Quasi-Optimal Estimates for Finite Element Approximations Using Orlicz Norms*

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Abstract. We consider the approximation by linear finite elements of the solution of the Dirichlet problem \(-\Delta u = f\). We obtain a relation between the error in the infinite norm and the error in some Orlicz spaces. As a consequence, we get quasi-optimal uniform estimates when \(u\) has second derivatives in the Orlicz space associated with the exponential function. This estimate contains, in particular, the case where \(f\) belongs to \(L^\infty\) and the boundary of the domain is regular. We also show that optimal order estimates are valid for the error in this Orlicz space provided that \(u\) be regular enough.

1. Introduction. Consider the problem of finding \(u\) such that

\[
\begin{aligned}
-\Delta u &= f & \text{in } \Omega, \\
\quad u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

where \(\Omega\) is a bounded domain contained in \(\mathbb{R}^n\) and \(f\) is a given function.

We shall use standard notation for the Sobolev spaces \(W^{k,p}_p(\Omega)\) and \(H^k(\Omega) = W^k_2\) with the norms

\[
\|f\|_{k,p,\Omega} = \sum_{j \leq k} |f|_{j,p,\Omega},
\]

where

\[
|f|_{j,p,\Omega} = \sum_{|\alpha| = j} \|D^\alpha f\|_{L^r(\Omega)}.
\]

We shall write \(\|f\|_{k,p} = \|f\|_{k,p,\Omega}\) and \(|f|_{k,p} = |f|_{k,p,\Omega}\) when there is no confusion.

The letter \(C\) will denote a constant, not necessarily the same at each occurrence.

For simplicity we will consider \(\Omega\) to be a convex polyhedral domain, but the results are valid in more general domains as in [9].

Let \(\{T_h\}\) be a quasi-regular family of triangulations of \(\Omega\) and denote by \(u_h\) the \(H^1_0\)-projection of \(u\) into the space of piecewise linear functions \(M_h \subset H^1_0\), that is,

\[
\int_{\Omega} \nabla u_h \nabla v_h \, dx = \int_{\Omega} f v \, dx, \quad v_h \in M_h.
\]

It is well known (see [1]) that

\[
|u - u_h|_{0,2} \leq C h^2 |u|_{2,2} \quad \text{and} \quad |u - u_h|_{1,2} \leq C h |u|_{2,2}.
\]
Many authors have studied estimates for $u - u_h$ in $W^{1,p}_\Omega$-norms and $L^p$-norms. In [8] the following optimal estimate for the gradient of the error in $L^p$ is obtained,

$$|u - u_h|_{1,p} \leq C h \|u\|_{2,p} \quad \text{for } 1 < p \leq \infty.$$ 

Then by the usual duality argument (see [1]) they get

$$|u - u_h|_{0,p} \leq C h^2 \|u\|_{2,p} \quad \text{for } 2 \leq p \leq \infty,$$

provided that $\Omega$ is a convex polygonal domain or $\partial \Omega$ is smooth.

As is known, this duality argument cannot be applied for $p = \infty$.

A quasi-optimal estimate for the error in $L^\infty$ was obtained in [9], where it is proved that

$$|u - u_h|_{0,\infty} \leq C h^2 \log \frac{1}{h} \|u\|_{2,\infty}.$$

Moreover, in [4] an example is given that shows that the logarithm in this estimate cannot be removed.

We will work here with Orlicz spaces defined in the following way. Given a convex function $\phi: \mathbb{R} \to \mathbb{R}^+$, $\phi(0) = 0$, let

$$L^\phi(\Omega) = \left\{ f \ | \exists b > 0 \ | \int_\Omega \phi \left( \frac{|f|}{b} \right) \, dx < \infty \right\}.$$ 

$L^\phi$ is a Banach space with the norm

$$\|f\|_{L^\phi} = \inf \left\{ b > 0 \ | \int_\Omega \phi \left( \frac{|f(x)|}{b} \right) \, dx \leq 1 \right\}.$$

We will call $W^{k,p}_\phi$ the space of functions in $L^\phi$ with derivatives up to the order $k$ in $L^\phi$, and we will use analogous notation as in the $L^p$ case for the norms and seminorms.

When the boundary of $\Omega$ is regular and $1 < p < \infty$ [3],

$$\|u\|_{2,p} \leq C |f|_{0,p},$$

and consequently,

$$|u - u_h|_{0,p} \leq C h^2 |f|_{0,p}.$$

As is well known, the regularity result mentioned above is not true for $p = \infty$, but if $f \in L^\infty$ the solution $u \in W^{2,1}_\phi$, where $\phi_1(t) = e^t - t - 1$. Moreover, the second derivatives of $u$ are in the space of functions with bounded mean oscillation BMO (same proof as in the $L^p$ case [3], using the result of [6]) and this space is contained in $L^{\phi_1}$ when the domain is bounded, [5]. Then it is natural to seek an estimate for $|u - u_h|_{0,\infty}$ when $u$ has second derivatives in $L^{\phi_1}$.

In this paper we obtain a relation between the error in $L^\infty$ and the error in some Orlicz spaces that implies in particular the following quasi-optimal estimate,

$$|u - u_h|_{0,\infty} < C h^2 \left( \log \frac{1}{h} \right)^2 \|u\|_{2,\phi_1}.$$

This estimate contains as a particular case the following one proved in [9],

$$|u - u_h|_{0,\infty} < C h^2 \left( \log \frac{1}{h} \right)^2 |f|_{0,\infty}.$$
A similar estimate was obtained also in [7] but with a higher power of the logarithm and with the BMO norm of the second derivatives in the right-hand side.

Our result is more general because BMO is strictly contained in $L^{\phi_1}$ (for example, in $\Omega = (-1, 1)$ the function

$$f(x) = \begin{cases} \log x, & x > 0, \\ 0, & x < 0, \end{cases}$$

is in $L^{\phi_1}$ but not in BMO).

Error estimates for problems where $u$ has other kinds of singularities can be obtained by our theorem. As examples, consider $\Omega = \{x \in R^2 | |x| < 1/e\}$ and

$$u(x) = |x|^2 \left( \log \frac{1}{|x|} \right)^{1/n} - 1/e^2, \quad n \in N.$$  

In this case, $D^\alpha u \in L^\phi(\Omega)$ for $|\alpha| = 2$, where $\phi(t) = e^{t^n} - t^n - 1$, and then we will get the following estimate,

$$|u - u_h|_{0, \infty} \leq C h^2 \left( \log \frac{1}{h} \right)^{1+1/n} \|u\|_{2, \phi}.$$  

Finally, we show in the two-dimensional case that

$$|u - u_h|_{0, \phi_1} \leq C h^2 \|u\|_{2, \infty},$$

provided that $\partial \Omega$ is smooth or $\Omega$ is a Lipschitz convex domain. In this way we show that the logarithm factor can be removed if we replace the $L^\infty$-norm on the left by a slightly weaker Orlicz norm.

2. Error Estimates.

**Lemma 1.** If $v \in M_h$ the following inverse inequality holds,

$$|v|_{0, \infty} \leq C \phi^{-1}(1/h^n) |v|_{0, \phi}.$$  

**Proof.** Let $T \in \mathcal{T}_h$ such that $|v|_{0, \infty, T} = |v|_{0, \infty}$. By usual scaling arguments one can see that

$$|v|_{0, \infty, T} \leq C (1/h^n) \int_T |v(x)| \, dx.$$  

Let $\psi$ be the complementary function of $\phi$; then we can apply the Hölder inequality for Orlicz spaces, and we have

$$|v|_{0, \infty, T} \leq C (1/h^n) |v|_{0, \phi} |\chi|_{0, \psi},$$

where $\chi$ is the characteristic function of $T$. But $|\chi|_{0, \psi} = b$, where $b$ satisfies

$$\int_T \psi(1/b) \, dx = 1,$$

so $b = 1/\psi^{-1}(1/|T|)$ and then, using the inequality $t \leq \phi^{-1}(t)\psi^{-1}(t)$, we get

$$b \leq |T| \phi^{-1}(1/|T|) \leq Ch^n \phi^{-1}(1/h^n),$$

and (2) and (3) imply (1). \qed
**Lemma 2.** Let $g$ be a continuous function such that $\frac{\partial g}{\partial x_j} \in L^q(Q)$, where $Q \subset \mathbb{R}^n$ is an open set with Lipschitz boundary. Assume that

$$\mu(t) = \int_0^t \phi^{-1}(1/s^n) \, ds$$

is finite. Then,

$$|g(x + y) - g(x)| \leq C \|x\| \|y\|.$$  

*Proof.* Taking an extension, we can assume that $g$ is in $W^{1}_\phi(R^n)$. Let $\eta \in C^\infty_0$ such that $\int \eta = 1$ and $0 \leq \eta(x) \leq 1$, $\eta_t(x) = t^{-n} \eta(x/t)$ and $v(x,t) = g \ast \eta_t(x)$; then

$$(\partial v/\partial x_j)(x,t) = \int (\partial g/\partial x_j)(y) \eta_t(x - y) \, dy,$$

and applying the Hölder inequality, we have

$$(\partial v/\partial x_j)(x,t) \leq 2 \|\partial g/\partial x_j\|_{0,\phi} \|\eta_t\|_{0,\psi}.$$  

Set $b = t^{-n}/\phi^{-1}(t^{-n})$; then, since $\eta(x/t) \leq 1$ and $\psi$ is convex, we have

$$\int \psi(t^{-n} \eta(x/t)/b) \, dx = \int \psi(\phi^{-1}(t^{-n}) \eta(x/t)) \, dx \leq \int \eta(x/t) t^{-n} \, dx = 1.$$  

Consequently,

$$|\eta_t|_{0,\psi} \leq t^{-n}/\phi^{-1}(t^{-n}) \leq \phi^{-1}(t^{-n}),$$

and by (5),

$$|((\partial v/\partial x_j)(x,t)| \leq 2 \|\partial g/\partial x_j\|_{0,\phi} \phi^{-1}(t^{-n}).$$

A similar estimate for $\partial v/\partial t$ can be obtained in the following way. First observe that

$$\partial \eta_t/\partial t = - \sum_{i=1}^n \partial(x_i \eta)/\partial x_i;$$

then,

$$(\partial v/\partial t)(x,t) = (g \ast \partial \eta_t/\partial t)(x) = - \sum_{i=1}^n (g \ast \partial(x_i \eta)/\partial x_i)$$

$$= - \sum_{i=1}^n \partial g/\partial x_i \ast (x_i \eta),$$

and now we are in the same situation as before, with $\eta$ replaced by $x_i \eta$. In the same way we can prove that

$$|(x_i \eta)|_{0,\psi} \leq \phi^{-1}(t^{-n}) \max\{\|x_i \eta\|_{L^1}, \|x_i \eta\|_{L^\infty}\}$$

and then,

$$|((\partial v/\partial t)(x,t)| \leq C \|g\|_{1,\phi} \phi^{-1}(t^{-n}),$$

where $C$ depends on $\eta$. 

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Now (4) follows easily, writing
\[ g(x + y) - g(x) = \left[ g(x + y) - v(x + y, |y|) \right] + \left[ v(x + y, |y|) - v(x, |y|) \right] + \left[ v(x, |y|) - g(x) \right] \]
and estimating each summand separately. □

Now we restrict ourselves to functions of the form \( \phi(t) = \sum_{j=2}^{\infty} a_j t^j \) with \( a_j \geq 0 \), because our main example is of this form. For this class of functions it is easy to prove results about the error for Lagrange interpolation in the \( \phi \)-norm. In fact, using the known estimates for \( L^p \)-norms and the series expansion of \( \phi \), we get the following result,
\[ |u - I_h u|_{\phi, j} \leq Ch^{2 - j} \| u \|_{2, \phi}, \quad j = 0, 1, \]
where \( I_h u \) is the Lagrange interpolation of \( u \). Then we can state the following corollary of Lemma 2.

**Corollary 1.** Let \( \phi(t) = \sum_{j=2}^{\infty} a_j t^j \), \( a_j \geq 0 \), be an Orlicz function; then
\[ |u - I_h u|_{0, \infty} \leq Ch \mu(h) \| u \|_{2, \phi}. \]

We can now give a theorem which compares the error in \( L^\infty \) and \( L^p \)-norms.

**Theorem 1.** If \( \phi \) satisfies the condition of Corollary 1 and \( \mu \) is the function associated with \( \phi \) in Lemma 2, then there exists a constant \( C \) such that
\[ |u - u_h|_{0, \infty} \leq Ch \mu(h) \left[ \| u \|_{2, \phi} + \frac{|u - u_h|_{0, \phi}}{h^2} \right]. \]

**Proof.** By Lemma 1 and Corollary 1 we have
\[ |u - u_h|_{0, \infty} \leq |u - I_h u|_{0, \infty} + |I_h u - u_h|_{0, \infty} \leq C \left[ h \mu(h) \| u \|_{2, \phi} + \phi^{-1}(1/h^n)|I_h u - u_h|_{0, \phi} \right]. \]
But \( |I_h u - u|_{0, \phi} \leq Ch^2 \| u \|_{2, \phi} \) and then,
\[ |u - u_h|_{0, \infty} \leq C \left[ h \mu(h) \| u \|_{2, \phi} + h^2 \phi^{-1}(h^{-n}) \| u \|_{2, \phi} + \phi^{-1}(h^{-n})|u - u_h|_{0, \phi} \right]. \]
Noting that \( h \phi^{-1}(h^{-n}) \leq \mu(h) \), we obtain the result. □

**Corollary 2.** There exists a constant \( C \) such that
\[ |u - u_h|_{0, \infty} \leq Ch \left( \log h^{-1} \right) \mu(h) \| u \|_{2, \phi} \]
and, in particular,
\[ |u - u_h|_{0, \infty} < Ch^2 \left( \log h^{-1} \right)^2 \| u \|_{2, \phi}. \]

**Proof.** By the known estimates [9], [1]
\[ |u - u_h|_{0, \infty} \leq Ch^2 \log h^{-1} \| u \|_{2, \infty} \quad \text{and} \quad |u - u_h|_{0, 2} \leq Ch^2 \| u \|_{2, 2} \]
we get by interpolation
\[ |u - u_h|_{0, p} \leq Ch^2 \log h^{-1} \| u \|_{2, p} \quad \text{for } 2 \leq p < \infty, \]
with \( C \) independent of \( p \). Using the expansion in power series of \( \phi \), we get
\[ |u - u_h|_{0, \phi} < Ch^2 \log h^{-1} \| u \|_{2, \phi}, \]

hence, by Theorem 1, we get (6).
When \( \phi = \phi_1 \) it is easily shown that \( \mu_1(h) \leq Ch \log h^{-1} \) for small \( h \) and this proves (7). \( \square \)

We will show in the following theorem that as a consequence of the estimates for \( |u - u_h|_{1,\infty} \) [8] we have optimal-order estimates in the \( \phi_1 \)-norm if \( u \in W^{2,2}_\infty \).

**Theorem 2.** Let \( \Omega \subset \mathbb{R}^2 \) be such that \( \partial \Omega \) is smooth or \( \Omega \) is convex with Lipschitz boundary. Then there exists a constant \( C \) such that

\[
|u - u_h|_{0,\phi_1} \leq Ch^2 \|u\|_{2,\infty}.
\]

**Proof.** In [8] it is proved that

\[
|u - u_h|_{1,p} \leq Ch \|u\|_{2,p}, \quad 2 \leq p < \infty.
\]

On the other hand, if \( v \in H^1_0(\Omega) \) and \( -\Delta v = g \),

\[
\|v\|_{2,q} \leq \frac{C}{q-1} \|g\|_q \quad \text{for} \quad 1 < q \leq 2.
\]

(8)

In fact, if \( \Omega \) has a smooth boundary (for instance \( C^{1,1} \)), (8) can be shown by the classical proof [3], examining carefully the constants involved. In the case of a Lipschitz convex domain this result was proven recently by T. Wolff in unpublished work. Indeed, he has proven a weak type inequality for \( L^1 \) that, together with the known result for \( q = 2 \), implies (8) by usual interpolation methods.

By the known duality argument of Aubin-Nitsche [1], and using (8), we get

\[
|u - u_h|_{0,p} \leq Ch^2 \|u\|_{2,p}, \quad 2 \leq p < \infty,
\]

with \( C \) independent of \( p \).

But, in general, if we have two functions \( g_1 \) and \( g_2 \) such that

\[
|g_1|_{0,p} \leq C_1 p |g_2|_{0,p}, \quad 2 \leq p < \infty,
\]

then,

\[
|g_1|_{0,\phi_1} \leq C_1 C_2 |g_2|_{0,\infty},
\]

where \( C_2 \) depends only on \( \Omega \). In fact,

\[
\int_\Omega \phi_1 \left( \frac{|g_1(x)|}{K |g_2|_{0,\infty}} \right) dx = \int_\Omega \sum_{j=2}^{\infty} \frac{|g_1(x)|^j}{K^j |g_2|_{0,\infty}^j} \frac{1}{j!} dx \\
= \sum_{j=2}^{\infty} \frac{1}{j! K^j |g_2|_{0,\infty}^j} \int_\Omega |g_1(x)|^j dx \leq \sum_{j=2}^{\infty} \frac{C_1 j^j |g_2|_{0,\infty}^j}{j! K^j |g_2|_{0,\infty}^j} \\
\leq \sum_{j=2}^{\infty} \left( \frac{C_1}{K} \right)^j \frac{j^j}{j!} \frac{1}{|\Omega|}.
\]

and the last series is convergent and less than 1 if we choose \( K = C_1 C_2 \) with \( C_2 \) sufficiently large, depending only on \( \Omega \). \( \square \)


