The Stability in $L_p$ and $W^1_p$ of the $L_2$-Projection onto Finite Element Function Spaces

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Abstract. The stability of the $L_2$-projection onto some standard finite element spaces $V_h$, considered as a map in $L_p$ and $W^1_p$, $1 \leq p \leq \infty$, is shown under weaker regularity requirements than quasi-uniformity of the triangulations underlying the definitions of the $V_h$.

0. Introduction. The purpose of this paper is to show the stability in $L_p$ and $W^1_p$, for $1 \leq p \leq \infty$, of the $L_2$-projection onto some standard finite element subspaces. Special emphasis is placed on requiring less than quasi-uniformity of the triangulations entering in the definitions of the subspaces.

In the one-dimensional case, which is discussed in Section 1 below, we first give a new proof of a result of T. Dupont (cf. de Boor [2]) showing $L_\infty$ stability without any restriction on the defining partitions, thus extending an earlier result by Douglas, Dupont and Wahlbin [6] for the quasi-uniform case. We then use the technique developed to show the stability in $W^1_p$, in the case $p > 1$, under a quite weak assumption on the partition, depending on $p$. We also show that some restriction on the partition is needed for stability if $p > 1$. We remark that the known $L_p$ stability result has been extended to higher degrees of regularity of the subspaces; see de Boor [3] and references therein.

In the case of a two-dimensional polygonal domain, discussed in Section 2, we demonstrate $L_p$ and $W^1_p$ stability results for the $L_2$-projection onto standard piecewise polynomial spaces of Lagrangian type. The requirements on the triangulations involved are more severe than in the one-dimensional case, but allow nevertheless a considerable degree of nonuniformity. The proofs are based on a technique used by Descloux [5] to show $L_\infty$ stability in the quasi-uniform case (cf. also Douglas, Dupont and Wahlbin [7]).

Results such as the above are of interest, for instance, in the analysis of Galerkin finite element methods for parabolic problems. Thus Bernardi and Raugel [1] use the $W^1_2$ stability of the $L_2$-projection to prove quasi-optimality of the Galerkin solution with respect to the energy norm, and Schatz, Thomée and Wahlbin [8] apply the $L_\infty$ stability in a similar way (in the quasi-uniform case).

1. The One-Dimensional Case. In this section we shall study the orthogonal projection $\pi = \pi_h$ with respect to $L_2(0,1)$ onto the subspace

$$V_h = \{ x \in C(0,1); x|_{I_j} \in P_k, \; j = 0, \ldots, N; \; x(0) = x(1) = 0 \},$$
where $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$ is a partition of $[0,1]$ and $I_j = (x_j, x_{j+1})$.

We shall first demonstrate the following result, in which $\| \cdot \|_p$ denotes the norm in $L_p(0,1)$.

**Theorem 1.** There is a constant $C$ depending only on $k$ such that

$$\|\pi_2 u\|_p \leq C \|u\|_p \quad \forall \ u \in L_p(0,1), \ 1 \leq p \leq \infty.$$ 

We shall then turn to estimates in $W^{1,p}(0,1) = \{ \psi \in L^p(0,1) ; \psi' = d\psi/dx \in L^p(0,1) ; \psi(0) = \psi(1) = 0 \}$ and show, with $h_i = x_{i+1} - x_i$,

**Theorem 2.** Let $1 \leq p \leq \infty$ and assume, for $p > 1$, that the partition is such that $h_i/h_j \leq C_0 \alpha^{\frac{i-j}{k}}$, where $1 \leq \alpha < (k+1)^{p/(p-1)}$. Then

$$\left\| (\pi_2 u)' \right\|_p \leq C \|u'\|_p \quad \forall \ u \in \dot{W}^{1,p}_p(0,1),$$

where $C$ depends on $k$, and for $p > 1$ also on $C_0$, $\alpha$, and $p$.

For the proofs of these results we introduce the spaces

$$V^2_h = \{ \chi \in V_h ; \chi(x_i) = 0, i = 1, \ldots, N \}$$

and $V^1_h$, the orthogonal complement of $V^2_h$ in $V_h$ with respect to the usual inner product in $L^2(0,1)$. For $k = 1$ we have $V^2_h = \{0\}$ and $V^1_h = V_h$. We also introduce the orthogonal projections $\pi_j$ onto $V^j_h$, $j = 1, 2$, and obtain at once

(1.1) \( \pi = \pi_1 + \pi_2 \) \quad (\pi = \pi_1 \text{ for } k = 1). \]

We note that $\pi_2$ is determined locally on each $I_j$ by the equations

(1.2) \( (\pi_2 u, q)_{I_j} = (u, q)_{I_j} \) \quad for \( q \in P^0_k(I_j) = \{ q \in P_k ; q(x_j) = q(x_{j+1}) = 0 \}, \)

where $\langle \cdot, \cdot \rangle_{I_j}$ is the standard inner product in $L^2(I_j)$, and that a function in $V^1_h$ is completely determined by its values at the interior nodes, so that $\dim V^1_h = N$.

For $u \in C[0,1]$ with $u(0) = u(1) = 0$ we shall also use the piecewise linear interpolant $r_h u \in V_h$ and note that, for $1 \leq p \leq \infty$,

(1.3) \( \left\| (r_h u)' \right\|_p \leq \|u'\|_p, \)

and, denoting the norm in $L^p(I_i)$ by $\| \cdot \|_{p,I_i}$

(1.4) \( \|u - r_h u\|_{p,I_i} \leq \frac12 h_i \|u'\|_p. \)

**Lemma 1.** There is a constant $C$ depending only on $k$ such that, for $1 \leq p \leq \infty$,

(1.5) \( \|\pi_2 u\|_p \leq C \|u\|_p, \quad u \in L_p(0,1), \)

and

(1.6) \( \left\| (\pi_2 (u - r_h u))' \right\|_p \leq C \|u'\|_p, \quad u \in \dot{W}^{1,p}_p(0,1). \)

**Proof.** We consider first (1.5) for $p = 1$ and set $\tilde{u}_h = \pi_2 u$. It follows, by taking $q = \tilde{u}_h$ in (1.2), that

\[
\| \tilde{u}_h \|^2_{2,I_i} \leq \|u\|_{1,I_i} \|\tilde{u}_h\|_{\infty,I_i}.
\]
Hence \( \|u_k\|_{1,I} \leq C_1\|u\|_{1,I} \), where
\[
C_1 = \max_{q \in P_1^k(I)} \frac{\|q\|_{1,I}\|q\|_{\infty,I}}{\|q\|_{2,I}^2}.
\]
Using the change of variables \( y = (x - x_i)/h_i \), it is easily seen that \( C_1 \) is independent of the interval \( I_i \) and thus depends only on \( k \). Analogously, we obtain
\[
(1.7) \quad \|\pi_2 u\|_{p,I} \leq C_1\|u\|_{p,I},
\]
for \( p = \infty \), and then for general \( p \) by the Riesz-Thorin theorem [9]. The desired result now follows by taking \( p \)th powers and summing.

To prove (1.6), we note that
\[
\|\pi_2(u - r_h u')\|_{p,I} \leq C_2\|u - r_h u\|_{p,I}, \quad \text{where} \quad C_2 = \max_{q \in P_2^k(0,1)} \frac{\|q\|_p}{\|q\|_p},
\]
and, by (1.7) and (1.4),
\[
\|\pi_2(u - r_h u)\|_{p,I} \leq C_1\|u - r_h u\|_{p,I} \leq \frac{1}{2} C_1 h_i\|u'\|_{p,I},
\]
from which (1.6) follows with \( C = \frac{1}{2} C_1 C_2 \).

In order to study the projection \( \pi_1 \), we shall construct a basis for \( V_h^1 \). For this purpose let us define \( \psi \in P_k^0 \) by
\[
\psi(0) = 0, \quad \psi(1) = 1, \quad (\psi, q) = \int_0^1 \psi q \, dx = 0 \quad \forall q \in P_k^0.
\]

For each nodal point \( x_i \), we associate the function \( \psi_i \) defined by
\[
\psi_i(x) = \psi \left( \frac{x - x_{i-1}}{h_{i-1}} \right) \quad \text{on} \quad I_{i-1},
\]
\[
= \psi \left( \frac{x_{i+1} - x}{h_i} \right) \quad \text{on} \quad I_i,
\]
\[
= 0 \quad \text{on} \quad \mathcal{O} \left( I_{i-1} \cup I_i \right).
\]

It is then easily seen that \( \{\psi_i\}_{i=1}^N \subset V_h^1 \) and that these functions thus form a basis.

For \( u \) given, and \( w = \pi_1 u = \sum_{i=1}^N w_i \psi_i \), we then have
\[
\sum_{i=1}^N w_i (\psi_i, \psi_j) = (u, \psi_j) = u_j, \quad j = 1, \ldots, N,
\]
or in matrix form, with \( G = ((\psi_i, \psi_j)) \), \( W = (w_1, \ldots, w_N)^T \) and \( U = (u_1, \ldots, u_N)^T \),
\[
(1.8) \quad GW = U.
\]

We note that the Gram matrix \( G \) is tridiagonal. We shall need to compute its nonzero elements.

**Lemma 2.** We have
\[
\|\psi_i\|^2 = \frac{1}{k(k + 2)} (h_{i-1} + h_i)
\]
and
\[
(\psi_i, \psi_{i+1}) = \frac{(-1)^{k-1}}{k(k + 1)(k + 2)} h_i.
\]
Proof. By transformation of variables it suffices to show that
\[ \int_0^1 \psi(x)^2 \, dx = \frac{1}{k(k + 2)} \]
and
\[ \int_0^1 \psi(x)\psi(1 - x) \, dx = \frac{(-1)^{k-1}}{k(k + 1)(k + 2)}. \]
The definition of \( \psi \) implies easily
\[ \psi(x) = \frac{(-1)^{k-1}}{k!} \frac{1}{x(1 - x)} \frac{d^{k-1}}{dx^{k-1}} [x^{k+1}(1 - x)^k]. \]
Further, since \( \psi(x) - x \) and \( \psi(1 - x) - (1 - x) \in P_k^0 \), we find
\[ \int_0^1 \psi(x)(\psi(x) - x) \, dx = \int_0^1 \psi(x)(\psi(1 - x) - (1 - x)) \, dx = 0. \]
Hence, integrating by parts \( k - 1 \) times, we have
\[ \int_0^1 \psi(x)^2 \, dx = \frac{(-1)^{k-1}}{k!} \int_0^1 \frac{1}{1 - x} \frac{d^{k-1}}{dx^{k-1}} [x^{k+1}(1 - x)^k] \, dx \]
\[ = \frac{1}{k!} \int_0^1 x^{k+1}(1 - x)^k \frac{d^{k-1}}{dx^{k-1}} \frac{1}{1 - x} \, dx \]
\[ = \frac{1}{k} \int_0^1 x^{k+1} \, dx = \frac{1}{k(k + 2)} \]
and
\[ \int_0^1 \psi(x)\psi(1 - x) \, dx = \frac{(-1)^{k-1}}{k} \int_0^1 x^{k+1}(1 - x)^k \frac{d^{k-1}}{dx^{k-1}} \frac{1}{x} \, dx \]
\[ = \frac{(-1)^{k-1}}{k} \int_0^1 x(1 - x)^k \, dx = \frac{(-1)^{k-1}}{k(k + 1)(k + 2)}, \]
which completes the proof.

Let us introduce the diagonal matrix \( D \) with the same diagonal elements as \( G \), i.e.,
\[ d_i = \|\psi_i\|^2 = \frac{1}{k(k + 2)} (h_{i-1} + h_i). \]
We may then write \( G \) in the form \( G = D(I + K) \), where \( K \) is a tridiagonal matrix with diagonal elements zero and bidiagonal entries
\[ k_{i,i-1} = \frac{\langle \psi_i, \psi_{i+1} \rangle}{\|\psi_i\|^2} = \frac{(-1)^{k-1}}{k + 1} \frac{h_{i-1}}{h_{i-1} + h_i}, \]
\[ (1.9) \]
\[ k_{i,i+1} = \frac{(-1)^{k-1}}{k + 1} \frac{h_i}{h_{i-1} + h_i}. \]
The equation (1.8) now takes the form
\[ (I + K)W = D^{-1}U. \]
We are now ready to prove Theorem 1. By Lemma 1 it remains only to prove
\[(1.11) \| \pi_1 u \|_p \leq C \| u \|_p, \quad u \in L_p(0,1),\]
and we begin by showing this for \( p = \infty \). This will be done by showing (here and below we denote by \(| \cdot |_p\) the standard \( l_p \)-norms for \( N \)-vectors)
\[(1.12) \| \pi_1 u \|_\infty \leq C |W|_\infty,\]
then
\[(1.13) |W|_\infty \leq C |D^{-1} U|_\infty,\]
and finally
\[|D^{-1} U|_\infty \leq C \| u \|_\infty.\]

To see that (1.12) holds, we note that, since for no \( x \) in \( (0,1) \) more than two \( \psi_i(x) \) are nonzero, we have
\[|w_\psi(x)| = \max_{i=1}^N |w_i(x)| \leq 2 \| \psi \|_\infty |W|_\infty.\]

In view of (1.10), in order to show (1.13), we only need to show that \((I + K)^{-1}\) is bounded in \( l_\infty \). But this follows at once from the fact that, by (1.9),
\[|K|_\infty = \max_{i,j} |k_{ij}| = \frac{1}{k + 1} < 1,\]
and hence
\[|{(I + K)^{-1}}^{-1}|_\infty \leq \frac{1}{1 - 1/(k + 1)} = \frac{k + 1}{k}.
\]

Finally,
\[|D^{-1} U|_\infty = \max_j \frac{|(u, \psi_j)|}{\| \psi_j \|^2} \leq C_1 \| u \|_\infty,\]
where
\[C_1 = \max_j \frac{\| \psi_j \|_1}{\| \psi_j \|_2} = \frac{\| \psi \|_1}{\| \psi \|_2},\]
where the latter equation follows by transformation of the subintervals onto \([0,1]\).

This completes the proof of (1.11) for \( p = \infty \). For \( p = 1 \) the result follows at once by duality and for \( 1 < p < \infty \) by the Riesz-Thorin theorem. The proof of Theorem 1 is now complete.

We now turn to the proof of Theorem 2. We may write
\[\pi u = \pi_1(u - r_h u) + \pi_2(u - r_h u) + r_h u.\]
In view of Lemma 1 and (1.3) the last two terms are bounded, as desired, and it remains to consider \( w = \pi_1\varepsilon \) where \( \varepsilon = u - r_h u \). Letting \( W = (w_1, \ldots, w_N)^T \) where \( w_i = w(x_i) \), and \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)^T \) where \( \varepsilon_i = (\varepsilon, \psi_i) \), we find that \( W \) solves (1.8) with \( U \) replaced by \( \varepsilon \). We shall show, with \( D \) the diagonal matrix introduced above
and \( p' = p/(p - 1) \),
\[
\|w'\|_p \leq C|D^{-1/p'}W|_p,
\]
then
\[
|D^{-1/p'}W|_p \leq C|D^{-1-1/p'}\epsilon|_p,
\]
and finally
\[
|D^{-1-1/p'}\epsilon|_p \leq C\|u'\|_p,
\]
which together complete the proof.

We have first
\[
\|w'\|_p^p = \sum_{i=0}^N \left| \sum_{j=0}^i w_j \psi_j + w_{i+1} \psi_{i+1} \right|^p dx
\]
\[
\leq 2^{p/p'} \sum_{i=1}^N |w_i|^{p} \left( h_{i-1}^{-p+1} + h_i^{-p+1} \right) \|\psi_i\|_p^p
\]
\[
\leq C \sum_{i=1}^N d_i^{-p+1} |w_i|_p^p = C|D^{-1/p'}W|_p^p,
\]
where we have used
\[
d_i^{p-1} \leq C(h_{i-1} + h_i)^{p-1} \leq C(h_{i-1}^{-p+1} + h_i^{-p+1})^{-1}.
\]
The proof of (1.15) is also straightforward. We have, by Hölder’s inequality,
\[
|\epsilon_i| = |(\epsilon, \psi_i)| \leq \|\epsilon\|_p, |\psi_i\|_p, \|\psi_{i-1}\|_p, \|\psi_i\|_p, \|\psi_{i+1}\|_p
\]
\[
\leq C \left( h_i^{1/p'} \|\epsilon\|_p, h_i^{1/p'} \|\psi_{i-1}\|_p, h_i^{1/p'} \|\psi_i\|_p, h_i^{1/p'} \|\psi_{i+1}\|_p \right),
\]
and hence by (1.4),
\[
|\epsilon_i| \leq C \left( h_i^{1/p'} \|u'\|_p, h_i^{1/p'} \|u'\|_p \right)
\]
\[
\leq C d_i^{1/p'} \|u'\|_p, \|u'\|_p,
\]
whence (1.15) follows immediately.

It remains to show (1.14). Recalling that \( W \) satisfies (1.8), and hence (1.10), with \( U \) replaced by \( \epsilon \), we have
\[
\left( D^{-1/p'}(I + K)D^{1/p'} \right)D^{-1/p'}W = D^{-1-1/p'}\epsilon,
\]
and it thus suffices to show that \( I + D^{-1/p'}KD^{1/p'} \) has a bounded inverse in \( l_p \) under the assumptions of the theorem. For this purpose we estimate the powers of the second term. Since \( K' \) is \((2l + 1)\)-diagonal and has nonnegative elements, we have
\[
|D^{-1/p'}K'D^{1/p'}|_p \leq \max_{|i-j| \leq 2l} \left( d_i/d_j \right)^{1/p'} |K'|_p.
\]
Here,
\[
d_i/d_j = (h_{i-1} + h_i)/(h_{j-1} + h_j) \leq C_0^2 a^{2l+1} \quad \text{for } |i - j| \leq 2l.
\]
Further, again since \( K' \) is \((2l + 1)\)-diagonal, we have
\[
|K'|_1 \leq (2l + 1)|K'|_\infty \leq (2l + 1)|K'|_\infty \leq \frac{2l + 1}{(k + 1)^l},
\]
and, using once more the Riesz-Thorin theorem,
\[ |K'|_p \leq (2l + 1)^{1/p} \frac{1}{(k + 1)^l} \quad \text{for } 1 \leq p \leq \infty. \]

Altogether we find, under the assumptions made,
\[ \left| \left( I + D^{-1/p'}K^lD^{1/p'} \right)^{-1} \right|_p \leq 1 + \sum_{l=1}^{\infty} |D^{-1/p'}K^lD^{1/p'}|_p \]
\[ \leq 1 + \left( C_0^2 \alpha \right)^{1/p'} \sum_{l=1}^{\infty} (2l + 1)^{1/p} \left( \frac{\alpha^{2/p'}}{k + 1} \right)^l < \infty, \]

which completes the proof.

We conclude by remarking that in Theorem 1 and in the case \( p = 1 \) of Theorem 2 no restriction is made concerning the partitions used, and that quite strong mesh refinements are permitted for \( p > 1 \) in Theorem 2. The following example shows, however, that some restriction is needed in the latter case: Consider the partition with only one interior point \( x_1 = 1 - \epsilon \), so that \( h_0/h_1 = (1 - \epsilon)/\epsilon \). Let \( k = 1 \) and \( u(x) = x(1 - x) \). Then \( su = \beta \psi_1 \), where \( \beta \) is determined by the equation \( \beta \| \psi_1 \|^2 = (u, \psi_1) \), or, after an easy calculation, \( \beta = \frac{1}{4}(1 + \epsilon(1 - \epsilon)) \). In this case,
\[ \| (su)' \|_p = \beta \left\{ \int_0^{1 - \epsilon} \epsilon (1 - \epsilon)^{-p} \, dx + \int_1^{1 - \epsilon} \epsilon^{1-p} \, dx \right\}^{1/p} \geq \frac{1}{4} \epsilon^{-1/p'}, \]
which tends to \( \infty \) with \( 1/\epsilon \) if \( p > 1 \).

2. The Two-Dimensional Case. In this section we shall consider the orthogonal projection onto a finite element subspace of \( L^2(\Omega) \) where \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \). For simplicity we assume that \( \Omega \) is polygonal and consider a family of triangulations \( \mathcal{T}_h \) of \( \Omega \) into closed triangles \( K \) with disjoint interiors such that no vertex of any triangle lies on the interior of an edge of another triangle. We shall use the approximating spaces
\[ V_h = \{ v \in C(\Omega) ; v|_K \in P_k, v|_{\partial \Omega} = 0 \}. \]

In order to express our assumptions concerning the partition of \( \Omega \), we shall introduce some notation. For a given \( K_0 \) we let \( R_j(K_0) \) be the set of triangles which are "\( j \) triangles away from \( K_0 \)". Defined by setting \( R_0(K_0) = K_0 \) and then, recursively, for \( j \geq 1 \), \( R_j(K_0) \) the union of the closed triangles in \( \mathcal{T}_h \) which are not in \( \bigcup_{i < j} R_i(K_0) \), but which have at least one vertex in \( R_{j-1}(K_0) \). Thus \( R_j(K_0) \) is the union of the triangles which may be reached by a connected path \( Q_1, \ldots, Q_j \) with \( Q_1 \) a vertex of \( K_0 \), \( Q_j \) a vertex of \( K \) and \( Q_iQ_{i+1} \) an edge of the triangulation for \( 1 \leq i < j \), and not by any shorter such path. Setting \( l(K_0, K) = j \) for \( K \in R_j(K_0) \) it follows, in particular, that \( l(K_0, K) \) is symmetric in \( K \) and \( K_0 \). We also define \( n_j(K_0) \) to be the number of triangles in \( R_j(K_0) \).

Letting \( a_K \) denote the area of \( K \), we shall assume below that, with some positive constants \( C_1, C_2, \alpha, \beta, r \) with \( \alpha \geq 1, \beta \geq 1 \), we have uniformly for small \( h \),
\[ a_K/a_{K_0} \leq C_1 \alpha^{l(K, K_0)} \quad \forall K, K_0 \in \mathcal{T}_h, \]
\[ (2.1) \]
and
\[ n_j(K) \leq C_2 j^r \beta^j \quad \forall K \in \mathcal{T}_h, j \geq 1. \]
\[ (2.2) \]
When all triangles have angles bounded below, independently of \( h \), then \( a_K \) is bounded above and below by \( c h_K^2 \), where \( h_K \) is the diameter of \( K \). The case when the triangulations are quasi-uniform then corresponds to \( \alpha = 1 \). Note that by (2.1) we have

\[
\text{area}(R_j(K_0)) \geq c n_j(K_0) a_{K_0}^{-j},
\]

and, if the angles are bounded below,

\[
\text{area}(R_j(K_0)) \leq \text{area}\left( \bigcup_{i<j} R_i(K_0) \right) \leq C \left( \sum_{i=0}^{j} h_{K_0}^{i/2} \right)^2,
\]

whence

\[
n_j(K_0) \leq Cj^2 \quad \text{if } \alpha = 1,
\]

\[
\leq C\alpha^{-j} \quad \text{if } \alpha > 1.
\]

In particular, if the angles are bounded below, (2.1) with \( \alpha > 1 \) implies (2.2) with \( r = 0, \beta = \alpha^2 \). However, in practice this is a very crude estimate. In fact, for any triangulation which is a deformation of a quasi-uniform one, (2.2) holds with \( \beta = 1, \ r = 2 \).

The results of this section are based on the following variant of a lemma by Descloux [5] concerning the orthogonal projection \( \pi \) in \( L_2(\Omega) \) onto \( V_h \).

**Lemma 3.** Let \( 1 \leq p \leq \infty \). There are positive constants \( \gamma < 1 \) and \( C \) such that, if \( \text{supp} \, \nu_0 \subset K_0 \),

\[
(2.3) \quad \| \pi \nu_0 \|_{2, K} \leq C \gamma^{(k, K_0)} a_{K_0}^{1/2 - 1/p} \| \nu_0 \|_p \quad \forall K, \ K_0 \in \mathcal{T}_h,
\]

where \( \gamma \) depends only on \( k \) and \( C \) only on \( k \) and \( p \).

**Proof.** Letting \( D_j = \bigcup_{j \leq j} R_j(K_0) \) denote the union of triangles which may only be reached by paths of length at least \( j \), we shall want to show that for some \( \kappa > 0 \),

\[
(2.4) \quad \| \pi \nu_0 \|^2_{2, D_j} \leq \kappa \| \pi \nu_0 \|^2_{2, R_j} \quad \text{for } j \geq 1.
\]

Assuming this for a moment, we denote the left side by \( q_j \) and thus find

\[
q_j \leq \kappa (q_{j-1} - q_j) \quad \text{for } j \geq 1,
\]

whence

\[
q_j \leq \frac{\kappa}{1 + \kappa} q_{j-1} \leq \left( \frac{\kappa}{1 + \kappa} \right)^j q_0 \leq \gamma^j \| \pi \nu_0 \|_2^2,
\]

where \( \gamma = (\kappa/(1 + \kappa))^{1/2} \). Here, since \( \text{supp} \, \nu_0 \subset K_0 \), we find, with \( (\cdot, \cdot)_R \) the standard inner product in \( L_2(R) \) with \( R \) omitted for \( R = \Omega \), and \( p' \) the conjugate exponent \( p' = p/(p - 1) \),

\[
\| \pi \nu_0 \|_2 = \max_{\chi \in S_k} \frac{(\nu_0, \chi)}{\| \chi \|_2} \leq \max_{q \in P_{q_0}} \frac{(\nu_0, q)_{K_0} \| q \|_{2, K_0}^2}{\| q \|_{2, K_0}^2} \leq \| \nu_0 \|_{p, K_0} \max_{q \in P_{q_0}} \| q \|_{p', K_0}^2,
\]

and hence by the standard transformation to a reference triangle, with \( \delta \) depending on \( p \) and \( k \),

\[
\| \pi \nu_0 \|_2 \leq \delta \alpha_{K_0}^{1/p' - 1/2} \| \nu_0 \|_{p, K_0}.
\]
Altogether, we have, if \( j = l(K, K_0) \),
\[
\| \pi v_0 \|_{2,K} \leq \| \pi v_0 \|_{2,R_j} \leq q^{j/2}_1 \leq \delta \gamma^j a_{K_0}^{1/2-1/p} \| v_0 \|_{p,K},
\]
which is the desired result with \( C = \delta / \gamma \).

It remains to show (2.4). Since \( \text{supp} \, v_0 \subset K_0 \), we have
\[
(\pi v_0, \chi) = 0 \quad \text{for} \quad \chi \in V_h, \, \text{supp} \, \chi \subset D_{j-1} = D_j \cup R_j, \, \text{if} \, j \geq 1.
\]

Let \( \omega = \pi v_0 \) and define for any \( \omega \in S_h \) a new function \( \tilde{\omega}_j \) in \( S_h \) by setting \( \tilde{\omega}_j = \omega \) on \( D_j \) and \( \tilde{\omega}_j = 0 \) on \( \Sigma_{j-1} = \bigcup_{K \in S_h, \, K \cap D_j = \emptyset} K \), the union of triangles, all vertices of which may be reached from \( K_0 \) by paths of length at most \( j - 1 \). To define \( \tilde{\omega}_j \) on the remaining triangles \( K \), which are then included in \( R_j(k_0) \) but not in \( \Sigma_{j-1} \), we introduce for such a \( K \) the Lagrangian nodes (having barycentric coordinates \((i_1/k, i_2/k, i_3/k)\) with \( i_1, i_2 \) and \( i_3 \) nonnegative integers) and set \( \tilde{\omega}_j = \omega \) at all such nodes which do not belong to \( \Sigma_{j-1} \) or to an edge joining two vertices in \( \Sigma_{j-1} \), and \( \tilde{\omega}_j = 0 \) at the other nodes. With \( \chi = \tilde{\omega}_j \), (2.5) takes the form
\[
(\omega, \tilde{\omega}_j) = \| \omega \|^2_{2,D_j} + (\omega, \tilde{\omega}_j)_{R_j} = 0,
\]
whence
\[
(\omega, \tilde{\omega}_j) = \| \omega \|^2_{2,D_j} \leq (\omega, \tilde{\omega}_j)_{R_j}.
\]

In order to estimate the latter quantity, we consider again a triangle \( K \subset R_j \) with \( K \) not included in \( \Sigma_{j-1} \) and note that \( K \) has either one or two vertices in \( \Sigma_{j-1} \) and the remaining vertices in \( D_j \). For \( q \in P_k \) we let \( \tilde{q}_K \) be the polynomial in \( P_k \) which vanishes at the nodal points that are in \( \Sigma_{j-1} \) or on an edge joining two vertices in \( \Sigma_{j-1} \) and agrees with \( q \) at the other Lagrangian nodes. We thus have
\[
-(\omega, \tilde{\omega}_j)_K \leq \| \omega \|^2_{2,K} \max_{q \in P_k} \frac{-(q, \tilde{q}_K)_K}{\|q\|^2_{2,K}}.
\]

By transformation to a reference triangle we find that the latter maximum is independent of \( K \) in the two possible cases for the location of its vertices, so that, after summation,
\[
-(\omega, \tilde{\omega}_j)_{R_j} = - \sum_{K \subset R_j} (\omega, \tilde{\omega}_j)_K \leq \kappa \| \omega \|^2_{2,R_j},
\]
Together with (2.6), this completes the proof of (2.4) and hence of the lemma.

The constant \( \kappa \) may thus be expressed in terms of the reference triangle \( \tilde{K} \) with vertices \( Q_1, Q_2 \) and \( Q_3 \) as
\[
\kappa = \max_{j=1,2} \max_{q \in P_k} \frac{-(q, \tilde{q}_{K,j})_{\tilde{K}}}{\|q\|^2_{2,\tilde{K}}},
\]
where \( \tilde{q}_{K,1} = 0 \) at \( Q_1 \), \( \tilde{q}_{K,1} = q \) at the other nodes and \( \tilde{q}_{K,2} = 0 \) at the vertices of \( (Q_1Q_2, 2) \) and \( = q \) at the other vertices.

We are now ready for our stability estimate for \( \pi \) in \( L_p(\Omega) \). Here and below, \( \alpha, \beta \) and \( \gamma \) are the parameters in (2.1), (2.2) and (2.3).
Theorem 3. Let $1 < p < \infty$ and assume that the numbers $\alpha$, $\beta$ and $\gamma$ are such that

\begin{equation}
\gamma \beta \alpha^{1/2-1/p} < 1.
\end{equation}

Then

$$
\|\pi u\| \leq C\|u\|_p \quad \forall u \in L_p(\Omega),
$$

where $C$ depends only on $C_1$, $C_2$, $\alpha$, $\beta$, $r$, $k$ and $p$.

Proof. We have in the usual way, for each $K \in \mathcal{T}_h$,

\begin{equation}
\|\pi u\|_{p,K} \leq C a_K^{-1/2+1/p} \|\pi u\|_{2,K}.
\end{equation}

Here, writing $u = \sum_{K \in \mathcal{T}_h} u \mid_{K}$, and using Lemma 3, we find

$$
\|\pi u\|_{2,K} \leq \sum_{K \in \mathcal{T}_h} \|\pi (u \mid_{K'})\|_{2,K} \leq C \sum_{K \in \mathcal{T}_h} \gamma^{(K,K')}(a_K^{-1/2-1/p}) \|u\|_{p,K'},
$$

so that, using also (2.8) and (2.1),

$$
\|\pi u\|_{p,K} \leq C \sum_{K \in \mathcal{T}_h} \gamma^{(K,K')}(a_K^{-1/2-1/p}) \|u\|_{p,K'}.
$$

Introducing the vectors $X = \{x_K = \|\pi u\|_{p,K}; K \in \mathcal{T}_h\}$ and $Y = \{y_K = \|u\|_{p,K}; K \in \mathcal{T}_h\}$ and the symmetric matrix $M = (m_{K,K'})$ with $m_{K,K'} = \delta^{(K,K')}$, where $\delta = \gamma a^{1/2-1/p}$, we conclude for the corresponding $l_p$-vector and associate matrix norms $|\cdot|_p$

$$
||\pi u||_p = |X|_p \leq |M||Y|_p = |M||u|_p.
$$

It remains to bound the matrix norm $|M|_p$. We have by the Riesz-Thorin theorem and the symmetry of $M$,

$$
|M|_p \leq |M|_1^{1/p} |M|_\infty^{1-1/p} = |M|_\infty = \max_K \sum_{K'} \delta^{(K,K')}.
$$

Using now also the hypothesis (2.2) we find

$$
|M|_p \leq \max_K \sum_{j=0}^\infty n_j(K) \delta^j \leq C \sum_{j=0}^\infty j^r(\beta \delta)^j,
$$

where the latter sum is finite under assumption (2.7). This completes the proof.

We now show a stability estimate for the gradient of the $L^2$-projection.

Theorem 4. Let $1 < p < \infty$ and assume that the angles of $\mathcal{T}_h$ are bounded below, uniformly in $h$, and that $\alpha$, $\beta$, and $\gamma$ are such that

\begin{equation}
\gamma \beta \alpha^{1-1/p} < 1.
\end{equation}

Then

$$
\|\nabla \pi u\|_p \leq C\|\nabla u\|_p \quad \text{for } u \in W^1_p(\Omega).
$$

Proof. There exists a linear operator $r_h: \dot{W}^1_p(\Omega) \to V_h$ such that for $u \in \dot{W}^1_p(\Omega)$,

\begin{equation}
\|\nabla r_h u\|_p \leq C\|\nabla u\|_p
\end{equation}
and

\[(2.11) \quad \| u - r_h u \|_{p, K} \leq C h_K \| \nabla u \|_{p, K} \leq C a_K^{1/2} \| \nabla u \|_{p, K}. \]

For \( p > 2 \), \( u \in H^p_0(\Omega) \) implies \( u \in C(\bar{\Omega}) \), and \( r_h u \) may be chosen as an interpolant of \( u \) and \( K \) as \( K \), whereas for \( p \geq 2 \) a preliminary local regularization as in Clément [4] is needed and \( K \) may be chosen as \( K \cup R_1(K) \).

We may write

\[ \nabla \pi u = \nabla \pi \varepsilon + \nabla r_h u, \quad \text{where} \quad \varepsilon = u - r_h u, \]

and, in view of (2.10), it suffices to estimate \( \nabla \pi u \). We have the inverse estimate

\[ \| \nabla \pi \varepsilon \|_{p, K} \leq C a_K^{-1 + 1/p} \| \pi \varepsilon \|_{2, K}, \]

and, as in the proof of Theorem 3,

\[ \| \pi \varepsilon \|_{2, K} \leq C \sum_{K' \in \mathcal{T}_h} \gamma^{(K, K')} a_{K'}^{1/2 - 1/p} \| \varepsilon \|_{p, K'}. \]

Hence, using also (2.1) and (2.11),

\[ \| \nabla \pi \varepsilon \|_{p, K} \leq C \sum_{K' \in \mathcal{T}_h} \gamma^{(K, K')} (a_{K'}/a_K)^{1-1/p} a_K^{-1/2} \| \varepsilon \|_{p, K'}, \]

\[ \leq C \sum_{K' \in \mathcal{T}_h} (\gamma a^{1-1/p})^{(K, K')} \| \nabla u \|_{p, K'}. \]

The proof is now completed as in Theorem 3.

It is clear that the assumptions (2.7) and (2.9) are satisfied in the quasi-uniform case. In order to see that they permit severely nonuniform triangulations, it is necessary to know that the constant \( \gamma \) is not too close to \( 1 \). For this purpose we recall that \( \gamma = (\kappa/(1 + \kappa))^{1/2} \) with \( \kappa = \kappa_k = \max(\kappa_1, \kappa_2) \), where with the notation of the proof of Lemma 3,

\[(2.12) \quad \kappa_{jk} = \max_{q \in P} \frac{-(q, \bar{q}_{K, j})_K}{\| q \|_{2, K}}, \quad j = 1, 2, k \geq 1.\]

Introducing the Lagrangian basis functions \( \{ \psi_j \}_{1}^{N_k} \) corresponding to the Lagrangian nodes \( \{ Q_j \}_{1}^{N_k} \) in \( K \), so that \( \psi_j(Q_j) = \delta_{ij} \), we have

\[ \| q \|_{2, K} = (A \xi, \xi), \quad q = \sum_{i=1}^{N_k} \xi_i \psi_i \in P_k, \]

where \( A \) is the matrix with elements \( a_{ij} = (\psi_i, \psi_j) \). Correspondingly, the quadratic form in the numerator in (2.12) may be obtained as

\[ (q, \bar{q}_{K, j}) = (B_j \xi, \xi), \quad j = 1, 2, \]

where \( B_j \) is a symmetric matrix obtained from \( A \) as follows: Let \( S \) be the set of indices \( i \) such that \( \bar{q}_{K, j} \) is forced to vanish at \( Q_i \), \( i \in S \), and let

\[ S' = \{ 1, 2, \ldots, N_k \} \setminus S. \]

Then \( \bar{q}_{K, j} = \sum_{i \in S} \xi_i \psi_j \) and hence \( (q, \bar{q}_{K}) = (B \xi, \xi) \), with \( B = (b_{ij}) \), where

\[ b_{ij} = 0 \quad \text{if} \ i, j \in S, \]

\[ = \frac{1}{2} a_{ij} \quad \text{if} \ i \in S, \ j \in S' \text{ or } j \in S, \ i \in S', \]

\[ = a_{ij} \quad \text{if} \ i, j \in S'. \]
For $i = 1$, $S = \{1\}$, and for $i = 2$, $S$ consists of the indices for which $Q_i$ are on $\overline{Q_1Q_2}$. With this notation, $\kappa_{jk}$ is the largest eigenvalue of the eigenvalue problem
\begin{equation}
- B_j \xi = \lambda A \xi.
\end{equation}

For $k = 1$ we have $N_1 = 3$ and
\begin{align*}
(A\xi, \xi) &= (\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3) a_K/6, \\
(B_1\xi, \xi) &= (\xi_1^2 + \xi_3^2 + \xi_1\xi_3 + \frac{1}{2}\xi_1\xi_2 + \frac{1}{2}\xi_1\xi_3) a_K/6, \\
(B_2\xi, \xi) &= (\xi_3^2 + \frac{1}{2}\xi_1\xi_3 + \frac{1}{2}\xi_2\xi_3) a_K/6.
\end{align*}

By completing squares we find easily that for both $j = 1$ and $2$, $\lambda = (\sqrt{6} - 2)/4$ is the smallest number such that
\[ \lambda (A\xi, \xi) + (B_j\xi, \xi) \geq 0 \quad \forall \xi \in \mathbb{R}^3. \]
Hence,
\[ \kappa_1 = \kappa_{11} = \kappa_{12} = (\sqrt{6} - 2)/4 = 0.112, \quad \gamma_1 = \sqrt{3} - \sqrt{2} = 0.318. \]

For $k = 2$ and $k = 3$ we have $N_2 = 6$ and $N_3 = 10$ nodal points, respectively. By numerical computation we have determined the largest eigenvalues of (2.13) in these cases and found
\[ \kappa_{12} = 0.048, \quad \kappa_{22} = 0.165, \quad \kappa_2 = 0.165, \quad \gamma_2 = 0.376, \]
and
\[ \kappa_{13} = 0.032, \quad \kappa_{23} = 0.142, \quad \kappa_3 = 0.142, \quad \gamma_3 = 0.353. \]