On Approximate Zeros and Rootfinding Algorithms for a Complex Polynomial

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Abstract. In this paper we give criteria for a complex number to be an approximate zero of a polynomial \( f \) for Newton's method or for the \( k \)-th-order Euler method. An approximate zero for the \( k \)-th-order Euler method is an initial point from which the method converges with an order \( (k + 1) \). Also, we construct families of Newton (and Euler) type algorithms which are surely convergent.

1. Introduction. Newton's method has long been used for solving a nonlinear equation \( f(z) = 0 \). The Newton method attempts to solve \( f(z) = 0 \) by an iteratively defined sequence \( z_{n+1} = z_n - f(z_n)/f'(z_n) \), for an initial point \( z_0 \). It indeed converges to a root at a fast rate, if it starts with a good initial point. However, not much is known about the region of convergence or of fast convergence, and it is difficult to obtain a priori knowledge of convergence.

In this paper we study the efficiency and the convergence properties of the Newton method and other generalized methods for solving a polynomial equation \( f(z) = 0 \). We have two main goals. First, we establish an estimate for a point \( z_0 \), which predicts fast convergence of the algorithms starting at \( z_0 \). Secondly, we develop a method which is guaranteed to converge, given an arbitrary initial point \( z_0 \).

Following Shub and Smale, we consider the following generalized version of the Newton method, called the modified \( k \)-th-order Euler method.

We recall from elementary complex analysis that for a polynomial \( f \) and \( z \in \mathbb{C} \) such that \( f'(z) \neq 0 \), there is a well-defined local inverse branch \( f_{z,-1} \) of \( f \) such that \( f_{z,-1}(f(z)) = z \).

Definition 1.1. For an integer \( k \) and a complex number \( h \), the Euler method iteratively defines a sequence \( z_{n+1} = E_{k,h,f}(z_n) = T_k f_{z_n^{-1}}((1 - h)f(z_n)) \) for an initial point \( z_0 \), where \( T_k \) is the \( k \)-th-order truncation of \( f_{z_n^{-1}} \) considered as a power series about \( f(z_n) \).

For brevity we denote \( E_{k,h,f} \) by \( E_k \) if there is no confusion. Note that \( E_{1,1,f} \) gives the Newton method.

We define an approximate zero of \( f \) for \( E_k \) as follows.
**Definition 1.2.** $z_0$ is an approximate zero of $f$ for $E_k$ if

\[ \frac{|f(z_n)|}{|f(z_0)|} \leq \left(\frac{1}{2}\right)^{(k+1)n}, \]

\[ |z_n - \xi| \leq c \left(\frac{1}{2}\right)^{(k+1)n} |z_0 - \xi|, \]

where $c$ is a constant, $\xi$ is a root and $z_{n+1} = E_{k,1,f}(z_n) \to \xi$. □

Note the fast convergence of $z_n$ to $\xi$. This notion is due to Smale [12].

For a polynomial $f$ and $z \in \mathbb{C}$, let

\[ a_{f,z} = \max_{j \geq 2} \left| \frac{f(z)}{f'(z)} \right|^{1/(j-1)}. \]

We show that $z$ is an approximate zero of $f$ for all $E_k$ if $a_{f,z} \leq \frac{1}{48}$ (see Theorem 4.4). Recently, Smale [14] has obtained a similar result for $k = 1$ with a better estimate (a constant $\alpha_0$, in his notation, between $\frac{1}{8}$ and $\frac{1}{2}$) for a more general class of polynomial maps $f : \mathbb{C}^n \to \mathbb{C}^n$.

The estimate for $a_{f,z}$ plays an important role even when $z$ is not an approximate zero of $f$. It suggests the next iterate in the construction of algorithms which produces sure convergence.

We construct two families of modified Euler methods $A_k$ and $B_k$, which always converge to a root or a critical point of $f$ (see Theorems 5.A and 5.B). $\theta$ is called a critical point of $f$ if $f'(\theta) = 0$. As in the work of Shub and Smale ([9], [10]), the idea is to approximate the solution curve $\phi_t(z_0)$ to the Newton vector field $F(z) = -f(z)/f'(z)$ where $\phi_0(z_0) = z_0$. Note that $f(\phi_t(z_0)) = e^{-tf(z_0)}$, a straight line through $f(z_0)$. Hence one can approximate a root by approximating $f^{-1}(e^{-tf(z_0)}; t \to \infty)$. To do so, Shub and Smale use the modified Euler method with a fixed step size $h$ in $E_{k,h,f}$ together with a probabilistic estimate on the set of initial points. In our algorithms we use a varying step size $h$ at each point $z$, where $h$ is given in terms of $a_{f,z}$ and hence related to the radius of convergence of $f^{-1}$. In particular, we show that for any polynomial $f$ and initial point $z_0$, $B_k$ always produces a sequence $z_n$ converging to a root unless there is a critical value of $f$ on the ray $(0, f(z_0)]$ (see Theorem 5.B). Recently, Shub and Smale [11] have shown that an algorithm similar to $A_1$ converges to a root for almost all polynomials and for almost all initial points $z_0$.

We have run some experiments on the algorithm $A_1$ and other similar algorithms with a starting point 0 and with a supplementary algorithm of Shub and Smale [10] for degenerate cases such as $a_{f,z} \geq 50d^2$. This corresponds to the case where $z$ is near a critical point. Among $(100 \cdot d^2)$ randomly selected polynomials of each degree $d \leq 100$ with complex coefficients $|a_d| \leq 1$, the average number of iterations to locate an approximate zero or to locate $z$ such that $|f(z)| \leq 10^{-4}$ is found to be less than 200. Our experimental result is independent of the degree $d$.

2. Preliminaries. In this section we discuss some preliminary material needed in the later sections on the local behavior of analytic functions.

The main tools used in Section 3 are from the theory of schlicht functions. $f$ is called a schlicht function if $f(0) = 0$, $f'(0) = 1$ and it is univalent on $D_1(0)$, the unit disk at 0. A univalent function is a one-to-one complex analytic function.
To each \( z \in \mathbb{C} \) and complex polynomial \( f \) such that \( f(z) \neq 0 \) and \( f'(z) \neq 0 \), one associates a normalized polynomial \( \sigma \) by means of

\[
\sigma(w) = w + \sigma_2 w^2 + \cdots + \sigma_d w^d, \quad \text{where} \quad \sigma_j = \left( \frac{-f(z)}{f'(z)} \right)^{j-1} \frac{f^{(j)}(z)}{j! f'(z)}.
\]

Let \( R_{f,z} \) be the radius of convergence of \( f_z^{-1} \), considered as a power series at \( f(z) \).

For \( f(z) \neq 0 \), let \( H_{f,z} = R_{f,z}/|f(z)| \); see Figure 2.1. The following lemma is extracted from the work of Shub and Smale (see [9, p. 113]).

**Lemma 2.1.** Let \( \sigma^{-1} \) be the inverse branch of \( \sigma \) taking 0 to 0. Then

1. \( \sigma^{-1}(0) = 0, \quad \sigma^{-1}'(0) = 1. \)
2. Let \( x = f_z^{-1}((1 - h)f(z)) \) with \( |h| < H_{f,z} \). Then
   \[
   \frac{f(x)}{f(z)} = 1 - \sigma \circ \varepsilon, \quad \text{where} \quad \varepsilon = \frac{x - z}{F(z)} \quad \text{and} \quad F(z) = \frac{-f(z)}{f'(z)}.
   \]
3. \( f_z^{-1}((1 - h)f(z)) = z + F(z)\sigma^{-1}(h). \)
4. \( T_k f_z^{-1}((1 - h)f(z)) = z + F(z)T_k \sigma^{-1}(h). \)
5. The radius of convergence of \( \sigma^{-1} \) at 0 is \( R_{\sigma,0} = H_{f,z}. \)
6. \( \frac{1}{H} \sigma^{-1}(Hh) \) is schlicht, \( H \equiv H_{f,z}. \)

**Proof.** (1) is immediate. (2) is from Proposition 2 in [9]. For (3), (4) and (5), see [9, p. 114] and [10, p. 153]. (6) is a trivial consequence of (1) and (5). \( \square \)

Using Lemma 2.1, we may reformulate Definition 1.1 of \( E_{k,h,f} \) as follows.

**Definition 2.2.** \( E_{k,h,f}(z) = z + F(z)T_k \sigma^{-1}(h). \)

We will need the following properties.

**Lemma 2.3** (De Branges' Theorem: Bieberbach conjecture). Let \( g(z) = z + g_2 z^2 + g_3 z^3 + \cdots \) be schlicht. Then \( |g_k| \leq k. \)

**Proof.** See [2]. \( \square \)
LEMMA 2.4 (Shub and Smale). (1) Let $g$ be schlicht. Then $|g(h) - T_k g(h)| \leq (k + 1)r^{k+1}/(1 - r)^2$, where $r = |h| < 1$.

(2) Let $g$ be univalent on $D_H(0)$, $g(0) = 0$ and $g'(0) = 1$. Then for $h$ with $r = |h|/H < 1$,

$$|g(h) - T_k g(h)| \leq \frac{H(k + 1)r^{k+1}}{(1 - r)^2}.$$

Proof. From Lemma 2.3 we have

$$|g(h) - T_k g(h)| \leq \sum_{j=k+1}^{\infty} j r^j \leq r \left( \frac{r^{k+1}}{1 - r} \right)' \leq \frac{(k + 1)r^{k+1}}{(1 - r)^2}.$$

For the second statement, note that $\frac{1}{H} g(Hh)$ is schlicht, and then use (1). □

LEMMA 2.5 (Koebe Distortion Theorem). Let $g$ be schlicht. Then for $|h| = r < 1$,

(1) $\frac{r}{(1 + r)^2} \leq |g(h)| \leq \frac{r}{(1 - r)^2},$

(2) $\frac{1 - r}{(1 + r)^3} \leq |g'(h)| \leq \frac{1 + r}{(1 - r)^3}.$

Proof. See [4, Vol. 2, pp. 351 and 353]. □

By rescaling, we obtain immediately the following

COROLLARY 2.6. Let $g$ be univalent on $D_H(0)$ and $g(0) = 0$, $g'(0) = 1$. Let $r = |h|/H < 1$. Then

(1) $\frac{|h|}{(1 + r)^2} \leq |g(h)| \leq \frac{|h|}{(1 - r)^2},$

(2) $\frac{1 - r}{(1 + r)^3} \leq |g'(h)| \leq \frac{1 + r}{(1 - r)^3}.$

Proof. Note that $\frac{1}{H} g(Hh)$ is schlicht. Now use Lemma 2.5. □

COROLLARY 2.7. Let $x = f_z^{-1}((1 - h)f(z)) \equiv z + F(z)\sigma^{-1}(h)$ for $|h| < H_{f,z}$. Then we have

(1) $f'(x) = f'(z)\sigma'(\varepsilon) \equiv \frac{f'(z)}{\sigma^{-1}(h)}$, where $\varepsilon = \frac{x - z}{F(z)}$.

(2) $|f'(x)| \left( \frac{1 - r}{1 + r} \right)^3 \leq |f'(z)| \leq |f'(z)| \left( \frac{1 + r}{1 - r} \right)^3$, where $r = \frac{|h|}{H_{f,z}}$.

Proof. Recall from Lemma 2.1(2) that $f(x) = f(z)(1 - \sigma(\varepsilon))$ and $\sigma(\varepsilon) = h \equiv (f(z) - f(x))/f(z)$. Hence (1) is immediate by taking derivatives of $f$. (2) follows from Corollary 2.6(2) since $\sigma^{-1}(0) = 0$, $\sigma^{-1}(0) = 1$ and $\sigma^{-1}$ is univalent in $D_H(0)$. □

We close this section with the following lemma.

LEMMA 2.8. (1) $R_{f,z} = |f(z) - f(\theta^*)| \geq \min_{f'(\theta) = 0} |f(z) - f(\theta)|$ for some critical point $\theta^*$ of $f$.

(2) Let $x = f_z^{-1}((1 - h)f(z))$ with $|h|/H_{f,z} < 1$. Then $R_{f,z} \geq R_{f,z} - |f(z) - f(x)|$. 
(3) Let \( g = f - y \) be a translation of \( f \) by \( y \in \mathbb{C} \). Then \( E_{k,h',g}(x) = E_{k,h,f}(x) \), where \( h' = hf(z)/g(z) \).

Proof. For (1), see Lemma 3 in [12].

For (2), we note that by the uniqueness of analytic maps, we have \( f_z^{-1} \equiv f_z^{-1} \) on their common domain of definitions. In particular, \( f_z^{-1} \) is analytically continued for all \( w \) such that \( |w - f(z)| < R_{f,z} \). Since \( |w - f(z)| < |w - f(z)| + |f(z) - f(z)| < R_{f,z} \), \( f_z^{-1} \) is analytic for all \( w \) such that \( |w - f(z)| < R_{f,z} - |f(z) - f(z)| \). Hence \( R_{f,z} \geq R_{f,z} - |f(z) - f(z)| \).

For (3), note that \( g_z^{-1}(w - y) \) is well defined where \( f_z^{-1}(w) \) is well defined and \( g_z^{-1}(w - y) = f_z^{-1}(w) \). As power series at \( f(z) \) and \( g(z) \) respectively, we have

\[
Tk f_z^{-1}((1 - h)f(z)) = Tk g_z^{-1}((1 - h')g(z)) \quad \text{and} \quad E_{k,h',g}(z) = E_{k,h,f}(z). \quad \square
\]

3. Koebe Distortion Theorem and Euler Iteration. We recall that

\( f_z^{-1}(1 - h)f(z)) = f_z^{-1}(1 - h')g(z) \) for all \( w \) such that \( |w - f(z)| < R_{f,z} \) and \( R_{f,z} \) is the radius of convergence of \( f_z^{-1} \). In particular, we show that \( E_{k,h,f} \) approximates \( f_z^{-1} \) for all values on the disk of convergence as \( k \uparrow \infty \). The main goal of this section is to prove Theorem 3.2 below.

We recall that \( H_{f,z} = R_{f,z}/|f(z)| \), where \( R_{f,z} \) denotes the radius of convergence of \( f_z^{-1} \) at \( f(z) \).

THEOREM 3.1. Let \( x = f_z^{-1}((1 - h)f(z)) \). Assume that

\[
r = \frac{|h|}{H_{f,z}} < 1 \quad \text{and} \quad t \leq |h|(1 - r)^3 \bigg/ (1 + r)^3.
\]

Then \( D_t|F|(x) \subset f_z^{-1}(D_f(z)|f(z))) \), where

\[
s = \min \left\{ t \left(1 + r \right)^3 \bigg/ \left(1 - r \right)^3, \ H_{f,z}(1 - r) \right\} \quad \text{and} \quad F = -\frac{f(z)}{f'(z)}.
\]

The proof will be given later. \( \square \)

Let

\[
B_k(r) = (k + 1) \left(1 + r \right)^3 \bigg/ (1 - r)^3.
\]

and \( r_k \) be the smallest positive solution to \( B_k(r) = 1 \). Note that \( B_k(r) \) is increasing on \([0, r_k] \). The condition that \( |h| < r_k H_{f,z} \) is crucial for \( E_{k,h,f} \) to approximate \( f_z^{-1} \).

THEOREM 3.2. Let \( z' = E_{k,h,f}(z) \) with \( r = |h|/H_{f,z} < r_k \). Then we have

\[
z' = f_z^{-1}((1 - h')f(z)) \quad \text{and} \quad f(z')/f(z) = 1 - h + \epsilon, \quad \text{where} \quad |\epsilon| = |h - h'| \leq \min\{|h|B_k(r), H_{f,z}(1 - r)\}.
\]

A table of approximate values of \( r_k \) is given below.

**TABLE 3.1**

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>177</th>
<th>3303</th>
<th>47400</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_k )</td>
<td>.148</td>
<td>.225</td>
<td>.282</td>
<td>.329</td>
<td>.367</td>
<td>.495</td>
<td>.9</td>
<td>.99</td>
<td>.999</td>
</tr>
</tbody>
</table>

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Remark. We note that $\tau_k \uparrow 1$ as $k \uparrow \infty$. In [9], Shub and Smale showed that Theorem 3.2 holds for all $r \leq \gamma_k$ where $\gamma_k \uparrow 0.175$ as $k \uparrow \infty$.

Proof of Theorem 3.2. Recall that $z' = E_{k,h,f}(z) = z + F(z)T_k\sigma^{-1}(h)$, where $F(z) = -f(z)/f'(z)$. Let $\varphi(z') = (1 - h')f(z)$. Let $x = f_{-1}^{-1}((1 - h)f(z)) = z + F(z)\sigma^{-1}(h)$. Then $|z' - x| = |F| |\sigma^{-1}(h) - T_k\sigma^{-1}(h)|$. Since $\sigma^{-1}$ is univalent on $D_H(0)$, we have by Lemma 2.4,

$$t \equiv |\sigma^{-1}(h) - T_k\sigma^{-1}(h)| \leq \frac{H_{f,z}(k + 1)r^{k+1}}{(1 - r)^2}$$

$$= |h|B_k(r)\frac{(1 - r)^3}{(1 + r)^3} \leq |h|\frac{(1 - r)^3}{(1 + r)^3} \text{ for } r < r_k.$$ 

Now, by Theorem 3.1, we have $z' \in f_{-1}^{-1}(D_{f(z)}(f(z)))$ and

$$|f(z') - f(x)| = |(1 - h')f(z) - (1 - h)f(z)| = |f(z)||h' - h| \leq |f(z)|s.$$ 

Hence,

$$|\varepsilon| = |h' - h| \leq s = \text{Min}\{|h|B_k(r), H_{f,z}(1 - r)|$$

and we have $z' = f_{-1}^{-1}((1 - h')f(z))$ and $f(z')/f(z) = 1 - h'$ for some $h'$ with $|h' - h| \leq s$. □

We need the following lemmas to prove Theorem 3.1.

**Lemma 3.3.** (1) Let $g$ be univalent on $D_R(0)$. Then $D_{Rr}(g(z)) \subset g(D_{Rs}(z)) \subset D_{Ru}(g(z))$, for any $s < 1$, $t = s|g'(z)|/(1 + s)^2$ and $u = s|g'(z)|/(1 - s)^2$.

(2) Suppose that $g$ is univalent on $D_H(0)$, $g(0) = 0$ and $g'(0) = 1$. Let $z \in D_H(0)$, where $r = |z|/H$. Then for $s \leq 1 - r$ we have $D_{Ht}(g(z)) \subset g(D_{Hs}(z)) \subset D_{Hu}(g(z))$, where $t = ((1 - r)^3/(1 + r)^3) \cdot s/(1 - r + s)^2$ and $u = ((1 + r)/(1 - r)) \cdot s/(1 - r - s)^2$.

\[\text{FIGURE 3.2. } w = g(z)\]

**Proof.** (1) Let

$$\psi(h) = \frac{1}{Rg'(z)}(g(z + Rh) - g(z)).$$

Then it is easy to see that $\psi$ is schlicht. Hence by Lemma 2.5, $\delta/(1 + \delta)^2 \leq |\psi(h)| \leq \delta/(1 - \delta)^2$ for $|h| = \delta$, so that we have

$$D_{R\delta}(g(z)) \subset g(D_{R\delta}(z)) \subset D_{R\eta}(g(z)).$$
where \( R\mu = \delta R|g'(z)|/(1 + \delta)^2 \), \( R\eta = \delta R|g'(z)|/(1 - \delta)^2 \). By setting \( t = \mu \), \( s = \delta \) and \( u = \eta \), (1) is established.

For (2), note that \( g \) is univalent on \( DR(z) \), where \( R = H(1 - r) \), and hence \( (1 - r)/(1 + r)^3 \leq |g'(z)| \leq (1 + r)/(1 - r)^3 \) by Corollary 2.6. Let \( s = (1 - r)\delta \). Then we have

\[
R\mu \geq \frac{\delta H(1-r)}{(1+\delta)^2} \frac{1-r}{(1+r)^3} \geq \frac{Hs}{(1-r+s)^2(1+r)^3} \equiv Ht,
\]

\[
R\eta \leq \frac{\delta H(1-r)}{(1-\delta)^2} \frac{1+r}{(1-r)^3} \leq \frac{Hs}{(1-r-s)^2(1-r)} \equiv Hu,
\]

where

\[
t = \frac{(1-r)^3}{(1+r)^3} \frac{s}{(1-r+s)^2} \quad \text{and} \quad u = \frac{1+r}{1-r} \frac{s}{(1-r-s)^2}.
\]

Hence we have \( D_{Ht}(g(z)) \subseteq g(D_{Hs}(z)) \subseteq D_{Hu}(g(z)) \). □

Lemma 3.3 gives the following Quarter Theorem at an arbitrary point \( z \in D_H(0) \).

**Corollary 3.4.** Let \( g \) be univalent on \( D_H(0) \) and \( g(0) = 0 \) and \( g'(0) = 1 \). For \( z \in D_H(0) \), let \( r = |z|/H \). Then \( D_{Ht}(g(z)) \subseteq g(D_{H(1-r)}(z)) \), where \( t = \frac{1}{4}(1-r)^2/(1+r)^3) \).

**Proof.** Use \( s = 1 - r \) in Lemma 3.3(2). □

In Lemma 3.3(2) we will also need to estimate \( s \) as a function of \( t \).

**Corollary 3.5.** Suppose \( t \leq r((1-r)^3/(1+r)^3) \), where \( r = |z|/H < 1 \). Then \( D_{Ht}(g(z)) \subseteq g(D_{Hs}(z)) \), where \( s = \min\{s((1+r)^3/(1-r)^3), 1-r\} \).

**Proof.** Since \( r(1-r) \leq \frac{1}{4} \), we have \( t \leq \frac{1}{4}((1-r)^2/(1+r)^3) \). Hence by Corollary 3.4 we have \( D_{Ht}(g(z)) \subseteq D_{H(1-r)}(g(z)) \). Let \( t' = ((1-r)^3/(1+r)^3) \cdot s/(1-r+s)^2 \). Since \( s \leq 1-r \), Lemma 3.3(2) shows that \( D_{Ht'}(g(z)) \subseteq g(D_{Hs}(z)) \). However, since \( s \leq t((1-r)^3/(1+r)^3) \leq r \), we have

\[
t' = \frac{(1-r)^3}{(1+r)^3} \frac{s}{(1-r+s)^3} \geq \frac{(1-r)^3}{(1+r)^3} s = t.
\]

Consequently, \( D_{Ht}(g(z)) \subseteq D_{Ht'}(g(z)) \subseteq g(D_{Hs}(z)) \), as claimed. □

The proof of Theorem 3.1 now follows easily from Corollary 3.5.

**Proof of Theorem 3.1.** Suppose that \( z = f^{-1}_z((1-h')f(z)) \) where \( |z'-x| \leq |F|t \).

Since

\[
z' = f^{-1}_z((1-h')f(z)) = z + F(z)\sigma^{-1}(h') = z + F(z)\sigma^{-1}(h) + F(z)(\sigma^{-1}(h') - \sigma^{-1}(h)) = x + F(z)(\sigma^{-1}(h') - \sigma^{-1}(h)),
\]

we have

\[
|z'-x| = |F| |\sigma^{-1}(h') - \sigma^{-1}(h)| \leq |F|t = |F|Ht', \quad \text{where} \quad t' = \frac{t}{H} \leq \frac{|h| (1-r)^3}{H (1+r)^3} = \frac{r(1-r)^3}{(1+r)^3},
\]

by the hypothesis. Hence, by Corollary 3.5, we have \( |h' - h| \leq Hs' \), where \( s' = \min\{t'((1+r)^3/(1-r)^3), 1-r\} \). Now by setting \( s = Hs' \) we have the claim. □
4. Domain of Injectivity and a Notion of an Approximate Zero. The main goal of this section is to give a criterion to determine an approximate zero of a polynomial \( f \) for the modified Euler method. Hereafter we will denote \( E_{k,h,f} \) by \( E_k \) if there is no confusion.

**Definition.** \( z_0 \) is an approximate zero of \( f \) for \( E_k \) if

\[
\frac{|f(z_n)|}{|f(z_0)|} \leq \left( \frac{1}{2} \right)^{(k+1)n},
\]

\[
|z_n - \xi| \leq c \left( \frac{1}{2} \right)^{(k+1)n} |z_0 - \xi|,
\]

where \( z_n = E_{k,1,f}(z_0) \rightarrow \xi \) and \( c \) is a constant.

We will need the following estimate of the domain of injectivity, which itself is quite interesting.

**Theorem 4.1.** Let \( g(z) = z + a_2z^2 + \cdots \) be a power series and \( \psi \) be the compositional inverse of \( g \) taking 0 to 0. Let \( a = \sup \{ |a_i| \} \). Then \( \psi \) is well defined, analytic and one-to-one on \( DR(0) \), where \( (3 - \sqrt{8})/a \leq R \).

**Proof.** Suppose that \( |g(z) - z| < r \) on \( |z| = r \). Then 0 is the only root of \( g \) in \( Dr(0) \) by Rouché's Theorem. It follows that (see [1, Theorem 11, p. 131]) the inverse map \( \psi \) is well defined on \( g(Dr(0)) \). In particular, \( \psi \) is well defined on \( DR(0) \), where \( R = \min_{|z|=r} |g(z)| \). Now,

\[
|g(z)| = |z||1 + a_2z + a_3z^2 + \cdots| \geq r|1 - ((ar) + (ar)^2 + (ar)^3 + \cdots)| \geq r \left( 1 - \frac{ar}{1-ar} \right) \quad \text{on } |z| = r.
\]

But \( r(1 - ar/(1 - ar)) \) achieves the maximum \( (3 - \sqrt{8})/a \) when \( r = (2 - \sqrt{2})/2a \). Also note that

\[
|g(z) - z| = |a_2z^2 + a_3z^3 + \cdots| = |z||a_2z + a_3z^2 + \cdots| \leq r \frac{ar}{1-ar} < r, \quad \text{on } |z| = \frac{2 - \sqrt{2}}{2a}.
\]

Hence \( \psi \) is well defined and injective on \( DR(0) \), where \( R = (3 - \sqrt{8})/a \approx 1/5.83a > 1/6a \). \( \square \)

**Remark 4.2.** The corresponding upper bound \( R \leq 4/a \) is obtained in [12, p. 9, Extended Loewner's Theorem]. For a polynomial \( f \) and \( z \in \mathbb{C} \) we define

\[
a_{f,z} \equiv \max_{j \geq 2} \frac{\left| f(z) \right|}{|f'(z)|} \left( \frac{\left| f^{(j)}(z) \right|}{j!f''(z)} \right)^{1/(j-1)}.
\]

We apply Theorem 4.1 to a polynomial.

**Corollary 4.3.** Let \( f \) be a polynomial of degree \( d \) and \( z \) be a complex number such that \( f'(z) \neq 0 \), and \( f(z) \neq 0 \). Let \( f_z^{-1} \) be the inverse branch of \( f \) such that \( f_z^{-1}(f(z)) = z \). Then \( f_z^{-1} \), as a power series at \( f(z) \) has a radius of convergence \( R_{f,z} \) satisfying \( (3 - \sqrt{8})/a \leq R_{f,z}/|f(z)| \leq 4/a \).
Proof. Let $\sigma$ be the polynomial associated with $f$ as in Lemma 2.1. Since the radius of convergence of $\sigma^{-1}$ at 0 is $H_{f,z} = R_{f,z}/|f(z)|$ by Lemma 2.1, we have the claim by the previous theorem. \( \square \)

We now come to one of the main results.

**Theorem 4.4.** If $a_f, z_0 \leq 1/48$, then $z_0$ is an approximate zero of $f$ for $E_k$ for all $k$. In other words, we have

\[
\frac{|f(z_n)|}{|f(z_0)|} \leq \left(\frac{1}{2}\right)^{(k+1)^n},
\]

\[
|z_n - \xi| \leq 4 \left(\frac{1}{2}\right)^{(k+1)^n} |z_0 - \xi|,
\]

where $\xi$ is a root and $z_{n+1} = E_{k,1,f}(z_n) \to \xi$.

**Proof.** We will proceed with the proof by an induction on $n$. For simplicity, we denote $R_n = R_{f,z_n}$, $f_n = f(z_n)$, $f'_n = f'(z_n)$, $H_n = R_n/|f_n|$ and $F_n = -f(z_n)/f'(z_n)$.

**Claim 1.** $|f_1|/|f_0| \leq (\frac{1}{2})^{k+1}$, for all $k$.

We note that $a_f, z_0 \leq 1/48$ implies by Corollary 4.3 that

\[
\frac{1}{H_0} = \frac{|f_0|}{R_0} < \frac{1}{483} < \frac{1}{8.23} < 0.122 < \gamma_k
\]

for all $k$ (see Table 3.1). Hence we apply Theorem 3.2 with $h = 1$, and we have

\[
z_1 = f^{-1}_{z_0}(f(z_1)) \quad \text{and} \quad \frac{|f_1|}{|f_0|} \leq B_k \left( \frac{1}{H_0} \right) \leq \left(\frac{1}{2}\right)^{k+1},
\]

by noting that

\[
B_k \left( \frac{1}{H_0} \right) < B_k(0.122) = \left(\frac{k+1}{1-0.122}\right) \frac{(0.122)^k}{(1-0.122)^5} < \left(\frac{1}{2}\right)^{k+1} \quad \text{for } k \geq 2.
\]

For $k = 1$, we recall from Lemma 2.1 that $f(z_1)/f(z_0) = 1 - \sigma \circ \varepsilon$, where $\varepsilon = (z_1 - z_0)/F_0 = 1$. Since

\[
|1 - \sigma(1)| = |\sigma_2 + \sigma_3 + \cdots + \sigma_d| \leq \frac{a}{1-a} \leq \frac{1}{47} \leq \left(\frac{1}{2}\right)^2,
\]

we have that $|f_1|/|f_0| \leq (\frac{1}{2})^{k+1}$ for all $k$ as claimed. It is useful for the next claim to note that $|f_1|/|f_0| \leq 1/8$.

**Claim 2.** Suppose $|f_n|/|f_0| \leq (\frac{1}{2})^{(k+1)^n}$. Then $|f_{n+1}|/|f_0| \leq (\frac{1}{2})^{(k+1)^{n+1}}$. First note that $R_n \geq R_0 - |f_n| - |f_0|$ by Lemma 2.8(2). Since $R_0/|f_0| \geq 8.23$ and $|f_n|/|f_0| \leq 1/8$ for all $n$ and $k$, we have $R_n \geq 8.23|f_0| - \frac{9}{8}|f_0| \geq 7|f_0|$ for all $n$ and $k$. Hence we have

\[
\frac{1}{H_n} = \frac{|f_n|}{R_n} = \frac{|f_0|}{R_n} |f_n| \leq \frac{1}{7} |f_n| \leq \frac{1}{7} \left(\frac{1}{2}\right)^{(k+1)^n} \leq \gamma_k \left(\frac{1}{2}\right)^{(k+1)^n}
\]

for all $k$ and $n$. Now, applying Theorem 3.2 with $h = 1$, we have $|f_{n+1}|/|f_n| \leq B_k(1/H_n)$. Since

\[
B_k(r) = (k+1) \frac{(1+r)^3 r^k}{(1-r)^5} < B_k(r_k) \left(\frac{r}{r_k}\right)^k < \left(\frac{r}{r_k}\right)^k
\]
for \( r < r_k \) we have
\[
\frac{|f_{n+1}|}{|f_n|} \leq B_k \left( \frac{1}{H_n} \right) \leq \left( \frac{1}{2} \right)^{(k+1)n} = \left( \frac{1}{2} \right)^{k(k+1)n}.
\]
Hence we have
\[
\frac{|f_{n+1}|}{|f_0|} = \frac{|f_{n+1}|}{|f_n|} \frac{|f_n|}{|f_0|} \leq \left( \frac{1}{2} \right)^{(k+1)n} \left( \frac{1}{2} \right)^{(k+1)n} = \left( \frac{1}{2} \right)^{(k+1)n+1}
\]
as claimed.

Claim 3. \(|z_n - \xi| \leq 4(\frac{1}{2})^{(k+1)n}|z_0 - \xi|\).

First note that \( z_n \) defined here has \( H_n > 8 > 1/r_k \) (see the proof of Claim 2). Hence \( \sigma^{-1} \) is well defined at all \( z_n \), and we have \( \xi = z_n + F(z_n)\sigma^{-1}(1) \) and \( z_n - \xi = F(z_n)\sigma^{-1}(1) \), where \( \sigma \) is the polynomial associated with \( f \) and \( z_n \).

By Corollary 2.6(1) we note that
\[
\frac{|F(z_n)|}{(1 + 1/H_n)^2} \leq |z_n - \xi| \leq \frac{|F(z_n)|}{(1 - 1/H_n)^2}.
\]
Hence we have
\[
|z_n - \xi| \leq \frac{|F(z_n)|}{(1 - 1/H_n)^2} (1 + 1/H_0)^2 |z_0 - \xi| = \frac{|f_n|}{|f_0|} \frac{|f_0|}{(1 - 1/H_n)^2} |z_0 - \xi|.
\]
We note that \((1 + 1/H_0)^2/(1 - 1/H_n)^2 \leq 1.5\), since \( 1/H_0 \leq 0.122 \) and \( 1/H_n \leq 1/28 \) (see the proof of Claim 2). Further, we claim that \(|f_0'|/|f_n'| \leq (1 + 1/7)^3/(1 - 1/7) \leq 1.8\), so that we have \(|z_n - \xi| \leq 4(\frac{1}{2})^{(k+1)n}|z_0 - \xi|\). To see this, note that \( z_n = f_{z_0}^{-1}((1 - h) f(z_0)) \) for \(|h| = |f(z_n) - f(z_0)|/|f_0| \leq 9/8 \) and \(|h|/H_0 \leq (9/8)/8.23 = 1/7\). Now apply Corollary 2.7(2) with \( r = 1/7\); we have \(|f_0'|/|f_n'| \leq (1 + r)/(1 - r)^3 \leq 1.8\). Hence we have completed Claim 3. \(\square\)

5. Algorithms. The main goal of this section is to construct new algorithms to find a root of a polynomial. Applied to any polynomial \( f \), these new algorithms always converge to a root or a critical point of \( f \). The underlying idea is that, for an initial point \( z_0 \), one analytically continues \( f_{z_0}^{-1} \) toward 0 in a radial direction as long as it is possible. The idea used to determine the approximate zero in Section 3 is also useful.

As mentioned in Section 1, the radius of convergence (or equivalently, \( a_{f,z} \)) plays an important role as a successive overrelaxation parameter in our algorithms.

Recall that
\[
a_{f,z} = \max_{j \geq 2} \left| \frac{f(z)}{f'(z)} \right| \left| \frac{f^{(j)}(z)}{j!f'(z)} \right|^{1/(j-1)}
\]
from Section 4.

Now we describe the algorithms.

ALGORITHM \( A_k \). For a polynomial \( f \) and a complex number \( z_0 \in C \), define iteratively,
\[
z_{n+1} = E_{k,h_n,f}(z_n), \quad \text{where} \quad h_n = \operatorname{Min} \left( 1, \frac{1}{48a_{f,z_n}} \right).
\]
For example, if \( k = 1 \) we have \( z_{n+1} = z_n - h_n(f(z_n)/f'(z_n)) \).

**ALGORITHM B_k.** For a polynomial \( f \) and \( z_0 \in \mathbb{C} \), let \( w_0 = f(z_0) \). Define iteratively

\[
z_{n+1} = E_{k,1,g_n}(z_n),
\]

where \( g_n = f - w_{n+1} \), \( w_{n+1} = (1 - h_n)w_n \), and \( h_n = \text{Min}(1, 1/1800a_{f,z_n}) \).

*Remark.* Note that

\[
z_{n+1} = E_{k,1,g_n}(z_n) = E_{k,h,f}(z_n),
\]

where \( h = (f(z_n) - w_{n+1})/f(z_n) \) by Lemma 2.8(3). For example, if \( k = 1 \) we have

\[
z_{n+1} = z_n - (f(z_n) - w_{n+1})/f'(z_n).
\]

**THEOREM 5.A.** \( z_n \) in Algorithm \( A_k \) always converges to a root or a critical point of \( f \).

*Proof.* Note that once \( a_{f,z_n} \leq 1/48 \) (i.e., \( h_n = 1 \)) then \( z_n \) is an approximate zero of \( f \) and converges to a root of \( f \) by Theorem 4.4. We may assume that \( a_{f,z_n} > 1/48 \) and hence \( h_n = 1/1800a_{f,z_n} \). Applying Theorem 3.2 with \( h_n \), we obtain \( |f(z_{n+1})|/|f(z_n)| \leq 1 - h_n \), where \( |h_n' - h_n| \leq B(k)h_n \leq 3/4h_n \) for all \( k \). Inductively one has \( f(z_N)/f(z_0) = \prod (1 - h_n') \), where \( |h_n' - h_n| \leq 3/4h_n \).

Notice that \( |f(z_n)|/|f(z_0)| \) converges always since it is decreasing. We will show that \( z_n \) converges to a critical point of \( f \), if \( |f(z_n)|/|f(z_0)| \) converges to a nonzero number. Recall from the theory of infinite products that this implies that \( \sum b_n \) is bounded, where \( 1 - b_n = |1 - h_n'| \). Note that \( \sum h_n \) and \( \sum |h_n'| \) are also bounded since \( h_n \geq h_n' \), \( -h_n' - h_n \) is \( 1/4 \) \( h_n \geq 1/16 \) \( h_n' \) by (1). Again by the theory of infinite products we have \( \prod (1 - h_n') \to w \), a nonzero complex number. This \( w \) is a critical value of \( f \) since \( h_n \to 0 \) and \( |f'(z_n)| \to 0 \) by the definitions of \( h_n \) and \( a_{f,z_n} \). Further, we claim that \( |z_{n+1} - z_n| \to 0 \) and \( z_n \) converges to a critical point \( \theta \). To see this, just note that

\[
a_{f,z} = \max_{j=2,\ldots,d} \frac{f(z)}{f'(z)} \left| \frac{f^{(j)}(z)}{j!f'(z)} \right|^{1/(j-1)} \geq \frac{f(z)}{f'(z)} \left| \frac{f^{(d)}(z)}{d!f'(z)} \right|^{1/(d-1)}.
\]

Hence

\[
|z_{n+1} - z_n| = h_n |f(z_n)|/|f'(z_n)| \leq \frac{1}{48} |f'(z_n)|^{1/(d-1)} \to 0.
\]

Since there are finite preimages of \( w \), we conclude that \( z_n \to \theta \) where \( w = f(\theta) \).

**THEOREM 5.B.** For any polynomial \( f \) and \( z_0 \in \mathbb{C} \), \( z_n \) in Algorithm \( B_k \) converges to a root or a critical point of \( f \). Further, \( z_n \) converges to a root unless there is a critical value of \( f \) on the ray \((0, f(z_0))\).

*Proof.* It is easy to see that once \( h_n = 1 \) (i.e., \( a_{f,z_n} \leq 1/1800 \leq 1/48 \)) then \( z_n \) is an approximate zero of \( f \) and hence \( z_n \) converges to a root of \( f \) by Theorem 4.4. Note that if \( H_n \geq 7200 \) then \( h_n = 1 \), and \( z_n \) is an approximate zero by Corollary 4.3. We will show inductively that either \( z_n \) is an approximate zero or \( z_n \) satisfies the bound

\[
\frac{w_n}{f(z_n)} = 1 + \varepsilon_n, \quad |\varepsilon_n| \leq \frac{H_n}{14400} \leq \frac{1}{2}.
\]
For simplicity, we denote $f_n = f(z_n)$, $R_n = R_{f,z_n}$, $H_n = R_n/|f_n|$. We claim that (1) completes the proof: Recall that $R_n = |f_n - f(\theta^*)|$ for some critical point $\theta^*$ by Lemma 2.8(1) and that $H_{f,z_n}/7200 \leq h_n \leq H_{f,z_n}/308$ for $h_n < 1$ by Corollary 4.3. Now in the case $h_n < 1$,

$$
\frac{|w_n - f(\theta^*)|}{|w_n|} = \frac{|f_n|}{|w_n|} \frac{|w_n - f_n| + |f_n - f(\theta^*)|}{|f_n|} \leq \frac{1}{|1 + \varepsilon_n|} (|\varepsilon_n| + H_n)
$$

$$
\leq 2 \left( \frac{H_n}{14400} + H_n \right) \text{ since } |\varepsilon_n| \leq \frac{1}{2}
$$

$$
\leq 2.5 H_n \leq 20000h_n.
$$

Using the same argument as in Theorem 5.3, $w_n = \prod_{m=1}^n (1 - h_m)$ converges to a nonzero number only if $h_n \to 0$ and hence only if $|w_n - f(\theta^*)| \to 0$. Since there is no critical value on $(0, w_0]$, this is possible only if $w_n \to f(\theta^*) = 0$. Again using the same argument as in Theorem 5.3, we conclude that $z_n \to \theta^*$ where $f(\theta^*) = 0$. Now we start an induction to show (1). Suppose $f_n/w_n = 1 + \varepsilon_n$, $|\varepsilon_n| \leq H_n/14400 \leq 1/2$. Then we will show that either $z_{n+1}$ is an approximate zero or it satisfies $f_{n+1}/w_{n+1} = 1 + \varepsilon_{n+1}$, $|\varepsilon_{n+1}| \leq H_{n+1}/14400 \leq 1/2$. Recall that $z_{n+1} = E_{k,h,f}(z_n)$, where $h = (f_n - w_{n+1})/f_n$ and $w_{n+1} = (1 - h_n)w_n$. Note that

$$
|h| = \frac{|f_n - w_{n+1}|}{|f_n|} = \frac{|f_n - (1 - h_n)w_n|}{|f_n|} = \frac{|f_n - (1 - h_n)(1 + \varepsilon_n)f_n|}{|f_n|} = |1 - (1 - h_n)(1 + \varepsilon_n)| = |h_n - \varepsilon_n(1 - h_n)| \leq h_n + |\varepsilon_n|
$$

$$
\leq \frac{H_n}{308} + \frac{H_n}{14400} \leq \frac{H_n}{300}.
$$

Applying Theorem 3.2 to $z_n$ with $h$, we have

$$
\frac{f_{n+1}}{f_n} = 1 - h + h\delta, \quad \text{ where } |\delta| \leq B_k \left( \frac{1}{300} \right) \leq \frac{1}{145} \text{ for all } k,
$$

$$
\frac{f_{n+1}}{w_{n+1}} = 1 + \frac{h\delta}{1-h}, \quad \text{ since } w_{n+1} = (1-h)f_n,
$$

and

$$
\frac{w_{n+1}}{f_{n+1}} = 1 + \varepsilon_{n+1} = \frac{1}{1 + \mu}, \quad \mu = \frac{h\delta}{1-h}.
$$

Note that

$$
H_{n+1} = \frac{R_{n+1}}{|f_{n+1}|} \geq \frac{|f_{n+1}|}{|f_{n+1}|} \frac{R_n - |f_n - f_{n+1}|}{|f_n|} \text{ by Lemma 2.8(2)}
$$

$$
\geq \frac{1}{|1 - h + h\delta|} |H_n - |h - h\delta||
$$

$$
\geq \frac{1}{|1 - h + h\delta|} \left( H_n - \frac{H_n}{300} \left( 1 + \frac{145}{145} \right) \right)
$$

$$
\geq \frac{296}{297} \frac{H_n}{|1 - h + h\delta|}.
$$

Now

$$
|\mu| = \left| \frac{h\delta}{1-h} \right| \leq \left| \frac{H_n}{300} \frac{145}{145} \right| \frac{1}{|1-h|} \leq \left| \frac{1 - h + h\delta}{1-h} \right| \frac{H_{n+1}}{296} \frac{1}{300} \frac{1}{145}
$$

$$
= \left| 1 + \mu \right| \frac{H_{n+1}}{43000}.
$$
Note that if $|\mu| \geq \frac{1}{4}$, then

$$H_{n+1} \geq \frac{|\mu|}{1 + |\mu|} 43000 \geq 7200$$

and hence $z_{n+1}$ is an approximate zero. If $z_{n+1}$ is not an approximate zero then $H_{n+1} < 7200$ and $|\mu| < \frac{1}{4}$. Hence we have

$$|\varepsilon_{n+1}| \leq 2|\mu| \leq 2 \cdot \frac{5}{4} \cdot \frac{H_{n+1}}{43000} \leq \frac{H_{n+1}}{30000} \leq \frac{H_{n+1}}{14400} \leq \frac{1}{2}. \quad \square$$

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