Q$ consists in the $p$-adic valuations $v_p$ corresponding to the primes $p$ of $\mathbb{Q}$ and the additive valuation $v_\infty = -\log|\cdot|$ corresponding to the unique archimedean absolute value $|\cdot|$ on $\mathbb{Q}$. Of course, the multiplicities in the sum formula (S) are all $\lambda_v = 1$.

5. Examples. We are now in a position to calculate the Néron-Tate height $\delta$ on the group $E(K)$ of rational points on an elliptic curve $E$ over the rational number field $K = \mathbb{Q}$. To this end, we use the defining formula (16) for $\delta$ with multiplicities $\lambda_v = 1$ to reduce the computation of $\delta$ to that of the local Néron-Tate heights $\delta_v$ on $E(K)$. For discrete valuations $v$ of $\mathbb{Q}$, the height $\delta_v$ is calculated by means of Theorem 1 in accordance with Remark 3, and for the archimedean absolute value $v_\infty = -\log|\cdot|$, the calculation of $\delta_{v_\infty}$ is performed on the basis of Theorem 2.

(i) Examples of Silverman. We illustrate our procedure by verifying the height calculations of Silverman [9].

(A) $E: y^2 + 21xy + 494y = x^3 + 26x^2$,
$P = (0, 0) \in E(\mathbb{Q})$,
$\Delta = -2^{13} \cdot 13^3 \cdot 19^2$.

Silverman obtains
$\delta(P) = 0.010,492,\ldots.$

We have
(a) $\delta_{v_\infty}(P) = 0.038,612,393,\ldots,$
(b) $P \not\in E_0(\mathbb{Q})$ for $p = 2, 13$ and $19$; and
$\delta_v(P) = 0$ for all primes $p \neq 2, 13$ or $19$.

Now
$13P \in E_0(\mathbb{Q})$ for $p = 2$,
$3P \in E_0(\mathbb{Q})$ for $p = 13$,
$2P \in E_0(\mathbb{Q})$ for $p = 19$.

One computes
$\Psi_2(P) = 2 \cdot 13 \cdot 19, \quad \Psi_3(P) = 2^3 \cdot 13^3 \cdot 19^2, \quad \Psi_{13}(P) = -2^{80} \cdot 13^{56} \cdot 19^{42}$
and
$x_{2P} = -2 \cdot 13, \quad x_{3P} = -2 \cdot 19, \quad x_{13P} = -2^4 \cdot 5 \cdot 13 \cdot 19.$

This leads to
$\delta_{v_2}(13P) = \frac{37}{12} \ln 2, \quad \delta_{v_3}(3P) = \frac{1}{4} \ln 13, \quad \delta_{v_{13}}(2P) = \frac{1}{6} \ln 19.$

Hence, by (14) of Proposition 4,
$\delta_{v_2}(P) = \frac{97}{156} \ln 2, \quad \delta_{v_3}(P) = -\frac{1}{12} \ln 13, \quad \delta_{v_{13}}(P) = -\frac{1}{12} \ln 19.$
By (16) this adds up to
\[ \delta(P) = 0.010,492,061, \ldots \]

(B) \[
E: y^2 + 11xy + 80y = x^3 + 8x^2, \\
P = (0,0) \in E(\mathbb{Q}), \\
\Delta = -2^{11} \cdot 5^2 \cdot 19.
\]

Silverman gets
\[ \delta(P) = 0.010,284,\ldots \]
and we obtain similarly to (A)
\[ \delta(P) = 0.010,284,005,\ldots \]

(ii) The Bremner-Cassels Curves. Our procedure turns out to be particularly useful for calculating the global Néron-Tate height on the elliptic curves
\[ E_p: y^2 = x^3 + px \]
for primes \( p \) of \( \mathbb{Q} \) such that \( p \equiv 5 \pmod{8} \), as they were considered by Bremner and Cassels [1]. The authors exhibit points \( P \in E(\mathbb{Q}) \) of infinite order on 43 curves of this type, where
\[
P = (x_P, y_P) \quad \text{with} \quad x_P = \frac{r^2}{s^2}, \quad y_P = \frac{r \cdot t}{s^3} \quad \text{for} \ r, s, t \in \mathbb{Z}
\]
such that
\[
g.c.d.(r, s) = 1; \quad r, t \not\equiv 0 \pmod{p}; \quad \text{and} \\
r \equiv t \equiv 1 \pmod{2}, \quad s \equiv 0 \pmod{2}.
\]
One easily checks that \( P \in E_0(\mathbb{Q}) \) for all primes \( p \) of \( \mathbb{Q} \) and all points \( P \in E(\mathbb{Q}) \) displayed in [1]. (Notice that \( 2y_P \) and \( 3x_P^2 + p \) are relatively prime.) This leads to

**Proposition 7.** For the points \( P \in E_p(\mathbb{Q}) \) of infinite order on the Bremner-Cassels curves in [1], the Néron-Tate height is
\[ \delta(P) = \delta_{\nu_p}(P) + \frac{1}{12} \ln |\Delta| + \ln |s|. \]

(iii) Modular Elliptic Curves. In [16, pp. 75–113], N. M. Stephens and J. Davenport list 68 modular elliptic curves \( E \) of rank 1 with a rational point \( P \in E(\mathbb{Q}) \) of infinite order. We computed the Néron-Tate heights of these points \( P \).\(^\dagger\) Comparison of the Néron-Tate height of the generator of the 63rd curve in their table with the Néron-Tate height of the point in Silverman’s second example (see (i) (B) above) shows that the two values agree. It turns out, as one easily checks, that the corresponding two curves are birationally isomorphic (see Table 1).

\(^\dagger\) We have compared the height values in our Table 1 with those in a corresponding (unpublished) table of Silverman containing up to six digits behind the period. They agree (except for the sixth digits of the curves 58A, 61A, 135A, 153A, 189C and for the fifth and sixth digit of the curve 185D).
Table 1

<table>
<thead>
<tr>
<th>Table Entry</th>
<th>Conditions</th>
<th>Global Height</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$a_1 = 0$, $a_2 = 0$, $a_3 = 1$, $a_4 = -1$, $a_6 = 0$</td>
<td>0.0255570412</td>
</tr>
<tr>
<td>2.</td>
<td>$a_1 = 0$, $a_2 = 1$, $a_3 = 1$, $a_4 = 0$, $a_6 = 0$</td>
<td>0.03140253544</td>
</tr>
<tr>
<td>3.</td>
<td>$a_1 = 1$, $a_2 = -1$, $a_3 = 1$, $a_4 = 0$, $a_6 = 0$</td>
<td>0.04649742319</td>
</tr>
<tr>
<td>4.</td>
<td>$a_1 = 0$, $a_2 = -1$, $a_3 = 1$, $a_4 = -2$, $a_6 = 2$</td>
<td>0.018787296368</td>
</tr>
<tr>
<td>5.</td>
<td>$a_1 = 1$, $a_2 = -1$, $a_3 = 0$, $a_4 = -1$, $a_6 = 1$</td>
<td>0.021215392</td>
</tr>
<tr>
<td>6.</td>
<td>$a_1 = 1$, $a_2 = 0$, $a_3 = 0$, $a_4 = -2$, $a_6 = 1$</td>
<td>0.0395985681</td>
</tr>
<tr>
<td>7.</td>
<td>$a_1 = 1$, $a_2 = 0$, $a_3 = 0$, $a_4 = -1$, $a_6 = 0$</td>
<td>0.187757</td>
</tr>
<tr>
<td>8.</td>
<td>$a_1 = 0$, $a_2 = 0$, $a_3 = 1$, $a_4 = 2$, $a_6 = 0$</td>
<td>0.049013989632</td>
</tr>
<tr>
<td>9.</td>
<td>$a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = -2$, $a_6 = 0$</td>
<td>0.048832105054</td>
</tr>
<tr>
<td>10.</td>
<td>$a_1 = 1$, $a_2 = 0$, $a_3 = 1$, $a_4 = -2$, $a_6 = 0$</td>
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</tr>
<tr>
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<td>$a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_6 = 0$</td>
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</tr>
<tr>
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</tr>
<tr>
<td>13.</td>
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<tr>
<td>14.</td>
<td>$a_1 = 0$, $a_2 = 0$, $a_3 = 1$, $a_4 = 1$, $a_6 = 0$</td>
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</tr>
<tr>
<td>15.</td>
<td>$a_1 = 0$, $a_2 = 1$, $a_3 = 1$, $a_4 = -7$, $a_6 = 5$</td>
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<tr>
<td>16.</td>
<td>$a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = -1$, $a_6 = 1$</td>
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<tr>
<td>17.</td>
<td>$a_1 = 1$, $a_2 = -1$, $a_3 = 1$, $a_4 = -2$, $a_6 = 0$</td>
<td>0.151285692281</td>
</tr>
</tbody>
</table>
Table 1 (continued)

| 18. | $a_1 = 0$ | $a_2 = 1$ | $a_3 = 1$ | $a_4 = -1$ | $a_6 = -1$ | $P = (-1 ; 0)$ | Global height: 0.082351726475 |
| 19. | $a_1 = 1$ | $a_2 = 1$ | $a_3 = 0$ | $a_4 = -2$ | $a_6 = 0$ | $P = (-1 ; 2)$ | Global height: 0.07162694647 |
| 20. | $a_1 = 1$ | $a_2 = 1$ | $a_3 = 0$ | $a_4 = -7$ | $a_6 = 5$ | $P = (2 ; 1)$ | Global height: 0.034456340202 |
| 21. | $a_1 = 0$ | $a_2 = 1$ | $a_3 = 0$ | $a_4 = 0$ | $a_6 = 4$ | $P = (0 ; 2)$ | Global height: 0.119959949363 |
| 22. | $a_1 = 1$ | $a_2 = -1$ | $a_3 = 1$ | $a_4 = 4$ | $a_6 = 6$ | $P = (0 ; 2)$ | Global height: 0.56516781309 |
| 23. | $a_1 = 1$ | $a_2 = 1$ | $a_3 = 0$ | $a_4 = 1$ | $a_6 = 1$ | $P = (0 ; 1)$ | Global height: 0.043953079838 |
| 24. | $a_1 = 0$ | $a_2 = -1$ | $a_3 = 1$ | $a_4 = -7$ | $a_6 = 10$ | $P = (4 ; 5)$ | Global height: 0.04489257808 |
| 25. | $a_1 = 1$ | $a_2 = 0$ | $a_3 = 1$ | $a_4 = 2$ | $a_6 = 0$ | $P = (1 ; 1)$ | Global height: 0.060421607704 |
| 26. | $a_1 = 0$ | $a_2 = 1$ | $a_3 = 1$ | $a_4 = -10$ | $a_6 = 10$ | $P = (1 ; 1)$ | Global height: 0.420260708766 |
| 27. | $a_1 = 0$ | $a_2 = 1$ | $a_3 = 0$ | $a_4 = -2$ | $a_6 = 1$ | $P = (1 ; 1)$ | Global height: 0.260265346941 |
| 28. | $a_1 = 0$ | $a_2 = 1$ | $a_3 = 0$ | $a_4 = 1$ | $a_6 = 1$ | $P = (0 ; 1)$ | Global height: 0.21616582287 |
| 29. | $a_1 = 0$ | $a_2 = -1$ | $a_3 = 1$ | $a_4 = -19$ | $a_6 = 39$ | $P = (1 ; 4)$ | Global height: 0.049979957634 |
| 30. | $a_1 = 1$ | $a_2 = 0$ | $a_3 = 1$ | $a_4 = -33$ | $a_6 = 68$ | $P = (2 ; 2)$ | Global height: 0.585232076797 |
| 31. | $a_1 = 0$ | $a_2 = -1$ | $a_3 = 1$ | $a_4 = 1$ | $a_6 = 0$ | $P = (0 ; 0)$ | Global height: 0.108047599334 |
| 32. | $a_1 = 0$ | $a_2 = 1$ | $a_3 = 0$ | $a_4 = -4$ | $a_6 = 0$ | $P = (-2 ; 2)$ | Global height: 0.115753996413 |
| 33. | $a_1 = 1$ | $a_2 = 1$ | $a_3 = 0$ | $a_4 = -1$ | $a_6 = 1$ | $P = (0 ; 1)$ | Global height: 0.08868409567 |
| 34. | $a_1 = 0$ | $a_2 = 1$ | $a_3 = 1$ | $a_4 = -12$ | $a_6 = 2$ | $P = (-3 ; 4)$ | Global height: 0.017243387509 |
Table 1 (continued)

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<th>a1</th>
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<th>a3</th>
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<th>a6</th>
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| 52 | \( a_1 = 0 \) | \( a_2 = -1 \) | \( a_3 = 1 \) | \( a_4 = -148 \) | \( a_6 = 748 \) |
| 175A | \( P = (7; 2) \) | \( \text{global height: } 0.332314998542 \) |

| 53 | \( a_1 = 0 \) | \( a_2 = -1 \) | \( a_3 = 1 \) | \( a_4 = -33 \) | \( a_6 = 93 \) |
| 175C | \( P = (-3; 12) \) | \( \text{global height: } 0.046286666901 \) |

| 54 | \( a_1 = 0 \) | \( a_2 = -1 \) | \( a_3 = 0 \) | \( a_4 = 3 \) | \( a_6 = 1 \) |
| 176A | \( P = (1; 2) \) | \( \text{global height: } 0.087531915526 \) |

| 55 | \( a_1 = 0 \) | \( a_2 = -1 \) | \( a_3 = 0 \) | \( a_4 = -4 \) | \( a_6 = 5 \) |
| 184B | \( P = (2; 1) \) | \( \text{global height: } 0.051533618406 \) |

| 56 | \( a_1 = 0 \) | \( a_2 = -1 \) | \( a_3 = 0 \) | \( a_4 = 0 \) | \( a_6 = 1 \) |
| 184C | \( P = (0; 1) \) | \( \text{global height: } 0.061565455601 \) |

| 57 | \( a_1 = 0 \) | \( a_2 = -1 \) | \( a_3 = 1 \) | \( a_4 = -5 \) | \( a_6 = 6 \) |
| 185A | \( P = (0; 2) \) | \( \text{global height: } 0.055139483611 \) |

| 58 | \( a_1 = 1 \) | \( a_2 = 0 \) | \( a_3 = 1 \) | \( a_4 = -4 \) | \( a_6 = -3 \) |
| 185B | \( P = (3; 2) \) | \( \text{global height: } 0.712632645336 \) |

| 59 | \( a_1 = 0 \) | \( a_2 = 1 \) | \( a_3 = 1 \) | \( a_4 = -156 \) | \( a_6 = 700 \) |
| 185D | \( P = (4; 12) \) | \( \text{global height: } 0.057028352204 \) |

| 60 | \( a_1 = 0 \) | \( a_2 = 0 \) | \( a_3 = 1 \) | \( a_4 = -3 \) | \( a_6 = 0 \) |
| 189A | \( P = (-1; 1) \) | \( \text{global height: } 0.0316066094417 \) |

| 61 | \( a_1 = 0 \) | \( a_2 = 0 \) | \( a_3 = 1 \) | \( a_4 = -24 \) | \( a_6 = 45 \) |
| 189C | \( P = (-3; 9) \) | \( \text{global height: } 0.931621776106 \) |

| 62 | \( a_1 = 1 \) | \( a_2 = 1 \) | \( a_3 = 0 \) | \( a_4 = 2 \) | \( a_6 = 2 \) |
| 190C | \( P = (1; 2) \) | \( \text{global height: } 0.065910740941 \) |

| 63 | \( a_1 = 1 \) | \( a_2 = -1 \) | \( a_3 = 1 \) | \( a_4 = -48 \) | \( a_6 = 147 \) |
| 190D | \( P = (13; 33) \) | \( \text{global height: } 0.010284005728 \) |

| 64 | \( a_1 = 0 \) | \( a_2 = -1 \) | \( a_3 = 0 \) | \( a_4 = -4 \) | \( a_6 = -2 \) |
| 1920 | \( P = (3; 2) \) | \( \text{global height: } 0.675801867206 \) |

| 65 | \( a_1 = 0 \) | \( a_2 = -1 \) | \( a_3 = 0 \) | \( a_4 = -2 \) | \( a_6 = 1 \) |
| 196A | \( P = (0; 1) \) | \( \text{global height: } 0.043017725683 \) |

| 66 | \( a_1 = 0 \) | \( a_2 = 0 \) | \( a_3 = 1 \) | \( a_4 = -5 \) | \( a_6 = 4 \) |
| 197A | \( P = (1; 0) \) | \( \text{global height: } 0.069433995882 \) |

| 67 | \( a_1 = 1 \) | \( a_2 = -1 \) | \( a_3 = 0 \) | \( a_4 = -18 \) | \( a_6 = 4 \) |
| 1981 | \( P = (-1; 5) \) | \( \text{global height: } 0.097521495699 \) |

| 68 | \( a_1 = 0 \) | \( a_2 = 1 \) | \( a_3 = 0 \) | \( a_4 = -3 \) | \( a_6 = -2 \) |
| 200C | \( P = (-1; 1) \) | \( \text{global height: } 0.146605513301 \) |
6. Lang's Conjectures. Silverman [9] used his above-cited examples of rank-one elliptic curves \( E \) over \( \mathbb{Q} \) to estimate the constants \( c_1, c_2 \) in S. Lang's Conjecture 2 (see [6]) about lower bounds for the Néron-Tate height \( \delta \) on nontorsion points in \( E(\mathbb{Q}) \). We wish to carry through a similar estimation with respect to Lang's Conjecture 3 (see [6]) for Selmer's [8] rank-two elliptic curves \( E \) over \( \mathbb{Q} \).

In Section 3, Remark 2, we observed that the Néron-Tate height \( \delta \) is a quadratic form on the rational point group \( E(\mathbb{Q}) \). This property of \( \delta \) is tantamount to the fact that the function

\[
\beta(P, Q) = \frac{1}{2} \{ \delta(P + Q) - \delta(P) - \delta(Q) \}
\]

<table>
<thead>
<tr>
<th>Table 2</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>( a_1 = 0 )</th>
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<th>( a_3 = 0 )</th>
<th>( a_4 = 0 )</th>
<th>( a_6 = -388800 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 = (76/1, 224/1) )</td>
<td>( P_2 = (124/1, 1232/1) )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The transformation with \( (r, s, t, u) = (0, 0, 0, 2) \) leads to

| \( a_1 = 0 \) | \( a_2 = 0 \) | \( a_3 = 0 \) | \( a_4 = 0 \) | \( a_6 = -6075 \) |

| \( P_1 = (19/1, 28/1) \) |

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<tr>
<th>( p )</th>
<th>the local height</th>
<th>( \text{decimal} )</th>
</tr>
</thead>
<tbody>
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<td>( 2 )</td>
<td>( (1/3) \ln(2) )</td>
<td>.231049060186</td>
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<tr>
<td>( 3 )</td>
<td>( (13/12) \ln(3) )</td>
<td>1.19016312723</td>
</tr>
<tr>
<td>( 5 )</td>
<td>( (1/3) \ln(5) )</td>
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</tr>
<tr>
<td>( \infty )</td>
<td>( -3.220039705773 )</td>
<td></td>
</tr>
</tbody>
</table>

The global height is 1.73765197128

| \( P_2 = (31/1, 154/1) \) |

<table>
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<tr>
<th>( p )</th>
<th>the local height</th>
<th>( \text{decimal} )</th>
</tr>
</thead>
<tbody>
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<td>( 2 )</td>
<td>( (1/3) \ln(2) )</td>
<td>.231049060186</td>
</tr>
<tr>
<td>( 3 )</td>
<td>( (13/12) \ln(3) )</td>
<td>1.19016312723</td>
</tr>
<tr>
<td>( 5 )</td>
<td>( (1/3) \ln(5) )</td>
<td>.536479304144</td>
</tr>
<tr>
<td>( \infty )</td>
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<td></td>
</tr>
</tbody>
</table>

The global height is 1.889072235727

| \( P_1 + P_2 = (241/4, -3689/8) \) |

<table>
<thead>
<tr>
<th>( p )</th>
<th>the local height</th>
<th>( \text{decimal} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2 )</td>
<td>( (4/3) \ln(2) )</td>
<td>.924196240746</td>
</tr>
<tr>
<td>( 3 )</td>
<td>( (13/12) \ln(3) )</td>
<td>1.19016312723</td>
</tr>
<tr>
<td>( 5 )</td>
<td>( (1/3) \ln(5) )</td>
<td>.536479304144</td>
</tr>
<tr>
<td>( \infty )</td>
<td>( 1.8319182537 )</td>
<td></td>
</tr>
</tbody>
</table>

The global height is 2.834030682983

Regulator : 3.125459338543
for \( P, Q \in E(\mathbb{Q}) \) represents a symmetric bilinear form on \( E(\mathbb{Q}) \). If \( E \) has rank two over \( \mathbb{Q} \) and \( P = P_1, Q = P_2 \) are two basis points of \( E(\mathbb{Q}) \), the quantity

\[
R = \left| \det \left( \beta(P_i, P_j) \right)_{i,j=1,2} \right| \in \mathbb{R}
\]

is called the **regulator** of the elliptic curve \( E \) over \( \mathbb{Q} \). In addition to the Néron-Tate height of the basis points \( P_1, P_2 \) of the rank-two curves \( E \) in Selmer’s tables [8], we have also computed their regulator \( R \). More detailed information about Selmer’s curves is to be found in [6]. To begin with, we list in detail two examples, namely the curves with \( A = 30 \) and \( A = 246 \) in [8] (see Table 2).
In analogy to Silverman [9], we now use these Selmer curves to estimate the constants in Lang's Conjecture 3. Suppose \( E \) over \( \mathbb{Q} \) is given in Weierstrass normal form

\[
(E) \quad y^2 = x^3 + ax + b \quad (a, b \in \mathbb{Z}).
\]

Following Lang [6], we define the height of \( E \) over \( \mathbb{Q} \) to be the number

\[
H(E) = \max\{|a|^3, |b|^2\},
\]

so that approximately

\[
h(E) \approx -\log H(E) \approx 6\mu_{v_\infty},
\]

where again \( v_\infty = -\log | \) denotes the additive archimedean valuation of \( \mathbb{Q} \). Let \( N \) stand for the conductor of \( E \) over \( \mathbb{Q} \) (see [11]).

Then we enunciate, in the case of rank-two curves, Lang's Conjecture 3. There is a basis \( \{P_1, P_2\} \) of \( E(\mathbb{Q}) \) modulo torsion such that \( \delta(P_1) \leq \delta(P_2) \) and

\[
\delta(P_1) \leq c_1 H(E)^{1/24} \cdot N^{\epsilon(N)/2} \cdot \log N \cdot (2/\sqrt{3})^{1/2},
\]

\[
\delta(P_2) \leq c_2 H(E)^{1/12} \cdot N^{\epsilon(N) \cdot \log N \cdot c}
\]

for some positive real constants \( c, c_1, c_2 \), where

\[
\lim_{N \to \infty} \epsilon(N) = 0.
\]

Now the constants \( c_1 \) and \( c_2 \) in Lang's Conjecture 3 satisfy the inequalities

\[
c_1 \geq \left( \frac{H(E)^{1/24} \cdot N^{\epsilon(N)/2} \cdot \log N \cdot (2/\sqrt{3})^{1/2}}{\delta(P_1)} \right)^{-1},
\]

\[
c_2 \geq \left( \frac{H(E)^{1/12} \cdot N^{\epsilon(N) \cdot \log N \cdot c}}{\delta(P_2)} \right)^{-1}.
\]

On choosing \( c = 1 \) and putting, in analogy to the example on p. 166 of [6],

\[
\epsilon(N) = (\log N \cdot \log \log N)^{-1/2},
\]

we obtain for the constants \( c_1 \) and \( c_2 \) the estimates

\[
c_1 \geq 0.021,784, \ldots, \quad c_2 \geq 0.002,709, \ldots.
\]

Here we let \( E \) range over the rank-two curves in [8] and take the maximal values for \( c_1 \) and \( c_2 \), which are attained at the curves with \( A = 246 \) and \( A = 30 \), respectively. For the sake of completeness, we include here the numerical estimates of the constants \( c_1 \) and \( c_2 \) for all values of \( A \) in Selmer's table [8] in order to show how \( c_1 \) and \( c_2 \) oscillate as \( A \) varies (see Table 3).

---

\[\text{This estimation is based on the assumption that the points in Selmer's table [8] are of minimal height. We wish to thank M. Reichert for verifying this on a Siemens PC MX-2 for Selmer's curves with } A = 30, 37, 65, 91, 110, 124, 126, 163, 182, 217, 254, 342, 468 \text{ and } 469. \text{ Only for } A = 254, \text{ the point } P_1 + P_2 \text{ is to be taken instead of } P_2 \text{ since it has a slightly smaller height value.}\]
### Table 3

| 19 | 0.00513126 | 0.00110389 |
| 30 | 0.012703 | 0.0020903 |
| 37 | 0.00483937 | 0.000962382 |
| 65 | 0.00345923 | 0.00182501 |
| 86 | 0.00629273 | 0.00227975 |
| 91 | 0.0036522 | 0.000453137 |
| 110 | 0.012277 | 0.00153855 |
| 124 | 0.00374265 | 0.00174993 |
| 126 | 0.00433063 | 0.00114267 |
| 127 | 0.00414683 | 0.000940948 |
| 132 | 0.0183099 | 0.00244798 |
| 153 | 0.0105335 | 0.0011188 |
| 163 | 0.00458141 | 0.000483074 |
| 182 | 0.00445021 | 0.000517017 |
| 183 | 0.00524747 | 0.00199503 |
| 201 | 0.00511671 | 0.00120915 |
| 203 | 0.0095474 | 0.0016455 |
| 209 | 0.00721788 | 0.000812224 |
| 210 | 0.012048 | 0.00121204 |
| 217 | 0.00308153 | 0.000319989 |
| 218 | 0.00327199 | 0.00201074 |
| 219 | 0.00500126 | 0.00171349 |
| 246 | 0.0217843 | 0.00193905 |
| 254 | 0.00531365 | 0.000936285 |
| 271 | 0.00370666 | 0.000782067 |
| 273 | 0.00472123 | 0.000663038 |
| 282 | 0.0182864 | 0.00183782 |
| 309 | 0.00457362 | 0.00272742 |
| 335 | 0.00274443 | 0.00165495 |
| 342 | 0.00352578 | 0.00103604 |
| 345 | 0.0142214 | 0.00141568 |
| 348 | 0.0175728 | 0.0019122 |
| 370 | 0.00282848 | 0.00117946 |
| 379 | 0.0038673 | 0.000711223 |
| 390 | 0.0111818 | 0.000859068 |
| 397 | 0.00346161 | 0.00056949 |
| 399 | 0.0042891 | 0.000624869 |
| 407 | 0.00267502 | 0.00118686 |
| 420 | 0.0121164 | 0.00107526 |
| 433 | 0.00518512 | 0.000529235 |
| 435 | 0.0139496 | 0.00147541 |
| 436 | 0.00482868 | 0.00160066 |
| 446 | 0.0041804 | 0.00145041 |
| 450 | 0.00415706 | 0.00168546 |
| 462 | 0.0108269 | 0.000877993 |
| 468 | 0.00388973 | 0.00098101 |
| 469 | 0.00339982 | 0.000304454 |
| 477 | 0.00744997 | 0.00111861 |
| 497 | 0.00513293 | 0.000907821 |
| 498 | 0.0171181 | 0.00149825 |

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