On Approximate Zeros and Rootfinding Algorithms for a Complex Polynomial

By Myong-Hi Kim

Abstract. In this paper we give criteria for a complex number to be an approximate zero of a polynomial \( f \) for Newton's method or for the \( k \)-th-order Euler method. An approximate zero for the \( k \)-th-order Euler method is an initial point from which the method converges with an order \( (k+1) \). Also, we construct families of Newton (and Euler) type algorithms which are surely convergent.

1. Introduction. Newton’s method has long been used for solving a nonlinear equation \( f(z) = 0 \). The Newton method attempts to solve \( f(z) = 0 \) by an iteratively defined sequence \( z_{n+1} = z_n - f(z_n)/f'(z_n) \), for an initial point \( z_0 \). It indeed converges to a root at a fast rate, if it starts with a good initial point. However, not much is known about the region of convergence or of fast convergence, and it is difficult to obtain a priori knowledge of convergence.

In this paper we study the efficiency and the convergence properties of the Newton method and other generalized methods for solving a polynomial equation \( f(z) = 0 \). We have two main goals. First, we establish an estimate for a point \( z_0 \), which predicts fast convergence of the algorithms starting at \( z_0 \). Secondly, we develop a method which is guaranteed to converge, given an arbitrary initial point \( z_0 \).

Following Shub and Smale, we consider the following generalized version of the Newton method, called the modified \( k \)-th-order Euler method.

We recall from elementary complex analysis that for a polynomial \( f \) and \( z \in \mathbb{C} \) such that \( f'(z) \neq 0 \), there is a well-defined local inverse branch \( f_z^{-1} \) of \( f \) such that \( f_z^{-1}(f(z)) = z \).

Definition 1.1. For an integer \( k \) and a complex number \( h \), the Euler method iteratively defines a sequence \( z_{n+1} = E_{k,h,f}(z_n) = T_k f_z^{-1}((1-h)f(z_n)) \) for an initial point \( z_0 \), where \( T_k \) is the \( k \)-th-order truncation of \( f_z^{-1} \) considered as a power series about \( f(z_0) \).

For brevity we denote \( E_{k,h,f} \) by \( E_k \) if there is no confusion. Note that \( E_{1,1,f} \) gives the Newton method.

We define an approximate zero of \( f \) for \( E_k \) as follows.
Definition 1.2. $z_0$ is an approximate zero of $f$ for $E_k$ if

\begin{align}
(1) & \quad \frac{|f(z_n)|}{|f(z_0)|} \leq \left(\frac{1}{2}\right)^{(k+1)n}, \\
(2) & \quad |z_n - \xi| \leq c \left(\frac{1}{2}\right)^{(k+1)n} |z_0 - \xi|,
\end{align}

where $c$ is a constant, $\xi$ is a root and $z_{n+1} = E_{k,1} f(z_n) \to \xi$. □

Note the fast convergence of $z_n$ to $\xi$. This notion is due to Smale [12].

For a polynomial $f$ and $z \in \mathbb{C}$, let

\[ a_{f,z} = \max_{j \geq 2} \left| \frac{f(z)}{f'(z)} \right| |f'(z)|^{1/(j-1)} \]

We show that $z$ is an approximate zero of $f$ for all $E_k$ if $a_{f,z} \leq \frac{1}{48}$ (see Theorem 4.4). Recently, Smale [14] has obtained a similar result for $k = 1$ with a better estimate (a constant $\alpha_0$, in his notation, between $\frac{1}{8}$ and $\frac{1}{2}$) for a more general class of polynomial maps $f: \mathbb{C}^n \to \mathbb{C}^n$.

The estimate for $a_{f,z}$ plays an important role even when $z$ is not an approximate zero of $f$. It suggests the next iterate in the construction of algorithms which produces sure convergence.

We construct two families of modified Euler methods $A_k$ and $B_k$, which always converge to a root or a critical point of $f$ (see Theorems 5.A and 5.B). A is called a critical point of $f$ if $f'(\theta) = 0$. As in the work of Shub and Smale ([9], [10]), the idea is to approximate the solution curve $\phi_t(z_0)$ to the Newton vector field $F(z) = -f(z)/f'(z)$ where $\phi_0(z_0) = z_0$. Note that $f(\phi_t(z_0)) = e^{-t} f(z_0)$, a straight line through $f(z_0)$. Hence one can approximate a root by approximating $f^{-1}(e^{-t} f(z_0); t \to \infty)$. To do so, Shub and Smale use the modified Euler method with a fixed step size $h$ in $E_{k,h,f}$ together with a probabilistic estimate on the set of initial points. In our algorithms we use a varying step size $h$ at each point $z$, where $h$ is given in terms of $a_{f,z}$ and hence related to the radius of convergence of $f_z^{-1}$. In particular, we show that for any polynomial $f$ and initial point $z_0$, $B_k$ always produces a sequence $z_n$ converging to a root unless there is a critical value of $f$ on the ray $(0, f(z_0)]$ (see Theorem 5.B). Recently, Shub and Smale [11] have shown that an algorithm similar to $A_1$ converges to a root for almost all polynomials and for almost all initial points $z_0$.

We have run some experiments on the algorithm $A_1$ and other similar algorithms with a starting point 0 and with a supplementary algorithm of Shub and Smale [10] for degenerate cases such as $a_{f,z} \geq 50d^2$. This corresponds to the case where $z$ is near a critical point. Among $(100 \cdot d^2)$ randomly selected polynomials of each degree $d \leq 100$ with complex coefficients $|a_i| \leq 1$, the average number of iterations to locate an approximate zero or to locate $\xi$ such that $|f(\xi)| \leq 10^{-4}$ is found to be less than 200. Our experimental result is independent of the degree $d$.

2. Preliminaries. In this section we discuss some preliminary material needed in the later sections on the local behavior of analytic functions.

The main tools used in Section 3 are from the theory of schlicht functions. $f$ is called a schlicht function if $f(0) = 0$, $f'(0) = 1$ and it is univalent on $D_1(0)$, the unit disk at 0. A univalent function is a one-to-one complex analytic function.
To each $z \in \mathbb{C}$ and complex polynomial $f$ such that $f(z) \neq 0$ and $f'(z) \neq 0$, one associates a normalized polynomial $\sigma$ by means of

$$\sigma(w) = w + \sigma_2 w^2 + \cdots + \sigma_d w^d, \quad \text{where} \quad \sigma_j = \left(\frac{-f(z)}{f'(z)}\right)^{j-1} \frac{f^{(j)}(z)}{j! f'(z)}.$$

Let $R_{f,z}$ be the radius of convergence of $f_z^{-1}$, considered as a power series at $f(z)$.

For $f(z) \neq 0$, let $R_{f,z} = R_{f,z}/|f(z)|$; see Figure 2.1. The following lemma is extracted from the work of Shub and Smale (see [9, p. 113]).

**Lemma 2.1.** Let $\sigma^{-1}$ be the inverse branch of $\sigma$ taking $0$ to $0$. Then

1. $\sigma^{-1}(0) = 0$, $\sigma^{-1}'(0) = 1$.
2. Let $x = f_z^{-1}((1 - h)f(z))$ with $|h| < H_{f,z}$. Then
   $$\frac{f(x)}{f(z)} = 1 - \sigma \circ \varepsilon, \quad \text{where} \quad \varepsilon = \frac{x - z}{F(z)} \quad \text{and} \quad F(z) = \frac{-f(z)}{f'(z)}.$$
3. $f_z^{-1}((1 - h)f(z)) = z + F(z)\sigma^{-1}(h)$.
4. $T_k f_z^{-1}((1 - h)f(z)) = z + F(z)T_k\sigma^{-1}(h)$.
5. The radius of convergence of $\sigma^{-1}$ at $0$ is $R_{\sigma,0} \equiv H_{f,z}$.
6. $\frac{1}{H}\sigma^{-1}(Hh)$ is schlicht, $H \equiv H_{f,z}$.

**Proof.** (1) is immediate. (2) is from Proposition 2 in [9]. For (3), (4) and (5), see [9, p. 114] and [10, p. 153]. (6) is a trivial consequence of (1) and (5). □

Using Lemma 2.1, we may reformulate Definition 1.1 of $E_{k,h,f}$ as follows.

**Definition 2.2.** $E_{k,h,f}(z) = z + F(z)T_k\sigma^{-1}(h)$.

We will need the following properties.

**Lemma 2.3** (De Branges’ Theorem: Bieberbach conjecture). Let $g(z) = z + g_2 z^2 + g_3 z^3 + \cdots$ be schlicht. Then $|g_k| \leq k$.

**Proof.** See [2]. □

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**Figure 2.1.** $R = R_{f,z}$, $H = H_{f,z}$

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Lemma 2.4 (Shub and Smale). (1) Let $g$ be schlicht. Then $|g(h) - T_k g(h)| \leq (k+1)r^{k+1}/(1-r)^2$, where $r = |h| < 1$.

(2) Let $g$ be univalent on $D_H(0)$, $g(0) = 0$ and $g'(0) = 1$. Then for $h$ with $r = |h|/H < 1$,

$$|g(h) - T_k g(h)| \leq \frac{H(k+1)r^{k+1}}{(1-r)^2}.$$

Proof. From Lemma 2.3 we have

$$|g(h) - T_k g(h)| \leq \sum_{j=k+1}^{\infty} j^r j^r \leq r \left( \frac{r^{k+1}}{1-r} \right) \leq \frac{(k+1)r^{k+1}}{(1-r)^2}.$$

For the second statement, note that $\frac{1}{H} g(Hh)$ is schlicht, and then use (1). □

Lemma 2.5 (Koebe Distortion Theorem). Let $g$ be schlicht. Then for $|h| = r < 1$,

(1) $$\frac{r}{(1+r)^2} \leq |g(h)| \leq \frac{r}{(1-r)^2},$$

(2) $$\frac{1-r}{(1+r)^3} \leq |g'(h)| \leq \frac{1+r}{(1-r)^3}.$$

Proof. See [4, Vol. 2, pp. 351 and 353]. □

By rescaling, we obtain immediately the following

Corollary 2.6. Let $g$ be univalent on $D_H(0)$ and $g(0) = 0$, $g'(0) = 1$. Let $r = |h|/H < 1$. Then

(1) $$\frac{|h|}{(1+r)^2} \leq |g(h)| \leq \frac{|h|}{(1-r)^2},$$

(2) $$\frac{1-r}{(1+r)^3} \leq |g'(h)| \leq \frac{1+r}{(1-r)^3}.$$

Proof. Note that $\frac{1}{H} g(Hh)$ is schlicht. Now use Lemma 2.5. □

Corollary 2.7. Let $x = f^{-1}_z((1-h)f(z)) \equiv z + F(z)\sigma^{-1}(h)$ for $|h| < H_{f,z}$. Then we have

(1) $$f'(x) = f'(z)\sigma'(\varepsilon) \equiv \frac{f'(z)}{\sigma^{-1'}(h)}, \quad \text{where } \varepsilon = \frac{x-z}{F(z)},$$

(2) $$|f'(z)|\frac{(1-r)^3}{1+r} \leq |f'(x)| \leq |f'(z)|\frac{(1+r)^3}{1-r}, \quad \text{where } r = \frac{|h|}{H_{f,z}}.$$

Proof. Recall from Lemma 2.1(2) that $f(x) = f(z)(1 - \sigma(\varepsilon))$ and $\sigma(\varepsilon) = h \equiv (f(z) - f(x))/f(z)$. Hence (1) is immediate by taking derivatives of $f$. (2) follows from Corollary 2.6(2) since $\sigma^{-1}(0) = 0$, $\sigma^{-1'}(0) = 1$ and $\sigma^{-1}$ is univalent in $D_H(0)$. □

We close this section with the following lemma.

Lemma 2.8. (1) $R_{f,z} = |f(z) - f(\theta^*)| \geq \min_{f'((\theta)=0} |f(z) - f(\theta)|$ for some critical point $\theta^*$ of $f$.

(2) Let $x = f^{-1}_z((1-h)f(z))$ with $|h|/H_{f,z} < 1$. Then $R_{f,z} \geq R_{f,z} - |f(z) - f(x)|$. 
(3) Let \( g = f - y \) be a translation of \( f \) by \( y \in \mathbb{C} \). Then \( E_{k,h',g}(x) = E_{k,h,f}(x) \), where \( h' = hf(z)/g(z) \).

**Proof.** For (1), see Lemma 3 in [12].

For (2), we note that by the uniqueness of analytic maps, we have \( f_{-1}^z \equiv f_{-1}^z \) on their common domain of definitions. In particular, \( f_{-1}^z \) is analytically continued for all \( w \) such that \( |w - f(z)| < R_{f,z} \). Since \( |w - f(z)| < |f(z) - f(x)| < R_{f,z} \), \( f_{-1}^z \) is analytic for all \( w \) such that \( |w - f(z)| < R_{f,z} - |f(z) - f(x)| \). Hence \( R_{f,z} \geq R_{f,z} - |f(z) - f(x)| \).

For (3), note that \( g_{-1}^z(w - y) \) is well defined where \( f_{-1}^z(w) \) is well defined and \( g_{-1}^z(w - y) = f_{-1}^z(w) \). As power series at \( f(z) \) and \( g(z) \) respectively, we have

\[
f_{-1}^z(f(z) - w) \equiv g_{-1}^z(f(z) - w - y) \equiv g_{-1}^z(g(z) - w),
\]

where \( w = hf(z) = h'g(z) \) and \( h' = h(f(z)/g(z)) \). Hence we also have

\[
T_k f_{-1}^z((1-h)f(z)) = T_k g_{-1}^z((1-h')g(z)) \quad \text{and} \quad E_{k,h',g}(z) = E_{k,h,f}(z). \quad \square
\]

### 3. Koebe Distortion Theorem and Euler Iteration

We recall that

\[
E_{k,h,f}(z) = T_k f_{-1}^z(1-h)f(z)).
\]

In this section we show that \( E_{k,h,f} \) approximates \( f_{-1}^z \) with a suitable \( h \), i.e., \( E_{k,h,f}(z) = f_{-1}^z(w) \) for \( w \) such that \( |w - f(z)| < R_{f,z} \) and \( R_{f,z} \) is the radius of convergence of \( f_{-1}^z \). In particular, we show that \( E_{k,h,f} \) approximates \( f_{-1}^z \) for all values on the disk of convergence as \( k \uparrow \infty \). The main goal of this section is to prove Theorem 3.2 below.

We recall that \( H_{f,z} = R_{f,z}/|f(z)| \), where \( R_{f,z} \) denotes the radius of convergence of \( f_{-1}^z \) at \( f(z) \).

**Theorem 3.1.** Let \( x = f_{-1}^z((1-h)f(z)) \). Assume that

\[
r = \frac{|h|}{H_{f,z}} < 1 \quad \text{and} \quad t \leq \frac{|h|(1-r)^3}{(1+r)^3}.
\]

Then \( D_{tf(z)}(x) \subseteq f_{-1}^z(D_{tf(z)}(f(z))) \), where

\[
s = \min \left\{ t \frac{(1+r)^3}{(1-r)^3}, \ H_{f,z}(1-r) \right\} \quad \text{and} \quad F = -f(z) f'(z).
\]

The proof will be given later. \( \square \)

Let

\[
B_k(r) = (k+1)(1+r)^3 \frac{r^k}{(1-r)^5}
\]

and \( r_k \) be the smallest positive solution to \( B_k(r) = 1 \). Note that \( B_k(r) \) is increasing on \([0, r_k] \). The condition that \( |h| < r_k H_{f,z} \) is crucial for \( E_{k,h,f} \) to approximate \( f_{-1}^z \).

**Theorem 3.2.** Let \( z' = E_{k,h,f}(z) \) with \( r = |h|/H_{f,z} < r_k \). Then we have

\[
z' = f_{-1}^z((1-h')f(z)) \quad \text{and} \quad f(z')/f(z) = 1 - h + \varepsilon, \quad \text{where} \quad |\varepsilon| = |h - h'| \leq \min\{|h|B_k(r), H_{f,z}(1-r)\}.
\]

A table of approximate values of \( r_k \) is given below.

<table>
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<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>177</th>
<th>3303</th>
<th>47400</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_k )</td>
<td>.148</td>
<td>.225</td>
<td>.282</td>
<td>.329</td>
<td>.367</td>
<td>.495</td>
<td>.9</td>
<td>.99</td>
<td>.999</td>
</tr>
</tbody>
</table>

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Remark. We note that \( r_k \uparrow 1 \) as \( k \uparrow \infty \). In [9], Shub and Smale showed that Theorem 3.2 holds for all \( r \leq \gamma_k \) where \( \gamma_k \uparrow 0.175 \) as \( k \uparrow \infty \).

Proof of Theorem 3.2. Recall that \( z' = E_{k,h,f}(z) = z + F(z)T_k^{-1}(h) \), where \( F(z) = -f(z)/f'(z) \). Let \( f(z') = (1 - h')f(z) \). Let \( x = f^{-1}_z((1 - h)f(z)) = z + F(z)T_k^{-1}(h) \). Then \( |z' - x| = |F| |\sigma^{-1}(h) - T_k^{-1}(h)| \). Since \( \sigma^{-1} \) is univalent on \( D_H(0) \), we have by Lemma 2.4,

\[
|\tau - \sigma^{-1}(h) - T_k^{-1}(h)| \leq \frac{H_{f,z}(k + 1)r^{k+1}}{(1 - r)^2}
\]

Now, by Theorem 3.1, we have \( z' \in f^{-1}_z(D_{f(z)\delta}(f(z))) \) and

\[
|f(z') - f(x)| = |(1 - h')f(z) - (1 - h)f(z)| = |f(z)||h' - h| \leq |f(z)|\delta.
\]

Hence,

\[
|\tau| = |h' - h| \leq \delta = \min\{|h|B_k(r), H_{f,z}(1 - r)\}
\]

and we have \( z' = f^{-1}_z((1 - h')f(z)) \) and \( f(z')/f(z) = 1 - h' \) for some \( h' \) with \( |h' - h| \leq \delta \).

We need the following lemmas to prove Theorem 3.1.

**Lemma 3.3.** (1) Let \( g \) be univalent on \( D_R(z) \). Then \( D_{Rt}(g(z)) \subset g(D_{Rs}(z)) \subset D_{Ru}(g(z)) \), for any \( s < 1, t = s|g'(z)|/(1 + s)^2 \) and \( u = s|g'(z)|/(1 - s)^2 \).

(2) Suppose that \( g \) is univalent on \( D_H(0) \), \( g(0) = 0 \) and \( g'(0) = 1 \). Let \( z \in D_H(0) \), where \( r = |z|/H \). Then for \( s \leq 1 - r \) we have \( D_{Ht}(g(z)) \subset g(D_{Hs}(z)) \subset D_{Hu}(g(z)) \), where \( t = ((1 - r)^3/(1 + r)^3) \cdot s/(1 - r + s)^2 \) and \( u = ((1 + r)/(1 - r)) \cdot s/(1 - r - s)^2 \).

**Figure 3.2.** \( w = g(z) \)

Proof. (1) Let

\[
\psi(h) = \frac{1}{Rg'(z)}(g(z + Rh) - g(z)).
\]

Then it is easy to see that \( \psi \) is schlicht. Hence by Lemma 2.5, \( \delta/(1 + \delta)^2 \leq |\psi(h)| \leq \delta/(1 - \delta)^2 \) for \( |h| = \delta \), so that we have

\[
D_{R\delta}(g(z)) \subset g(D_{R\delta}(z)) \subset D_{R\eta}(g(z)),
\]
where \( R\mu = \delta R|g'(z)|/(1 + \delta)^2 \), \( R\eta = \delta R|g'(z)|/(1 - \delta)^2 \). By setting \( t = \mu \), \( s = \delta \) and \( u = \eta \), (1) is established.

For (2), note that \( g \) is univalent on \( D_R(z) \), where \( R = H(1 - r) \), and hence \((1 - r)/(1 + r)^3 \leq |g'(z)| \leq (1 + r)/(1 - r)^3 \) by Corollary 2.6. Let \( s = (1 - r)\delta \). Then we have

\[
R\mu \geq \frac{\delta H(1 - r)}{(1 + \delta)^2} \frac{1 - r}{(1 + r)^3} = \frac{Hs}{(1 - r + s)^2} \frac{(1 - r)^3}{(1 + r)^3} \equiv Ht,
\]

\[
R\eta \leq \frac{\delta H(1 - r)}{(1 - \delta)^2} \frac{1 + r}{(1 - r)^3} \leq \frac{Hs}{(1 - r - s)^2} \frac{1 + r}{1 - r} \equiv Hu,
\]

where

\[
t = \frac{(1 - r)^3}{(1 + r)^3} \frac{s}{1 + r} \quad \text{and} \quad u = \frac{1 + r}{1 - r} \frac{s}{1 - r - s^2}.
\]

Hence we have \( D_Ht(g(z)) \subset g(D_{Hs}(z)) \subset D_{Hu}(g(z)) \).

Lemma 3.3 gives the following Quarter Theorem at an arbitrary point \( z \in D_H(0) \).

**Corollary 3.4.** Let \( g \) be univalent on \( D_H(0) \) and \( g(0) = 0 \) and \( g'(0) = 1 \). For \( z \in D_H(0) \), let \( r = |z|/H \). Then \( D_Ht(g(z)) \subset g(D_{H(1-r)}(z)) \), where \( t = \frac{1}{2}(1 - r)^2/(1 + r)^3 \).

**Proof.** Use \( s = 1 - r \) in Lemma 3.3(2). \( \square \)

In Lemma 3.3(2) we will also need to estimate \( s \) as a function of \( t \).

**Corollary 3.5.** Suppose \( t \leq r((1 - r)^3/(1 + r)^3) \), where \( r = |z|/H < 1 \). Then \( D_Ht(g(z)) \subset g(D_{Hs}(z)) \), where \( s = \text{Min}\{t((1 + r)^3/(1 - r)^3), 1 - r\} \).

**Proof.** Since \( r(1 - r) \leq \frac{1}{4} \), we have \( t \leq \frac{1}{4}((1 - r)^2/(1 + r)^3) \). Hence by Corollary 3.4 we have \( D_Ht(g(z)) \subset D_{H(1-r)}(g(z)) \). Let \( t' = ((1 - r)^3/(1 + r)^3) \cdot s/(1 - r + s)^2 \). Since \( s \leq 1 - r \), Lemma 3.3(2) shows that \( D_Ht'(g(z)) \subset g(D_{Hs}(z)) \). However, since \( s \leq t((1 - r)^3/(1 + r)^3) \leq r \), we have

\[
t' = \frac{(1 - r)^3}{(1 + r)^3} \frac{s}{1 - r + s^2} \geq \frac{(1 - r)^3}{(1 + r)^3} s = t.
\]

Consequently, \( D_Ht(g(z)) \subset D_Ht'(g(z)) \subset g(D_{Hs}(z)) \), as claimed. \( \square \)

The proof of Theorem 3.1 now follows easily from Corollary 3.5.

**Proof of Theorem 3.1.** Suppose that \( z = f_z^{-1}((1 - h')f(z)) \) where \( |z' - x| \leq |F|t \).

Since

\[
z' = f_z^{-1}((1 - h')f(z)) = z + F(z)\sigma^{-1}(h') = z + F(z)\sigma^{-1}(h) + F(z)(\sigma^{-1}(h') - \sigma^{-1}(h)) = x + F(z)(\sigma^{-1}(h') - \sigma^{-1}(h)),
\]

we have

\[
|z' - x| = |F||\sigma^{-1}(h') - \sigma^{-1}(h)| \leq |F|t = |F|Ht', \quad \text{where} \quad t' = \frac{t}{H} \leq \frac{|h|(1 - r)^3}{H(1 + r)^3} \leq \frac{r(1 - r)^3}{(1 + r)^3},
\]

by the hypothesis. Hence, by Corollary 3.5, we have \( |h' - h| \leq Hs' \), where \( s' = \text{Min}\{t'((1 + r)^3/(1 - r)^3), 1 - r\} \). Now by setting \( s = Hs' \) we have the claim. \( \square \)
4. Domain of Injectivity and a Notion of an Approximate Zero. The main goal of this section is to give a criterion to determine an approximate zero of a polynomial $f$ for the modified Euler method. Hereafter we will denote $E_{k,h,f}$ by $E_k$ if there is no confusion.

Definition. $z_0$ is an approximate zero of $f$ for $E_k$ if

\[ \frac{|f(z_n)|}{|f(z_0)|} \leq \left( \frac{1}{2} \right)^{(k+1)n}, \]

\[ |z_n - \xi| \leq c \left( \frac{1}{2} \right)^{(k+1)n} |z_0 - \xi|, \]

where $z_n = E_{k,1,f}(z_0) \to \xi$ and $c$ is a constant.

We will need the following estimate of the domain of injectivity, which itself is quite interesting.

**Theorem 4.1.** Let $g(z) = z + a_2z^2 + \cdots$ be a power series and $\psi$ be the compositional inverse of $g$ taking $0$ to $0$. Let $a = \sup z |a(z)|^{1/(i-1)}$. Then $\psi$ is well defined, analytic and one-to-one on $D_R(0)$, where $(3 - \sqrt{8})/a \leq R$.

**Proof.** Suppose that $|g(z) - z| < r$ on $|z| = r$. Then $0$ is the only root of $g$ in $D_r(0)$ by Rouché's Theorem. It follows that (see [1, Theorem 11, p. 131]) the inverse map $\psi$ is well defined on $g(D_r(0))$. In particular, $\psi$ is well defined on $D_R(0)$, where $R = \min |z| = r |g(z)|$. Now,

\[ |g(z)| = |z| |1 + a_2z + a_3z^2 + \cdots| \]

\[ \geq r |1 - ((a^2 + (ar)^2 + (ar)^3 + \cdots)| \]

\[ \geq r \left( 1 - \frac{ar}{1 - ar} \right) \text{ on } |z| = r. \]

But $r(1 - ar/(1 - ar))$ achieves the maximum $(3 - \sqrt{8})/a$ when $r = (2 - \sqrt{2})/2a$. Also note that

\[ |g(z) - z| = |a_2z^2 + a_3z^3 + \cdots| = |z| |a_2z + a_3z^2 + \cdots| \]

\[ \leq r \frac{ar}{1 - ar} < r, \quad \text{on } |z| = \frac{2 - \sqrt{2}}{2a}. \]

Hence $\psi$ is well defined and injective on $D_R(0)$, where $R = (3 - \sqrt{8})/a \approx 1/5.83a > 1/6a$. \hfill \Box

**Remark 4.2.** The corresponding upper bound $R \leq 4/a$ is obtained in [12, p. 9, Extended Loewner’s Theorem]. For a polynomial $f$ and $z \in \mathbb{C}$ we define

\[ a_{f,z} \equiv \max_{j \geq 2} \left| \frac{f(z)}{f'(z)} \right|^j. \]

We apply Theorem 4.1 to a polynomial.

**Corollary 4.3.** Let $f$ be a polynomial of degree $d$ and $z$ be a complex number such that $f'(z) \neq 0$, and $f(z) \neq 0$. Let $f^{-1}_z$ be the inverse branch of $f$ such that $f^{-1}_z(f(z)) = z$. Then $f^{-1}_z$, as a power series at $f(z)$ has a radius of convergence $R_{f,z}$ satisfying $(3 - \sqrt{8})/a \leq R_{f,z}/|f(z)| \leq 4/a$. 


Proof. Let \( \sigma \) be the polynomial associated with \( f \) as in Lemma 2.1. Since the radius of convergence of \( \sigma^{-1} \) at 0 is \( H_{f,z} = R_{f,z}/|f(z)| \) by Lemma 2.1, we have the claim by the previous theorem. \( \square \)

We now come to one of the main results.

**Theorem 4.4.** If \( a_{f,z_0} \leq 1/48 \), then \( z_0 \) is an approximate zero of \( f \) for \( E_k \) for all \( k \). In other words, we have

\[
\begin{align*}
|\frac{f(z_n)}{f(z_0)}| &\leq \left(\frac{1}{2}\right)^{k+1}, \\
|z_n - \xi| &\leq 4 \left(\frac{1}{2}\right)^{k+1} |z_0 - \xi|,
\end{align*}
\]

where \( \xi \) is a root and \( z_{n+1} = E_{k+1, f(z_n)} \rightarrow \xi \).

**Proof.** We will proceed with the proof by induction on \( n \). For simplicity, we denote \( R_n = R_{f,z_n}, f_n = f(z_n), f_n' = f'(z_n), H_n = R_n/|f_n| \) and \( F_n = -f(z_n)/f'(z_n) \).

**Claim 1.** \( \mid f_n/|f_0| \leq (\frac{1}{2})^{k+1} \), for all \( k \).

We note that \( a_{f,z_0} \leq 1/48 \) implies by Corollary 4.3 that

\[
\frac{1}{H_0} = \frac{|f_0|}{R_0} < \frac{1}{48} < \frac{1}{\sqrt{8}} < \frac{1}{8.23} < 0.122 < r_k
\]

for all \( k \) (see Table 3.1). Hence we apply Theorem 3.2 with \( h = 1 \), and we have

\[
z_1 = f^{-1}_z(f(z_1)) \quad \text{and} \quad \frac{|f_1|}{|f_0|} \leq B_k \left( \frac{1}{H_0} \right) \leq \left(\frac{1}{2}\right)^{k+1},
\]

by noting that

\[
B_k \left( \frac{1}{H_0} \right) < B_k(0.122) = \frac{(k+1)(1+0.122)^3(0.122)^k}{(1-0.122)^5} < \left(\frac{1}{2}\right)^{k+1} \quad \text{for } k \geq 2.
\]

For \( k = 1 \), we recall from Lemma 2.1 that \( f(z_1)/f(z_0) = 1 - \sigma \circ \varepsilon \), where \( \varepsilon = (z_1 - z_0)/F_0 = 1 \). Since

\[
|1 - \sigma(1)| = |\sigma_2 + \sigma_3 + \cdots + \sigma_d| \leq \frac{a}{1-a} \leq \frac{1}{47} \leq \left(\frac{1}{2}\right)^2,
\]

we have that \( |f_1/|f_0| \leq (\frac{1}{2})^{k+1} \) for all \( k \) as claimed. It is useful for the next claim to note that \( |f_1/|f_0| \leq 1/8 \).

**Claim 2.** Suppose \( |f_n/|f_0| \leq (\frac{1}{2})^{(k+1)n} \). Then \( |f_{n+1}/|f_0| \leq (\frac{1}{2})^{(k+1)(n+1)} \). First note that \( R_n \geq R_0 - |f_n| - |f_0| \) by Lemma 2.8(2). Since \( R_0/|f_0| \geq 8.23 \) and \( |f_n/|f_0| \leq 1/8 \) for all \( n \) and \( k \), we have \( R_n \geq 8.23|f_0| - \frac{9}{8}|f_0| \geq 7|f_0| \) for all \( n \) and \( k \). Hence we have

\[
\frac{1}{H_n} = \frac{|f_n|}{R_n} = \frac{|f_0| |f_n|}{R_n} \leq \frac{1}{7} \frac{|f_n|}{|f_0|} \leq \frac{1}{7} \left(\frac{1}{2}\right)^{(k+1)n} \leq r_k \left(\frac{1}{2}\right)^{(k+1)n} \quad \text{for all } k \text{ and } n.
\]

Now, applying Theorem 3.2 with \( h = 1 \), we have \( |f_{n+1}/|f_n| \leq B_k(1/H_n) \). Since

\[
B_k(r) = (k+1)\left(\frac{1+r}{1-r}\right)^3r^k < B_k(r_k) \left(\frac{r}{r_k}\right)^k,
\]

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for \( r < r_k \) we have

\[
\frac{|f_{n+1}|}{|f_n|} \leq B_k \left( \frac{1}{H_n} \right) \leq \left( \frac{1}{2} \right)^k (k+1)^n = \left( \frac{1}{2} \right)^k (k+1)^n.
\]

Hence we have

\[
\frac{|f_{n+1}|}{|f_0|} = \frac{|f_{n+1}|}{|f_n|} \frac{|f_n|}{|f_0|} \leq \left( \frac{1}{2} \right)^k (k+1)^n \left( \frac{1}{2} \right)^{(k+1)^n} = \left( \frac{1}{2} \right)^{(k+1)^n+1}
\]

as claimed.

Claim 3. \( |z_n - \xi| \leq 4(\frac{n+1}{2})(k+1)^n |z_0 - \xi| \).

First note that \( z_n \) defined here has \( H_n > 8 > 1/r_k \) (see the proof of Claim 2). Hence \( \sigma^{-1} \) is well defined at all \( z_n \), and we have \( \xi = z_n + F(z_n)\sigma^{-1}(1) \) and \( z_n - \xi = F(z_n)\sigma^{-1}(1) \), where \( \sigma \) is the polynomial associated with \( f \) and \( z_n \).

By Corollary 2.6(1) we note that

\[
\frac{|F(z_n)|}{(1 + 1/H_n)^2} \leq |z_n - \xi| \leq \frac{|F(z_n)|}{(1 - 1/H_n)^2}.
\]

Hence we have

\[
|z_n - \xi| \leq \frac{|F(z_n)|}{(1 - 1/H_n)^2} = \frac{|F(z_n)|}{(1 - 1/H_n)^2} |z_0 - \xi|
\]

\[
= \frac{|f_n|}{|f_0|} \frac{|f_0|}{(1 - 1/H_n)^2} |z_0 - \xi|.
\]

We note that \((1 + 1/H_0)^2/(1 - 1/H_n)^2 \leq 1.5\), since \( 1/H_0 \leq 0.122 \) and \( 1/H_n \leq 1/28 \) (see the proof of Claim 2). Further, we claim that \( |f_0'|/|f_0'| \leq (1 + 1/7)^3/(1 - 1/7) \leq 1.8 \), so that we have \( |z_n - \xi| \leq 4\left(\frac{n+1}{2}\right)^{(k+1)^n} |z_0 - \xi| \). To see this, note that \( z_n = f_{z_0}(1 - h) f(z_0) \) for \( |h| = |f(z_n) - f(z_0)|/|f_0| \leq 9/8 \) and \( |h|/H_0 \leq (9/8)/8.23 = 1/7 \). Now apply Corollary 2.7(2) with \( r = 1/7 \); we have \( |f_0'|/|f_0'| \leq (1 + r)/(1 - r)^3 \leq 1.8 \). Hence we have completed Claim 3.

5. Algorithms. The main goal of this section is to construct new algorithms to find a root of a polynomial. Applied to any polynomial \( f \), these new algorithms always converge to a root or a critical point of \( f \). The underlying idea is that, for an initial point \( z_0 \), one analytically continues \( f^{-1} \) toward 0 in a radial direction as long as it is possible. The idea used to determine the approximate zero in Section 3 is also useful.

As mentioned in Section 1, the radius of convergence (or equivalently, \( a_{f,z} \)) plays an important role as a successive overrelaxation parameter in our algorithms.

Recall that

\[
a_{f,z} = \max_{j \geq 2} \left| \frac{f(z)}{f'(z)} \right| \left| \frac{f^{(j)}(z)}{j! f^{(j-1)}(z)} \right|^{1/j(j-1)}.
\]

from Section 4.

Now we describe the algorithms.

**Algorithm A**. For a polynomial \( f \) and a complex number \( z_0 \in \mathbb{C} \), define iteratively,

\[
z_{n+1} = E_{k,h_n,f}(z_n), \quad \text{where } h_n = \min \left( 1, \frac{1}{48 a_{f,z_n}} \right).
\]
For example, if \( k = 1 \) we have \( z_{n+1} = z_n - h_n(f(z_n)/f'(z_n)) \).

**Algorithm B\(_k\).** For a polynomial \( f \) and \( z_0 \in \mathbb{C} \), let \( w_0 = f(z_0) \). Define iteratively

\[
 z_{n+1} = E_{k,1,g_n}(z_n),
\]

where \( g_n = f - w_{n+1}, w_{n+1} = (1 - h_n)w_n, \) and \( h_n = \text{Min}(1, 1/1800a_{f,z_n}). \)

**Remark.** Note that

\[
 z_{n+1} = E_{k,1,g_n}(z_n) = E_{k,h,f}(z_n),
\]

where \( h = (f(z_n) - w_{n+1})/f(z_n) \) by Lemma 2.8(3). For example, if \( k = 1 \) we have

\[
 z_{n+1} = z_n - (f(z_n) - w_{n+1})/f'(z_n).
\]

**Theorem 5.A.** \( z_n \) in Algorithm \( A_k \) always converges to a root or a critical point of \( f \).

**Proof.** Note that once \( a_{f,z_n} \leq 1/48 \) (i.e., \( h_n = 1 \)) then \( z_n \) is an approximate zero of \( f \) and converges to a root of \( f \) by Theorem 4.4. We may assume that \( a_{f,z_n} > 1/48 \) and hence \( h_n \equiv 1/48a_{f,z_n} < \frac{1}{3}H_{f,z_n} \) by Corollary 4.3. Applying Theorem 3.2 with \( h_n \), we obtain \( |f(z_{n+1})/f(z_n)| \leq 1 - h_n', \) where \( |h_n' - h_n| \leq B(h_n/8)h_n \leq 3/4h_n \) for all \( k \). Inductively one has \( f(z_N)/f(z_0) = \prod (1 - h_n') \), where \( |h_n' - h_n| \leq 3/4h_n \). Notice that \( |f(z_N)|/|f(z_0)| \) converges always since it is decreasing. We will show that \( z_n \) converges to a critical point of \( f \), if \( |f(z_n)|/|f(z_0)| \) converges to a nonzero number. Recall from the theory of infinite products that this implies that \( \sum b_n \) is bounded, where \( 1 - b_n = |1 - h_n'| \). Note that \( \sum h_n \) and \( \sum |h_n'| \) are also bounded since \( b_n \geq h_n - |h_n' - h_n| \geq \frac{1}{4}h_n \geq \frac{1}{16}|h_n'| \) by (1). Again by the theory of infinite products we have \( \prod (1 - h_n') \rightarrow w \), a nonzero complex number. This \( w \) is a critical value of \( f \) since \( h_n \rightarrow 0 \) and \( |f'(z_n)| \rightarrow 0 \) by the definitions of \( h_n \) and \( a_{f,z_n} \). Further, we claim that \( |z_{n+1} - z_n| \rightarrow 0 \) and \( z_n \) converges to a critical point \( \theta \). To see this, just note that

\[
 a_{f,z} = \max_{j=2,\ldots,d} \frac{f(z)}{f'(z)} \left| \frac{f^{(j)}(z)}{j!f'(z)} \right|^{1/(j-1)} \geq \frac{f(z)}{|f'(z)|} \left| \frac{1}{f'(z)} \right|^{1/(d-1)}.
\]

Hence

\[
 |z_{n+1} - z_n| = h_n \frac{|f(z_n)|}{|f'(z_n)|} \leq \frac{1}{48} |f'(z_n)|^{1/(d-1)} \rightarrow 0.
\]

Since there are finite preimages of \( w \), we conclude that \( z_n \rightarrow \theta \) where \( w = f(\theta) \).

**Theorem 5.B.** For any polynomial \( f \) and \( z_0 \in \mathbb{C} \), \( z_n \) in Algorithm \( B_k \) converges to a root or a critical point of \( f \). Further, \( z_n \) converges to a root unless there is a critical value of \( f \) on the ray \((0, f(z_0))\).

**Proof.** It is easy to see that once \( h_n = 1 \) (i.e., \( a_{f,z_n} \leq 1/1800 \leq 1/48 \)) then \( z_n \) is an approximate zero of \( f \) and hence \( z_n \) converges to a root of \( f \) by Theorem 4.4. Note that if \( H_n \geq 7200 \) then \( h_n = 1 \), and \( z_n \) is an approximate zero by Corollary 4.3. We will show inductively that either \( z_n \) is an approximate zero or \( z_n \) satisfies the bound

\[
 \frac{w_n}{f(z_n)} = 1 + \varepsilon_n, \quad |\varepsilon_n| \leq \frac{H_n}{14400} \leq \frac{1}{2}.
\]
For simplicity, we denote $f_n = f(z_n)$, $R_n = R_{f,z_n}$, $H_n = R_n/|f_n|$. We claim that (1) completes the proof: Recall that $R_n = |f_n - f(\theta^*)|$ for some critical point $\theta^*$ by Lemma 2.8(1) and that $H_{f,z_n}/7200 \leq h_n \leq H_{f,z_n}/308$ for $h_n < 1$ by Corollary 4.3. Now in the case $h_n < 1$,

$$
\frac{|w_n - f(\theta^*)|}{|w_n|} = \frac{|f_n| |w_n - f_n| + |f_n - f(\theta^*)|}{|f_n|} \leq \frac{1}{|1 + \varepsilon_n|} (|\varepsilon_n| + H_n)
$$

$$
\leq 2 \left( \frac{H_n}{14400} + H_n \right) \text{ since } |\varepsilon_n| \leq \frac{1}{2}
$$

$$
\leq 2.5 \frac{H_n}{14400} \leq 20000 h_n.
$$

Using the same argument as in Theorem 5.A, $w_n = \prod_{m=1}^n (1 - h_m)$ converges to a nonzero number only if $h_n \to 0$ and hence only if $|w_n - f(\theta^*)| \to 0$. Since there is no critical value on $[0, w_0]$, this is possible only if $w_n \to f(\theta^*) = 0$. Again using the same argument as in Theorem 5.A, we conclude that $z_n \to \theta^*$ where $f(\theta^*) = 0$. Now we start an induction to show (1). Suppose $f_n/w_n = 1 + \varepsilon_n$, $|\varepsilon_n| \leq H_n/14400 \leq 1/2$. Then we will show that either $z_{n+1}$ is an approximate zero or it satisfies $f_{n+1}/w_{n+1} = 1 + \varepsilon_{n+1}$, $|\varepsilon_{n+1}| \leq H_{n+1}/14400 \leq 1/2$. Recall that $z_{n+1} = E_{k,h,f}(z_n)$, where $h = (f_n - w_{n+1})/f_n$ and $w_{n+1} = (1 - h)w_n$. Note that

$$
|h| = \frac{|f_n - w_{n+1}|}{|f_n|} = \frac{|f_n - (1 - h_n)w_n|}{|f_n|} = \frac{|f_n - (1 - h_n)(1 + \varepsilon_n)f_n|}{|f_n|}
$$

$$
= \frac{|1 - (1 - h_n)(1 + \varepsilon_n)|}{|1 - (1 - h_n)(1 + \varepsilon_n)|} \leq h_n + |\varepsilon_n|
$$

$$
\leq \frac{H_n}{308} + \frac{H_n}{14400} \leq \frac{H_n}{300}.
$$

Applying Theorem 3.2 to $z_n$ with $h$, we have

$$
\frac{f_{n+1}}{f_n} = 1 - h + h\delta, \quad \text{where } |\delta| \leq B_k \left( \frac{1}{300} \right) \leq \frac{1}{145} \text{ for all } k,
$$

$$
\frac{f_{n+1}}{w_{n+1}} = 1 + \frac{h\delta}{1 - h}, \quad \text{since } w_{n+1} = (1 - h)f_n,
$$

and

$$
\frac{w_{n+1}}{f_{n+1}} = 1 + \varepsilon_{n+1} = \frac{1}{1 + \mu}, \quad \mu = \frac{h\delta}{1 - h}.
$$

Note that

$$
H_{n+1} = \frac{R_{n+1}}{|f_{n+1}|} \geq \left| \frac{f_n}{f_{n+1}} \right| \frac{R_n - |f_n - f_{n+1}|}{|f_n|} \text{ by Lemma 2.8(2)}
$$

$$
\geq \frac{1}{1 - h + h\delta} |H_n - |h - h\delta||
$$

$$
\geq \frac{1}{1 - h + h\delta} \left( H_n - \frac{H_n}{300} \left( 1 + \frac{145}{1} \right) \right)
$$

$$
\geq \frac{296}{297} \frac{H_n}{1 - h + h\delta}.
$$

Now

$$
|\mu| = \left| \frac{h\delta}{1 - h} \right| \leq \frac{H_n}{300} \frac{145}{1 - h} \leq \left| \frac{1 - h + h\delta}{1 - h} \right| \frac{H_{n+1}}{297} \frac{1}{300} \frac{1}{145}
$$

$$
= \left| 1 + \mu \right| \frac{H_{n+1}}{43000}.
$$
Note that if $|\mu| \geq \frac{1}{4}$, then
\[
H_{n+1} \geq \frac{|\mu|}{|1 + \mu|} 43000 \geq 7200
\]
and hence $z_{n+1}$ is an approximate zero. If $z_{n+1}$ is not an approximate zero then $H_{n+1} < 7200$ and $|\mu| < \frac{1}{4}$. Hence we have
\[
|\varepsilon_{n+1}| \leq 2|\mu| \leq 2 \cdot \frac{5}{4} \cdot \frac{H_{n+1}}{43000} < \frac{H_{n+1}}{30000} \leq \frac{H_{n+1}}{14400} < \frac{1}{2}.
\]

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Department of Mathematics
University of Southern California
Los Angeles, California 90089