On Weight Functions Admitting Chebyshev Quadrature

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Abstract. In this paper we prove the existence of Chebyshev quadrature for three new weight functions which are quite different from the two known examples given by Ullman [15] and Byrd and Stalla [2]. In particular, we indicate a simple method to construct weight functions for which there exist infinitely many Chebyshev quadrature rules.

1. Introduction. By a weight function \( w \) we mean a real-valued nonnegative function on \([-1, 1]\) for which the proper or improper Riemann integral exists and has positive value. We shall consider quadrature rules \( Q_n \) of the type

\[
Q_n[f] = \sum_{\nu=1}^{n} a_{\nu,n} f(x_{\nu,n}),
\]

having real nodes \( x_{\nu,n} \) and real weights \( a_{\nu,n} \).

A quadrature rule (1) is called a Chebyshev quadrature rule (in the strict sense) if the following holds:

\[
\begin{align*}
(2) & \quad a_{1,n} = a_{2,n} = \cdots = a_{n,n}, \\
(3) & \quad -1 < x_{1,n} < x_{2,n} < \cdots < x_{n,n} < 1, \\
(4) & \quad R_n[f] = 0 \quad \text{for all } f \in \mathcal{P}_n.
\end{align*}
\]

\( \mathcal{P}_n \) denotes the class of polynomials of degree \( \leq n \). We say that a weight function \( w \) admits Chebyshev quadrature if there exist Chebyshev quadrature rules \( Q_n \) for all positive integers \( n \).

The study of Chebyshev quadrature rules began in 1874 with the classical paper of Chebyshev [3]. Since then, there have been further investigations in the mathematical literature. For a review of recent advances in this field we refer to the paper of Gautschi [7].

Until 1966, the only known weight function admitting Chebyshev quadrature was the Chebyshev weight function

\[
w_1(x) = (1 - x^2)^{-1/2}.
\]
In 1966 Ullman [15] proved that the weight function
\begin{equation}
(6) \quad w_2(x) = w_1(x) \frac{1 + ax}{1 + a^2 + 2ax}, \quad |a| \leq \frac{1}{2}
\end{equation}
also admits Chebyshev quadrature. Recently, Byrd and Stalla [2] have shown that this result also holds for the weight function
\begin{equation}
(7) \quad w_3(x) = w_1(x) \frac{1}{2a + 1 + x}, \quad a \geq 1.
\end{equation}
There appears to be no other concrete example in the literature of weight functions admitting Chebyshev quadrature.

We now consider a weight function \(w\) as a product
\begin{equation}
(8) \quad w(x) = w_1(x)v(x).
\end{equation}
Kahaner [12] has shown that for \(w \in C(-1,1)\) to admit Chebyshev quadrature, a necessary condition on \(v\) is
\begin{equation}
(9) \quad v(x) \geq \frac{1}{2} c \quad \text{for all } x \in (-1,1),
\end{equation}
where
\begin{equation*}
c = \int_{-1}^{1} w(x) \, dx / \int_{-1}^{1} w_1(x) \, dx > 0.
\end{equation*}
Comparing the above weight functions \(w_1, w_2,\) and \(w_3,\) we see that in these cases \(v\) is a rational function continuous on \([-1,1]\).

In this paper we shall prove that the weight functions
\begin{align}
(10) \quad & w_4(x) = w_1(x)|x|^{-1/2}(1 + |x|)^{1/2}, \\
(11) \quad & w_5(x) = w_1(x)(1 + x)^{-1/4}(\sqrt{2} + (1 + x)^{1/2})^{1/2}, \\
(12) \quad & w_6(x) = w_1(x)(1 - x^2)^{-1/4}(1 + (1 - x^2)^{1/2})^{1/2}
\end{align}
also admit Chebyshev quadrature. Our method is quite different from those of Ullman [15] and Byrd and Stalla [2]. With regard to the still open problem of characterizing all weight functions admitting Chebyshev quadrature, it may be of interest that in these three cases the corresponding functions \(v\) in (8) have singularities either at an interior point or at one or both of the end points of the interval \([-1,1]\).

After establishing the existence of the Chebyshev quadrature rules for the weight functions (10), (11) and (12), and obtaining their nodes, we shall indicate a simple method for constructing weight functions for which there exist infinitely many Chebyshev quadrature rules. Such examples may also help path the way toward a solution of the above-mentioned problem.

2. Construction of the Chebyshev Rules. We first consider the Chebyshev weight function \(w_1\) given in (5). The corresponding Chebyshev rules \(Q^1_n\) have Gaussian degree of precision \(2n - 1\), i.e., \(R^1_n[f] = 0\) for all \(f \in \mathcal{P}_{2n-1}\) (see, e.g., Ghizzetti and Ossicini [11, p. 99 ff]). Transforming \(w_1\) and \(Q^1_n\) to the interval \([-1,0]\) as well as to the interval \([0,1]\) and compounding the two resulting weight functions and rules to the interval \([-1,1]\) gives the weight function \(w_1\) together with an equally weighted quadrature rule \(\hat{Q}_{2n}\) having \(2n\) nodes and degree of precision \(2n - 1\). The nodes of
the Chebyshev rule $Q^1_n$ are the zeros of the polynomial $T_n$, where $T_n$ denotes the Chebyshev polynomial of the first kind of degree $n$. Hence the nodes of $Q^1_{2n}$ are the zeros of the polynomial $\tilde{p}_{2n}$ given by

$$\tilde{p}_{2n}(x) = T_n(2x + 1)T_n(2x - 1).$$

We shall now show that the interpolatory quadrature rule (for definition see, e.g., [1, p. 16]) $Q^4_{2n}$, whose nodes are the zeros of

$$p_{2n}(x) = \tilde{p}_{2n}(x) - \frac{1}{2},$$

is a Chebyshev quadrature rule with $2n$ nodes for the weight function $w_4$. If $p_{2n} - \alpha$ ($\alpha \in \mathbb{R}$) has only real zeros, then the interpolatory quadrature formula, whose nodes are the zeros of $\tilde{p}_{2n} - \alpha$, is also equally weighted (see, e.g., [7, p. 103], [9], [4], [5]). So, the proof is completed if it is shown that

(i) $R^4_{2n}[q_{2n}] = 0$, $q_{2n}(x) := x^{2n}$,
(ii) all zeros of $p_{2n}$ are real, pairwise distinct and contained in the open interval $(-1, 1)$.

Because $Q^4_{2n}$ is an interpolatory quadrature rule, we have $R^4_{2n}[f] = 0$ for all $f \in \mathcal{P}_{2n-1}$ and therefore

$$2^{4n-2}R^4_{2n}[q_{2n}] = R^4_{2n}[p_{2n}]$$

$$= \int_{-1}^{1} w_4(x)[T_n(2x + 1)T_n(2x - 1) - \frac{1}{2}] \, dx$$

$$= -\pi + 2\int_{0}^{1} (x - x^2)^{-1/2}T_n(2x + 1)T_n(2x + 1) \, dx$$

$$= -\pi + 2\int_{-1}^{1} w_1(x)T_n(x)T_n(x + 2) \, dx$$

$$= -\pi + 2\int_{-1}^{1} w_1(x)T_n(x)2^{n-1}x^n \, dx$$

$$= -\pi + 2\int_{-1}^{1} w_1(x)\{T_n(x)\}^2 \, dx = 0,$$

using the known properties of $T_n$ (cf. here and in the following, e.g., Tricomi [14, p. 187 ff] or Paszkowski [13]) as well as the symmetry of $w_4$ and $p_{2n}$. This proves (i).

To prove (ii), we need consider only the interval $[0, 1]$ since $p_{2n}$ is symmetric. $T_n(2x - 1)$ has in $(0, 1)$ $n$ pairwise distinct real zeros. All $n - 1$ relative maxima of $|T_n(2x - 1)|$ as well as $T_n(1)$ have the value 1. Since $T_n(2x + 1) \geq 1$ for all $x \geq 0$, all relative maxima of $|\tilde{p}_{2n}|$ as well as $\tilde{p}_{2n}(1)$ have a value not less than 1. Therefore, $p_{2n}$ has all properties required in (ii).

To establish the Chebyshev quadrature rules $Q^4_{2n-1}$ for the weight function $w_4$ for all positive $n$, we consider first the Radau rules for the Chebyshev weight function $w_4$. They are given by (see, e.g., Ghizzetti and Ossicini [11, p. 101 ff.])

$$Q^+_n[f] = \frac{2\pi}{2n-1} \left\{ \frac{1}{2}f(1) + \sum_{\nu=1}^{n-1} f\left(\cos \frac{2\nu}{2n-1} \pi\right) \right\}$$

and

$$Q^-_n[f] = \frac{2\pi}{2n-1} \left\{ \frac{1}{2}f(-1) + \sum_{\nu=1}^{n-1} f\left(\cos \frac{2\nu}{2n-1} \pi\right) \right\}.$$
We note that the nodes of $Q_n^+$ are the zeros of $T_n - T_{n-1}$ and that the nodes of $Q_n^-$ are the zeros of $T_n + T_{n-1}$. Both quadrature rules have degree of precision $2n - 2$. Transforming $w_1$ and $Q_n^+$ to the interval $[-1, 0]$ and $w_1$ and $Q_n^-$ to the interval $[0, 1]$ and compounding the two resulting weight functions and rules to the interval $[-1, 1]$ yields the weight function $w_4$ together with an equally weighted quadrature rule $Q_{2n-1}^4$ having $2n - 1$ nodes and degree of precision not less than $2n - 2$. Because of the symmetry of $w_4$ and the symmetry of $Q_{2n-1}^4$ this quadrature rule has degree of precision $2n - 1$ and is therefore a Chebyshev quadrature rule. We have thus proven the following theorem.

**Theorem 1.** The weight function $w_4(x) = (1 - x^2)^{-1/2} |x|^{-1/2} (1 + |x|)^{1/2}$ admits Chebyshev quadrature. The nodes of the corresponding Chebyshev rules $Q_n^4$ are given by the zeros of $p_n^4$, where

\begin{align}
(17) \quad p_n^{4m}(x) &= T_m(2x - 1)T_m(2x + 1) - \frac{1}{2}, \\
(18) \quad xp_n^{4m}_x(x) &= [T_m(2x - 1) + T_m_x(2x - 1)] [T_m(2x + 1) - T_m_x(2x + 1)].
\end{align}

To establish Chebyshev quadrature for the weight functions $w_5$ and $w_6$, the lemma below is helpful (cf. also Gautschi [8, p. 482]).

**Lemma 1.** Let $w$ be a weight function on $[-1, 1]$ with $w(x) = w(-x)$ for all $x \in [-1, 1]$ and let $Q_{2n+1}$ be a quadrature rule with respect to $w$ given by

\begin{equation}
Q_{2n+1}[f] = a_0f(0) + \sum_{r=1}^{n} a_r [f(x_r) + f(-x_r)].
\end{equation}

with $0 < x_1 < \cdots < x_n$. Let $\tilde{w}(x) := w(\sqrt{x})/\sqrt{x}$ be a weight function on $[0, 1]$ and let $\tilde{Q}_{n+1}$ be the quadrature rule with respect to $\tilde{w}$ given by

\begin{equation}
\tilde{Q}_{n+1}[f] = a_0f(0) + \sum_{r=1}^{n} 2a_r f(x_r^2).
\end{equation}

Then $Q_{2n+1}$ has degree of precision $2m + 1$ if and only if $\tilde{Q}_{n+1}$ has degree of precision $m$.

This lemma is well known for $w(x) = 1$ and is used to derive Gauss rules on $[0, 1]$ with respect to the weight function $x^{-1/2}$. (Note that it is possible for $a_r$ to be zero.)

Application of Lemma 1 to the weight function $w_4$ and considering the rule $Q_{2n+1} := Q_{2n}^4$ (i.e., $a_0 = 0$) which, because of symmetry, has degree of precision $2n + 1$, gives the weight function

\begin{equation}
\tilde{w}(x) = x^{-3/4}(1 - x)^{-1/2}(1 + \sqrt{x})^{1/2}
\end{equation}
on the interval $[0, 1]$ and a corresponding equally weighted quadrature rule $\tilde{Q}_n$, whose nodes are the zeros of

\begin{equation}
\tilde{p}_n^4(x) = T_n(2\sqrt{x} - 1)T_n(2\sqrt{x} + 1) - \frac{1}{2}.
\end{equation}

By Lemma 1 the quadrature rule $\tilde{Q}_n$ has degree of precision $n$. The nodes of $Q_{2n}^4$ are all pairwise distinct and contained in $(-1, 1)$. So by (20), the $n$ nodes of $\tilde{Q}_n$ are also pairwise distinct and contained in $(0, 1)$. Transforming to the interval $[-1, 1]$ yields the following theorem.
Theorem 2. The weight function
\[ w_5(x) = (1 - x^2)^{-1/2}(1 + x)^{-1/4}(\sqrt{2} + (1 + x)^{1/2})^{1/2} \]
admits Chebyshev quadrature. The nodes for the corresponding Chebyshev rules \( Q_n^5 \) are given by the zeros of \( p_n^5 \), where
\[ p_n^5(x) = T_n(\sqrt{2x + 2} - 1)T_n(\sqrt{2x + 2} + 1) - \frac{1}{2}. \]

Using (21), let \( \tilde{w} \) be defined by
\[ \tilde{w}(x) = \tilde{w}(1 - x) = x^{-1/2}(1 - x)^{-3/4}(1 + \sqrt{1 - x})^{1/2}. \]
Since \( \tilde{w} \) is a weight function on \([0, 1]\) admitting Chebyshev quadrature, so is \( \tilde{w} \). By (22), the nodes of the corresponding Chebyshev rules \( \bar{Q}_n \) are the zeros of
\[ p_n^6(x) = T_n(2\sqrt{1 - x} - 1)T_n(2\sqrt{1 - x} + 1) - \frac{1}{2}. \]
Applying Lemma 1 \((a_0 = 0)\) to \( \tilde{w} \) and \( \bar{Q}_n \) gives the weight function \( w_6 \) on \([-1, 1]\) and the corresponding Chebyshev rule \( Q_{2n}^6 \).

To establish the Chebyshev quadrature rules \( Q_{2n-1}^6 \), we consider again the weight function \( w_1 \) and the corresponding Radau rules \( Q_n^+ \) and \( Q_n^- \) in (15) and (16). Transforming \( w_1 \) and \( Q_n^- \) to the interval \([-1, 0]\) and \( w_1 \) and \( Q_n^+ \) to the interval \([0, 1]\) and compounding yields the weight function \( w_6 \) in \([-1, 1]\) together with a quadrature rule \( Q_{2n}^+ \). Owing to symmetry, the rule \( Q_{2n}^+ \) has degree of precision \( 2n - 1 \). The \( 2n - 2 \) nodes in \((-1, 1)\) are equally weighted, for the nodes \(-1\) and \(1\) the weights are half as large as the weight of the other nodes. Applying Lemma 1 again gives on \([0, 1]\) the weight function \( \tilde{w} \) in (21) and a quadrature rule \( \tilde{Q}^*_n \) having degree of precision \( n - 1 \). \( \tilde{Q}^*_n \) has in \((0, 1)\) \( n - 1 \) equally weighted nodes; for the node \(1\) the weight is half as large as the weight of the other nodes. With the help of the transformation \( y = 1 - x \) we obtain on \([0, 1]\) the weight function \( \tilde{w} \) in (24) together with a corresponding quadrature rule \( \tilde{Q}^*_n \). Applying Lemma 1 again now yields the weight function \( w_6 \) and the Chebyshev quadrature rule \( Q_{2n-1}^6 \).

Theorem 3. The weight function
\[ w_6(x) = (1 - x^2)^{-3/4}(1 + (1 - x^2)^{1/2})^{1/2} \]
admits Chebyshev quadrature. The nodes of the corresponding Chebyshev rules \( Q_n^6 \) are given by the zeros of \( p_n^6 \), where
\[ p_{2m}^6(x) = T_m(2\sqrt{1 - x^2} - 1)T_n(2\sqrt{1 - x^2} + 1) - \frac{1}{2}, \]
\[ xp_{2m-1}^6(x) = \left[ T_m(2\sqrt{1 - x^2} - 1) - T_{m-1}(2\sqrt{1 - x^2} - 1) \right] \left[ T_m(2\sqrt{1 - x^2} + 1) + T_{m-1}(2\sqrt{1 - x^2} + 1) \right]. \]

In connection with the open problem of characterizing all weight functions admitting Chebyshev quadrature, we mention that the Jacobi weight function \((1 - x^2)^{-3/4}\) does not admit Chebyshev quadrature [6].

Remarks. (a) Note that the polynomial \( p_n^6 \) is symmetric on the interval \([-1, 1]\) and has even degree \( 2n \). Hence \( \tilde{p}_n^4, \ p_n^5, \ \tilde{p}_n, \) and \( p_n^6 \) are also polynomials because of the identities \( \tilde{p}_n^4(x) = p_n^4(\sqrt{x}), \ p_n^5(x) = p_n^5(\sqrt{2x + 2}), \ \tilde{p}_n = p_n^4(\sqrt{1 - x}), \) and \( p_n^6(x) = p_n^4(1 - x^2). \) Applying the same reasoning, \( p_{2n-1}^6 \) can also be shown to be a polynomial by virtue of the identity \( xp_{2n-1}^6(x) = q_{2n}(1 - x^2), \) where \( q_{2n} \) is defined by
\[ q_{2n}(x) = [T_n(2x + 1) + T_{n-1}(2x + 1)] - [T_n(2x - 1) + T_{n-1}(2x - 1)], \]
since again, \( q_{2n} \) is a symmetric polynomial of even degree \( 2n \).
(b) Since $T_n(x) = \cos(n \arccos x)$, the zeros of $p_{2m-1}^4$ and $p_{2m-1}^6$ can be obtained explicitly. Using the identity (see, e.g., [13, p. 22])

$$T_n(2x - 1) + T_{n-1}(2x - 1) = T_{2n}(\sqrt{x}) + T_{2n-2}(\sqrt{x}) = 2\sqrt{x} T_{2n-1}(\sqrt{x}),$$

we have that the zeros of the symmetric polynomial $p_{2m-1}^4$ agree with those of the function $t_{2m-1}$, where $t_{2m-1}$ is defined by

$$t_{2m-1}(x) = T_{2m-1}(\sqrt{|x|});$$

and by similar argument, that the zeros of $p_{2m-1}^6$ agree with those of $t_{2m-1}$, where $t_{2m-1}$ is defined by

$$t_{2m-1}(x) = T_{2m-1}(\sqrt{1 - (1 - x^2)^{1/2}}), \quad |x| \leq 1.$$

(c) Let $W$ be defined by $W(x) = w(-x)$. If $w$ admits Chebyshev quadrature, then so does $W$. This follows by transforming $w$ and $Q_n$ by a reflection at the origin on the interval $[-1,1]$. Applying this argument to $w_5$ shows that $\bar{w}_5$ also admits Chebyshev quadrature, where $\bar{w}_5$ is defined by

$$\bar{w}_5(x) = w_4(x)(1 - x)^{-1/4}(\sqrt{2} + (1 - x)^{1/2})^{1/2}.$$

3. Construction of Weight Functions Having Infinitely Many Chebyshev Quadrature Rules. Weight functions admitting Chebyshev quadrature are rare (Gautschi [7, p. 109]). Therefore, one may seek weight functions having infinitely many Chebyshev quadrature rules. Apart from the weight functions $w_1$, $w_2$, and $w_3$, the author has found in the literature only three other weight functions having this property. They are given by Geronimus [10] as follows:

$$w_a(x) = w_1(x) \frac{1 - a + 2ax^2}{(1 - a)^2 + 4ax^2}, \quad |a| < \frac{1}{2},$$

$$w_b(x) = \begin{cases} 
0 & \text{for all } x \in (-\alpha, \alpha), 0 < \alpha < 1, \\
\frac{w_1(x)(x^2 - \alpha^2)^{-1/2}}{|x|} & \frac{(1 - \alpha^2)(1 - a) + 2a(x^2 - \alpha^2)}{(1 - \alpha^2)(1 - a) + 4a(x^2 - \alpha^2)}, \\
|a| < \frac{1}{2},
\end{cases}$$

$$w_c(x) = w_1(x) \frac{1 + a^2 - 2ax}{1 + b^2 - 2bx},$$

$$a = \frac{2 + b^2 - \sqrt{(1 - b^2)(4 - b^2)}}{3b}, \quad |b| < 1, |a| < 1.$$

For each of these three weight functions, Chebyshev quadrature rules $Q_n$ exist for every even $n$. In the case of $w_b$ with $a = 0$ see also Gautschi [8, p. 483], where the Gaussian degree $2n - 1$ for even $n$ has been proved.

We now indicate a simple method for constructing other weight functions having infinitely many Chebyshev quadrature rules. Given a weight function $w_a$ on $[-1,1]$ of the form

$$w_a(x) = w_1(x) v_a(x)$$

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having a Chebyshev quadrature rule \( Q_n^a \), we transform \( w_a \) and \( Q_n^a \) to the interval \([0, 1]\) and apply Lemma 1. We obtain on \([-1, 1]\) the weight function

\[
(32) \quad w_b(x) = w_1(x) v_a(2x^2 - 1) = w_1(x) v_a(T_2(x))
\]

and a corresponding Chebyshev quadrature rule \( Q_{2n}^b \). Repeating this procedure and noting that

\[
(33) \quad T_n(T_m) = T_{nm},
\]

we obtain the following theorem.

**Theorem 4.** Let \( w_a(x) = w_1(x) v_a(x) \) be a weight function having a Chebyshev quadrature rule with \( n \) nodes and let \( k \in \mathbb{N} \). Then the weight function

\[
(34) \quad w_b(x) = w_1(x) v_a(T_2^k(x))
\]

has a Chebyshev quadrature rule with \( 2^kn \) nodes.

As a first example, we apply Theorem 4 to the weight function \( w_2 \) of Ullman given in (6). In the special case \( k = 1 \) we arrive at the weight function \( w_4 \) in (28) and the corresponding result of Geronimus [10].

Applying Theorem 4 to the weight function \( w_3 \) of Byrd and Stalla [2] for \( k = 1 \), we obtain that the weight function

\[
(35) \quad w_D(x) = \frac{1}{a + x^2}, \quad a \geq 1,
\]

has a Chebyshev quadrature rule \( Q_{2n}^D \) for every even \( n \in \mathbb{N} \).

In the case of weight functions \( w_4 \) and \( w_6 \) we arrive at the weight functions

\[
(36) \quad w_F(x) = w_1(x)|T_{2^k}(x)|^{-1/2}(1 + |T_{2^k}(x)|)^{1/2}, \quad k \in \mathbb{N},
\]

\[
(37) \quad w_F(x) = (1 - x^2)^{-3/4}|U_{2k-1}(x)|^{-1/2}\left\{1 + (1 - x^2)^{1/2}|U_{2k-1}(x)|\right\}^{1/2}, \quad k \in \mathbb{N},
\]

having for every \( n \in \mathbb{N} \) a Chebyshev quadrature rule with \( 2^kn \) nodes. \( (U_m \) denotes the Chebyshev polynomial of the second kind of degree \( m \).) With the help of \( w_F \) resp. \( w_E \) we see that for every \( k \in \mathbb{N} \) there exists a weight function admitting infinitely many Chebyshev quadrature rules, which has \( 2^k \) resp. \( 2^k - 1 \) pairwise distinct singularities in \((-1, 1)\).

Finally, we mention two generalizations of the above principle for the construction of weight functions admitting Chebyshev quadrature. Instead of transforming \( w_a \) and \( Q_n^a \) in (31) to the interval \([0, 1]\) we transform both to the interval \([\alpha, 1]\), \( 0 \leq \alpha < 1 \), and then apply Lemma 1. For example, in the case of the weight function \( w_2 \), we arrive at the weight function \( w_\beta \) in (29) and the corresponding result of Geronimus [10]. A further variation is given by the additional transformation \( y = \alpha + 1 - x \) before applying Lemma 1. If \( w_\delta \) is nonsymmetric on \([-1, 1]\), this also leads to new weight functions admitting Chebyshev quadrature.

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