On Approximate Zeros and Rootfinding Algorithms for a Complex Polynomial

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Abstract. In this paper we give criteria for a complex number to be an approximate zero of a polynomial \( f \) for Newton's method or for the \( k \)th-order Euler method. An approximate zero for the \( k \)th-order Euler method is an initial point from which the method converges with an order \( (k + 1) \). Also, we construct families of Newton (and Euler) type algorithms which are surely convergent.

1. Introduction. Newton's method has long been used for solving a nonlinear equation \( f(z) = 0 \). The Newton method attempts to solve \( f(z) = 0 \) by an iteratively defined sequence \( z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} \), for an initial point \( z_0 \). It indeed converges to a root at a fast rate, if it starts with a good initial point. However, not much is known about the region of convergence or of fast convergence, and it is difficult to obtain a priori knowledge of convergence.

In this paper we study the efficiency and the convergence properties of the Newton method and other generalized methods for solving a polynomial equation \( f(z) = 0 \). We have two main goals. First, we establish an estimate for a point \( z_0 \), which predicts fast convergence of the algorithms starting at \( z_0 \). Secondly, we develop a method which is guaranteed to converge, given an arbitrary initial point \( z_0 \).

Following Shub and Smale, we consider the following generalized version of the Newton method, called the modified \( k \)th-order Euler method.

We recall from elementary complex analysis that for a polynomial \( f \) and \( z \in \mathbb{C} \) such that \( f'(z) \neq 0 \), there is a well-defined local inverse branch \( f_z^{-1} \) of \( f \) such that \( f_z^{-1}(f(z)) = z \).

Definition 1.1. For an integer \( k \) and a complex number \( h \), the Euler method iteratively defines a sequence \( z_{n+1} = E_{k,h,f}(z_n) = T_k f_z^{-1}((1 - h)f(z_n)) \) for an initial point \( z_0 \), where \( T_k \) is the \( k \)th-order truncation of \( f_z^{-1} \) considered as a power series about \( f(z_n) \).

For brevity we denote \( E_{k,h,f} \) by \( E_k \) if there is no confusion. Note that \( E_{1,1,f} \) gives the Newton method.

We define an approximate zero of \( f \) for \( E_k \) as follows.

Received October 6, 1986; revised June 3, 1987 and December 28, 1987.
1980 Mathematics Subject Classification (1985 Revision). Primary 30E05, 65E05, 68Q15.
Definition 1.2. $z_0$ is an approximate zero of $f$ for $E_k$ if

\[
\frac{|f(z_n)|}{|f(z_0)|} \leq \left(\frac{1}{2}\right)^{(k+1)n},
\]

\[
|z_n - \xi| \leq c \left(\frac{1}{2}\right)^{(k+1)n} |z_0 - \xi|,
\]

where $c$ is a constant, $\xi$ is a root and $z_{n+1} = E_{k,1}(z_n) \rightarrow \xi$. \hfill \Box

Note the fast convergence of $z_n$ to $\xi$. This notion is due to Smale [12].

For a polynomial $f$ and $z \in \mathbb{C}$, let

\[
|f(z)|, |f'(z)|, \frac{|f(z)|}{|f'(z)|}, \max_{j \geq 2} \left| \frac{f^{(j)}(z)}{j!f'(z)^j} \right|^{1/(j-1)}
\]

We show that $2$ is an approximate zero of $f$ for all $E_k$ if $a_{f,z} \leq \frac{1}{48}$ (see Theorem 4.4). Recently, Smale [14] has obtained a similar result for $k = 1$ with a better estimate (a constant $\alpha_0$, in his notation, between $\frac{1}{8}$ and $\frac{1}{2}$) for a more general class of polynomial maps $f: \mathbb{C}^n \rightarrow \mathbb{C}^n$.

The estimate for $a_{f,z}$ plays an important role even when $z$ is not an approximate zero of $f$. It suggests the next iterate in the construction of algorithms which produces sure convergence.

We construct two families of modified Euler methods $A_k$ and $B_k$, which always converge to a root or a critical point of $f$ (see Theorems 5.A and 5.B). $\theta$ is called a critical point of $f$ if $f'(\theta) = 0$. As in the work of Shub and Smale ([9], [10]), the idea is to approximate the solution curve $\phi_t(z_0)$ to the Newton vector field $F(z) = -f(z)/f'(z)$ where $\phi_0(z_0) = z_0$. Note that $f(\phi_t(z_0)) = e^{-t}f(z_0)$, a straight line through $f(z_0)$. Hence one can approximate a root by approximating $f^{-1}(e^{-t}f(z_0))$ as $t \rightarrow \infty$. To do so, Shub and Smale use the modified Euler method with a fixed step size $h$ in $E_{k,h,f}$ together with a probabilistic estimate on the set of initial points. In our algorithms we use a varying step size $h$ at each point $z$, where $h$ is given in terms of $a_{f,z}$ and hence related to the radius of convergence of $f_z^{-1}$. In particular, we show that for any polynomial $f$ and initial point $z_0$, $B_k$ always produces a sequence $z_n$ converging to a root unless there is a critical value of $f$ on the ray $[0, f(z_0)]$ (see Theorem 5.B). Recently, Shub and Smale [11] have shown that an algorithm similar to $A_1$ converges to a root for almost all polynomials and for almost all initial points $z_0$.

We have run some experiments on the algorithm $A_1$ and other similar algorithms with a starting point $0$ and with a supplementary algorithm of Shub and Smale [10] for degenerate cases such as $a_{f,z} \geq 50d^2$. This corresponds to the case where $z$ is near a critical point. Among $(100 \cdot d^2)$ randomly selected polynomials of each degree $d \leq 100$ with complex coefficients $|a_1| \leq 1$, the average number of iterations to locate an approximate zero or to locate $\zeta$ such that $|f(\zeta)| \leq 10^{-4}$ is found to be less than 200. Our experimental result is independent of the degree $d$.

2. Preliminaries. In this section we discuss some preliminary material needed in the later sections on the local behavior of analytic functions.

The main tools used in Section 3 are from the theory of schlicht functions. $f$ is called a schlicht function if $f(0) = 0$, $f'(0) = 1$ and it is univalent on $D_1(0)$, the unit disk at 0. A univalent function is a one-to-one complex analytic function.
To each $z \in \mathbb{C}$ and complex polynomial $f$ such that $f(z) \neq 0$ and $f'(z) \neq 0$, one associates a normalized polynomial $\sigma$ by means of

$$\sigma(w) = w + \sigma_2 w^2 + \cdots + \sigma_d w^d,$$

where $\sigma_j = \left( \frac{-f(z)}{f'(z)} \right)^{j-1} \frac{f^{(j)}(z)}{j! f'(z)}$.

Let $R_{f,z}$ be the radius of convergence of $f_z^{-1}$, considered as a power series at $f(z)$.

For $f(z) \neq 0$, let $H_{f,z} = R_{f,z} / |f(z)|$; see Figure 2.1. The following lemma is extracted from the work of Shub and Smale (see [9, p. 113]).

**Lemma 2.1.** Let $\sigma^{-1}$ be the inverse branch of $\sigma$ taking 0 to 0. Then

1. $\sigma^{-1}(0) = 0$, $\sigma^{-1}'(0) = 1$.
2. Let $x = f_z^{-1}((1 - h)f(z))$ with $|h| < H_{f,z}$. Then

$$\frac{f(x)}{f(z)} = 1 - \sigma \circ \varepsilon, \quad \text{where } \varepsilon = \frac{x - z}{F(z)} \text{ and } F(z) = \frac{-f(z)}{f'(z)}.$$

3. $f_z^{-1}((1 - h)f(z)) = z + F(z)\sigma^{-1}(h)$.
4. $T_k f_z^{-1}((1 - h)f(z)) = z + F(z)T_k \sigma^{-1}(h)$.
5. The radius of convergence of $\sigma^{-1}$ at 0 is $R_{\sigma,0} = H_{f,z}$.
6. $\frac{1}{H} \sigma^{-1}(Hh)$ is schlicht, $H \equiv H_{f,z}$.

**Proof.** (1) is immediate. (2) is from Proposition 2 in [9]. For (3), (4) and (5), see [9, p. 114] and [10, p. 153]. (6) is a trivial consequence of (1) and (5). □

Using Lemma 2.1, we may reformulate Definition 1.1 of $E_{k,h,f}$ as follows.

**Definition 2.2.** $E_{k,h,f}(z) = z + F(z)T_k \sigma^{-1}(h)$.

We will need the following properties.

**Lemma 2.3** (De Branges' Theorem: Bieberbach conjecture). Let $g(z) = z + g_2 z^2 + g_3 z^3 + \cdots$ be schlicht. Then $|g_k| \leq k$.

**Proof.** See [2]. □
Lemma 2.4 (Shub and Smale). (1) Let \( g \) be schlicht. Then \(|g(h) - T_k g(h)| \leq \frac{(k+1)r^{k+1}}{(1-r)^2}\), where \( r = |h| < 1 \).

(2) Let \( g \) be univalent on \( D_H(0) \), \( g(0) = 0 \) and \( g'(0) = 1 \). Then for \( h \) with \( r = |h|/H < 1 \),

\[ |g(h) - T_k g(h)| \leq \frac{H(k+1)r^{k+1}}{(1-r)^2}. \]

Proof. From Lemma 2.3 we have

\[ |g(h) - T_k g(h)| \leq \sum_{j=k+1}^{\infty} j^r < r^k \leq \frac{r^{k+1}}{1-r} \leq \frac{(k+1)r^{k+1}}{(1-r)^2}. \]

For the second statement, note that \( \frac{1}{H} g(Hh) \) is schlicht, and then use (1). □

Lemma 2.5 (Koebe Distortion Theorem). Let \( g \) be schlicht. Then for \( |h| = r < 1 \),

(1) \[ \frac{r}{(1+r)^2} \leq |g(h)| \leq \frac{r}{(1-r)^2}, \]

(2) \[ \frac{1-r}{(1+r)^3} \leq |g'(h)| \leq \frac{1+r}{(1-r)^3}. \]

Proof. See [4, Vol. 2, pp. 351 and 353]. □

By rescaling, we obtain immediately the following

Corollary 2.6. Let \( g \) be univalent on \( D_H(0) \) and \( g(0) = 0 \), \( g'(0) = 1 \). Let \( r = |h|/H < 1 \). Then

(1) \[ \frac{|h|}{(1+r)^2} \leq |g(h)| \leq \frac{|h|}{(1-r)^2}, \]

(2) \[ \frac{1-r}{(1+r)^3} \leq |g'(h)| \leq \frac{1+r}{(1-r)^3}. \]

Proof. Note that \( \frac{1}{H} g(Hh) \) is schlicht. Now use Lemma 2.5. □

Corollary 2.7. Let \( x = f^{-1}_z((1-h)f(z)) \equiv z + F(z)\sigma^{-1}(h) \) for \( |h| < H_{f,z} \).

Then we have

(1) \[ f'(x) = f'(z)\sigma'(\epsilon) \equiv \frac{f'(z)}{\sigma^{-1'}(h)}, \quad \text{where} \quad \epsilon = \frac{x-z}{F(z)}, \]

(2) \[ |f'(z)|\frac{1-r}{1+r} \leq |f'(x)| \leq |f'(z)|\frac{1+r}{1-r}, \quad \text{where} \quad r = \frac{|h|}{H_{f,z}}. \]

Proof. Recall from Lemma 2.1(2) that \( f(x) = f(z)(1 - \sigma(\epsilon)) \) and \( \sigma(\epsilon) = h \equiv (f(z) - f(x))/f(z) \). Hence (1) is immediate by taking derivatives of \( f \). (2) follows from Corollary 2.6(2) since \( \sigma^{-1}(0) = 0 \), \( \sigma^{-1'}(0) = 1 \) and \( \sigma^{-1} \) is univalent in \( D_H(0) \). □

We close this section with the following lemma.

Lemma 2.8. (1) \( R_{f,z} = |f(z) - f(\theta^*)| \geq \min_{f'(\theta) = 0} |f(z) - f(\theta)| \) for some critical point \( \theta^* \) of \( f \).

(2) Let \( x = f^{-1}_z((1-h)f(z)) \) with \( |h|/H_{f,z} < 1 \). Then \( R_{f,z} \geq R_{f,z} - |f(z) - f(x)| \).
Let $g = f - y$ be a translation of $f$ by $y \in \mathbb{C}$. Then $E_{k,h',g}(x) = E_{k,h,f}(x)$, where $h' = hf(z)/g(z)$.

Proof. For (1), see Lemma 3 in [12]. For (2), we note that by the uniqueness of analytic maps, we have $f_z^{-1} \equiv f_z^{-1}$ on their common domain of definitions. In particular, $f_z^{-1}$ is analytically continued for all $w$ such that $|w - f(z)| < R_{f,z}$. Since $|w - f(z)| < |w - f(x)| + |f(x) - f(z)| < R_{f,z}$, $f_z^{-1}$ is analytic for all $w$ such that $|w - f(x)| < R_{f,z} - |f(z) - f(x)|$. Hence $R_{f,z} \geq R_{f,z} - |f(z) - f(x)|$.

For (3), note that $g_z^{-1}(w - y)$ is well defined where $f_z^{-1}(w)$ is well defined and $g_z^{-1}(w - y) = f_z^{-1}(w)$. As power series at $f(z)$ and $g(z)$ respectively, we have

$$f_z^{-1}(f(z) - w) \equiv g_z^{-1}(f(z) - w - y) \equiv g_z^{-1}(g(z) - w),$$

where $w = hf(z) = h'g(z)$ and $h' = h(f(z)/g(z))$. Hence we also have

$$E_{k,h}(f(z)) = E_{k,h'}(g(z))$$

and $E_{k,h',g}(z) = E_{k,h,f}(z)$. □

3. Koebe Distortion Theorem and Euler Iteration. We recall that $E_{k,h,f}(z) = T_k f_z((1 - h)f(z))$. In this section we show that $E_{k,h,f}$ approximates $f_z^{-1}$ with a suitable $h$, i.e., $E_{k,h,f}(z) = f_z^{-1}(w)$ for $w$ such that $|w - f(z)| < R_{f,z}$ and $R_{f,z}$ is the radius of convergence of $f_z^{-1}$. In particular, we show that $E_{k,h,f}$ approximates $f_z^{-1}$ for all values on the disk of convergence as $k \uparrow \infty$. The main goal of this section is to prove Theorem 3.2 below.

We recall that $H_{f,z} = R_{f,z}/|f(z)|$, where $R_{f,z}$ denotes the radius of convergence of $f_z^{-1}$ at $f(z)$.

**Theorem 3.1.** Let $x = f_z^{-1}((1 - h)f(z))$. Assume that

$$r = \frac{|h|}{H_{f,z}} < 1 \quad \text{and} \quad t \leq |h| \left( \frac{1 - r}{1 + r} \right)^3.$$

Then $D_t f(z) \subset f_z^{-1}(D_{s f(z)} f(z)))$, where

$$s = \min \left\{ \left( \frac{1 + r}{1 - r} \right)^3, \quad H_{f,z}(1 - r) \right\}$$

and $F = \frac{f(z)}{f'(z)}$.

The proof will be given later. □

Let $B_k(r) = (k + 1) \left( \frac{1 + r}{1 - r} \right)^3 r^k$

and $r_k$ be the smallest positive solution to $B_k(r) = 1$. Note that $B_k(r)$ is increasing on $[0, r_k]$. The condition that $|h| < r_k H_{f,z}$ is crucial for $E_{k,h,f}$ to approximate $f_z^{-1}$.

**Theorem 3.2.** Let $z' = E_{k,h,f}(z)$ with $r = |h|/H_{f,z} < r_k$. Then we have $z' = f_z^{-1}((1 - h')f(z))$ and $f(z')/f(z) = 1 - h + e$, where $|e| = |h - h'| \leq \min\{|h|B_k(r), H_{f,z}(1 - r)\}$.

A table of approximate values of $r_k$ is given below.

<table>
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<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>177</th>
<th>3303</th>
<th>47400</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_k$</td>
<td>0.148</td>
<td>0.225</td>
<td>0.282</td>
<td>0.329</td>
<td>0.367</td>
<td>0.495</td>
<td>0.9</td>
<td>0.99</td>
<td>0.999</td>
</tr>
</tbody>
</table>

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Remark. We note that $r_k \uparrow 1$ as $k \uparrow \infty$. In [9], Shub and Smale showed that Theorem 3.2 holds for all $r \leq \gamma_k$ where $\gamma_k \uparrow 0.175$ as $k \uparrow \infty$.

Proof of Theorem 3.2. Recall that $z' = E_{k,h}(z) = z + F(z)T_k\sigma^{-1}(h)$, where $F(z) = -f(z)/f'(z)$. Let $f(z') = (1 - h')f(z)$. Let $z = f_z^{-1}((1 - h)f(z)) = z + F(z)\sigma^{-1}(h)$. Then $|z' - x| = |F|\sigma^{-1}(h) - T_k\sigma^{-1}(h)|$. Since $\sigma^{-1}$ is univalent on $D_H(0)$, we have by Lemma 2.4,

$$t = |\sigma^{-1}(h) - T_k\sigma^{-1}(h)| \leq \frac{H_{f,z}(k + 1)r^{k+1}}{(1 - r)^2}$$

$$= |h|B_k(r)^{(1 - r)^3/(1 + r)^3} \leq |h|\frac{(1 - r)^3}{(1 + r)^3} \quad \text{for } r < r_k.$$  

Now, by Theorem 3.1, we have $z' \in f_z^{-1}(D_{f(z)}(s)(f(z)))$ and

$$|f(z') - f(z)| = |(1 - h')f(z) - (1 - h)f(z)| = |f(z)||h' - h| \leq |f(z)|s.$$  

Hence,

$$|\varepsilon| = |h' - h| \leq s = \text{Min}\{|h|B_k(r), H_{f,z}(1 - r)\}$$

and we have $z' = f_z^{-1}((1 - h')f(z))$ and $f(z')/f(z) = 1 - h'$ for some $h'$ with $|h' - h| \leq s$. 

We need the following lemmas to prove Theorem 3.1.

**Lemma 3.3.** (1) Let $g$ be univalent on $D_R(0)$. Then $D_{R_t}(g(z)) \subset g(D_{R_s}(z)) \subset D_{R_u}(g(z))$, for any $s < 1$, $t = s|g'(z)|/(1 + s)^2$ and $u = s|g'(z)|/(1 - s)^2$.

(2) Suppose that $g$ is univalent on $D_H(0)$, $g(0) = 0$ and $g'(0) = 1$. Let $z \in D_H(0)$, where $r = |z|/H$. Then for $s \leq 1 - r$ we have $D_{H_t}(g(z)) \subset g(D_{H_s}(z)) \subset D_{H_u}(g(z))$, where $t = ((1 - r)^3/(1 + r)^3) \cdot s/(1 - r + s)^2$ and $u = ((1 + r)/(1 - r)) \cdot s/(1 - r - s)^2$.

**Figure 3.2.** $w = g(z)$

*Proof. (1) Let

$$\psi(h) = \frac{1}{Rg'(z)}(g(z + Rh) - g(z)).$$

Then it is easy to see that $\psi$ is schlicht. Hence by Lemma 2.5, $\delta/(1 + \delta)^2 \leq |\psi(h)| \leq \delta/(1 - \delta)^2$ for $|h| = \delta$, so that we have

$$D_{R_u}(g(z)) \subset g(D_{R_\delta}(z)) \subset D_{R_\eta}(g(z)).$$
where $R\mu = \delta R|g'(z)|/(1 + \delta)^2$, $R\eta = \delta R|g'(z)|/(1 - \delta)^2$. By setting $t = \mu$, $s = \delta$ and $u = \eta$, (1) is established.

For (2), note that $g$ is univalent on $D_R(z)$, where $R = H(1 - r)$, and hence $(1 - r)/(1 + r)^3 \leq |g'(z)| \leq (1 + r)/(1 - r)^3$ by Corollary 2.6. Let $s = (1 - r)\delta$. Then we have

$$R\mu \geq \frac{\delta H(1 - r)}{(1 + \delta)^2} \frac{1 - r}{(1 + r)^3} = \frac{Hs}{(1 - r + s)^2} \frac{(1 - r)^3}{(1 + r)^3} \equiv Ht,$$

$$R\eta \leq \frac{\delta H(1 - r)}{(1 - \delta)^2} \frac{1 + r}{(1 - r)^3} \leq \frac{Hs}{(1 - r - s)^2} \frac{1 + r}{1 - r} \equiv Hu,$$

where

$$t = \frac{(1 - r)^3}{(1 + r)^3} \frac{s}{(1 - r + s)^2} \quad \text{and} \quad u = \frac{1 + r}{1 - r} \frac{s}{(1 - r - s)^2}.$$

Hence we have $D_{Ht}(g(z)) \subset g(D_{Hs}(z)) \subset D_{Hu}(g(z))$. □

Lemma 3.3 gives the following Quarter Theorem at an arbitrary point $z \in D_H(0)$.

**Corollary 3.4.** Let $g$ be univalent on $D_H(0)$ and $g(0) = 0$ and $g'(0) = 1$. For $z \in D_H(0)$, let $r = |z|/H$. Then $D_{Ht}(g(z)) \subset g(D_H(1 - r)(z))$, where $t = \frac{1}{4}((1 - r)^2/(1 + r)^3)$.

*Proof.* Use $s = 1 - r$ in Lemma 3.3(2). □

In Lemma 3.3(2) we will also need to estimate $s$ as a function of $t$.

**Corollary 3.5.** Suppose $t \leq r((1 - r)^3/(1 + r)^3)$, where $r = |z|/H < 1$. Then $D_{Ht}(g(z)) \subset g(D_{Hs}(z))$, where $s = \min\{r((1 + r)^3/(1 - r)^3), 1 - r\}$.

*Proof.* Since $r(1 - r) = \frac{1}{4}$, we have $t \leq \frac{1}{4}((1 - r)^2/(1 + r)^3)$. Hence by Corollary 3.4 we have $D_{Ht}(g(z)) \subset D_{H(1 - r)}(g(z))$. Let $t' = ((1 - r)^3/(1 + r)^3) \cdot s/(1 - r + s)^2$. Since $s \leq 1 - r$, Lemma 3.3(2) shows that $D_{H(1 - r)}(g(z)) \subset g(D_{Hs}(z))$. However, since $s \leq t((1 - r)^3/(1 + r)^3) \leq r$, we have

$$t' = \frac{(1 - r)^3}{(1 + r)^3} \frac{s}{(1 - r + s)^3} \geq \frac{(1 - r)^3}{(1 + r)^3} s = t.$$

Consequently, $D_{Ht}(g(z)) \subset D_{H(1 - r)}(g(z)) \subset g(D_{Hs}(z))$, as claimed. □

The proof of Theorem 3.1 now follows easily from Corollary 3.5.

*Proof of Theorem 3.1.* Suppose that $z = f^{-1}_z((1 - h')f(z))$ where $|z' - x| \leq |F|t$.

Since

$$z' = f^{-1}_z((1 - h')f(z)) = z + F(z)\sigma^{-1}(h') = z + F(z)\sigma^{-1}(h) + F(z)(\sigma^{-1}(h') - \sigma^{-1}(h)) = x + F(z)(\sigma^{-1}(h') - \sigma^{-1}(h)),$$

we have

$$|z' - x| = |F||\sigma^{-1}(h') - \sigma^{-1}(h)| \leq |F|t = |F|Ht', \quad \text{where} \quad t' = \frac{t}{H} \leq \frac{|h|}{H} \frac{(1 - r)^3}{(1 + r)^3} = \frac{r(1 - r)^3}{(1 + r)^3},$$

by the hypothesis. Hence, by Corollary 3.5, we have $|h' - h| \leq Hs'$, where $s' = \min\{t'((1 + r)^3/(1 - r)^3), 1 - r\}$. Now by setting $s = Hs'$ we have the claim. □
4. Domain of Injectivity and a Notion of an Approximate Zero. The main goal of this section is to give a criterion to determine an approximate zero of a polynomial \( f \) for the modified Euler method. Hereafter we will denote \( E_{k,h,f} \) by \( E_k \) if there is no confusion.

**Definition.** \( z_0 \) is an approximate zero of \( f \) for \( E_k \) if

\[
(1) \quad \frac{|f(z_n)|}{|f(z_0)|} \leq \left( \frac{1}{2} \right)^{(k+1)n},
\]

\[
(2) \quad |z_n - \xi| \leq c \left( \frac{1}{2} \right)^{(k+1)n} |z_0 - \xi|,
\]

where \( z_n = E_{k,1,f}^{-1}(z_0) \rightarrow \xi \) and \( c \) is a constant.

We will need the following estimate of the domain of injectivity, which itself is quite interesting.

**Theorem 4.1.** Let \( g(z) = z + a_2z^2 + \cdots \) be a power series and \( \psi \) be the compositional inverse of \( g \) taking 0 to 0. Let \( a = \sup \{ |a_k|^{1/(k-1)} \} \). Then \( \psi \) is well defined, analytic and one-to-one on \( D_R(0) \), where \((3 - \sqrt{8})/a \leq R\).

**Proof.** Suppose that \( |g(z) - z| < r \) on \( |z| = r \). Then 0 is the only root of \( g \) in \( D_r(0) \) by Rouché's Theorem. It follows that (see [1, Theorem 11, p. 131]) the inverse map \( \psi \) is well defined on \( g(D_r(0)) \). In particular, \( \psi \) is well defined on \( D_R(0) \), where \( R = \text{Min}_{|z| = r} |g(z)| \). Now,

\[
|g(z)| = |z||1 + a_2z + a_3z^2 + \cdots| \\
\geq r|1 - ((ar) + (ar)^2 + (ar)^3 + \cdots)| \\
\geq r \left( 1 - \frac{ar}{1-ar} \right) \quad \text{on} \quad |z| = r.
\]

But \( r \left( 1 - \frac{ar}{1-ar} \right) \) achieves the maximum \((3 - \sqrt{8})/a \) when \( r = (2 - \sqrt{2})/2a \). Also note that

\[
|g(z) - z| = |a_2z^2 + a_3z^3 + \cdots| = |z| |a_2z + a_3z^2 + \cdots| \\
\leq r \frac{ar}{1-ar} < r, \quad \text{on} \quad |z| = \frac{2 - \sqrt{2}}{2a}.
\]

Hence \( \psi \) is well defined and injective on \( D_R(0) \), where \( R = (3 - \sqrt{8})/a \approx 1/5.83a \leq 1/6a \). \( \square \)

**Remark 4.2.** The corresponding upper bound \( R \leq 4/a \) is obtained in [12, p. 9, Extended Loewner’s Theorem]. For a polynomial \( f \) and \( z \in \mathbb{C} \) we define

\[
a_{f,z} \equiv \max_{j \geq 2} \left| \frac{f(z)}{f'(z)} \right|^{|j/(j-1)|}.
\]

We apply Theorem 4.1 to a polynomial.

**Corollary 4.3.** Let \( f \) be a polynomial of degree \( d \) and \( z \) be a complex number such that \( f'(z) \neq 0 \), and \( f(z) \neq 0 \). Let \( f_z^{-1} \) be the inverse branch of \( f \) such that \( f_z^{-1}(f(z)) = z \). Then \( f_z^{-1} \), as a power series at \( f(z) \) has a radius of convergence \( R_{f,z} \) satisfying \((3 - \sqrt{8})/a \leq R_{f,z}/|f(z)| \leq 4/a \).
Proof. Let \( \sigma \) be the polynomial associated with \( f \) as in Lemma 2.1. Since the radius of convergence of \( \sigma^{-1} \) at 0 is \( H_{f,z} = R_{f,z}/|f(z)| \) by Lemma 2.1, we have the claim by the previous theorem. \( \square \)

We now come to one of the main results.

**Theorem 4.4.** If \( a_{f,z_0} \leq 1/48 \), then \( z_0 \) is an approximate zero of \( f \) for \( E_k \) for all \( k \). In other words, we have

\[
|f(z_n)| \leq \left( \frac{1}{2} \right)^{(k+1)^n},
\]

\[
|z_n - \xi| \leq 4 \left( \frac{1}{2} \right)^{(k+1)^n} |z_0 - \xi|
\]

where \( \xi \) is a root and \( z_{n+1} = E_{k,1,f}(z_n) \rightarrow \xi \).

Proof. We will proceed with the proof by induction on \( n \). For simplicity, we denote \( R_n = R_{f,z_n}, f_n = f(z_n), f'_n = f'(z_n), H_n = R_n/|f_n| \) and \( F_n = -f(z_n)/f'(z_n) \).

Claim 1. \( |f_1|/|f_0| \leq \left( \frac{1}{2} \right)^{k+1} \), for all \( k \).

We note that \( a_{f,z_0} \leq 1/48 \) implies by Corollary 4.3 that

\[
\frac{1}{H_0} = \frac{|f_0|}{R_0} < \frac{1}{48} \frac{1}{3 - \sqrt{8}} < \frac{1}{8.23} < 0.122 < r_k
\]

for all \( k \) (see Table 3.1). Hence we apply Theorem 3.2 with \( h = 1 \), and we have

\[
z_1 = f_{z_0}^{-1}(f(z_1)) \quad \text{and} \quad \frac{|f_1|}{|f_0|} \leq B_k \left( \frac{1}{H_0} \right) \leq \left( \frac{1}{2} \right)^{k+1},
\]

by noting that

\[
B_k \left( \frac{1}{H_0} \right) < B_k(0.122) = \frac{(k+1)(1 + 0.122)^3(0.122)^k}{(1 - 0.122)^5} < \left( \frac{1}{2} \right)^{k+1} \quad \text{for } k \geq 2.
\]

For \( k = 1 \), we recall from Lemma 2.1 that \( f(z_1)/f(z_0) = 1 - \sigma \circ \varepsilon \), where \( \varepsilon = (z_1 - z_0)/F_0 = 1 \). Since

\[
|1 - \sigma(1)| = |\sigma_2 + \sigma_3 + \cdots + \sigma_d| \leq \frac{a}{1 - a} \leq \frac{1}{47} \leq \left( \frac{1}{2} \right)^2,
\]

we have that \( |f_1|/|f_0| \leq \left( \frac{1}{2} \right)^{k+1} \) for all \( k \) as claimed. It is useful for the next claim to note that \( |f_1|/|f_0| \leq 1/8 \).

Claim 2. Suppose \( |f_n|/|f_0| \leq \left( \frac{1}{2} \right)^{(k+1)^n} \). Then \( |f_{n+1}|/|f_0| \leq \left( \frac{1}{2} \right)^{(k+1)^{n+1}} \).

First note that \( R_n \geq R_0 - |f_n| - |f_0| \) by Lemma 2.8(2). Since \( R_0/|f_0| \geq 8.23 \) and \( |f_n|/|f_0| \leq 1/8 \) for all \( n \) and \( k \), we have \( R_n \geq 8.23|f_0| - \frac{9}{8}|f_0| \geq 7|f_0| \) for all \( n \) and \( k \). Hence we have

\[
\frac{1}{H_n} = \frac{|f_n|}{R_n} = \frac{|f_0|}{R_n} \frac{|f_n|}{|f_0|} \leq \frac{1}{7} \frac{|f_n|}{|f_0|} \leq \frac{1}{7} \left( \frac{1}{2} \right)^{k+1} \left( \frac{1}{2} \right)^n \leq r_k \left( \frac{1}{2} \right)^{k+1}
\]

for all \( k \) and \( n \). Now, applying Theorem 3.2 with \( h = 1 \), we have \( |f_{n+1}|/|f_n| \leq B_k(1/H_n) \). Since

\[
B_k(r) = (k + 1) \frac{(1 + r)^3 r^k}{(1 - r)^5} < B_k(r_k) \left( \frac{r}{r_k} \right)^k < \left( \frac{r}{r_k} \right)^k
\]

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for \( r < r_k \) we have

\[
\frac{|f_{n+1}|}{|f_n|} \leq B_k \left( \frac{1}{H_n} \right) \leq \left( \frac{1}{2} \right)^{(k+1)n} = \left( \frac{1}{2} \right)^{k(k+1)n}.
\]

Hence we have

\[
\frac{|f_{n+1}|}{|f_0|} = \frac{|f_{n+1}|}{|f_n|} \frac{|f_n|}{|f_0|} \leq \left( \frac{1}{2} \right)^{(k+1)n} \left( \frac{1}{2} \right)^{(k+1)n} = \left( \frac{1}{2} \right)^{(k+1)n+1}
\]

as claimed.

**Claim 3.** \(|z_n - \xi| \leq 4(\frac{1}{2})^{(k+1)n}|z_0 - \xi|\).

First note that \( z_n \) defined here has \( H_n > 8 > 1/r_k \) (see the proof of Claim 2). Hence \( \sigma^{-1} \) is well defined at all \( z_n \), and we have \( \xi = z_n + F(z_n)\sigma^{-1}(1) \) and \( z_n - \xi = F(z_n)\sigma^{-1}(1) \), where \( \sigma \) is the polynomial associated with \( f \) and \( z_n \).

By Corollary 2.6(1) we note that

\[
\frac{|F(z_n)|}{(1 + 1/H_n)^2} \leq |z_n - \xi| \leq \frac{|F(z_n)|}{(1 - 1/H_n)^2}.
\]

Hence we have

\[
|z_n - \xi| \leq \frac{|F(z_n)|}{(1 - 1/H_n)^2} \leq \frac{|F(z_n)|}{(1 - 1/H_n)^2} \frac{(1+1/H_0)^2}{|F(z_0)|} |z_0 - \xi|
\]

We note that \((1+1/H_0)^2/(1 - 1/H_n)^2 \leq 1.5\), since \( 1/H_0 \leq 0.122 \) and \( 1/H_n \leq 1/28 \) (see the proof of Claim 2). Further, we claim that \(|f_0'\|/|f_n'| \leq (1 + 1/7)^3/(1 - 1/7) \leq 1.8\), so that we have \( |z_n - \xi| \leq 4(\frac{1}{2})^{(k+1)n}|z_0 - \xi|\). To see this, note that \( z_n = f_{z_0}^{-1}((1 - h)f(z_0)) \) where \(|h| = |f(z_n) - f(z_0)|/|f_0| \leq 9/8 \) and \( |h^2/H_0 \leq (9/8)/8.23 = 1/7 \). Now apply Corollary 2.7(2) with \( r = 1/7 \); we have \(|f_0'|/|f_n'| \leq (1 + r)/(1 - r)^3 \leq 1.8\). Hence we have completed Claim 3. \( \square \)

5. Algorithms. The main goal of this section is to construct new algorithms to find a root of a polynomial. Applied to any polynomial \( f \), these new algorithms always converge to a root or a critical point of \( f \). The underlying idea is that, for an initial point \( z_0 \), one analytically continues \( f_{z_0}^{-1} \) toward 0 in a radial direction as long as it is possible. The idea used to determine the approximate zero in Section 3 is also useful.

As mentioned in Section 1, the radius of convergence (or equivalently, \( a_{f,z} \)) plays an important role as a successive overrelaxation parameter in our algorithms.

Recall that

\[
a_{f,z} = \max_{j \geq 2} \left\{ \frac{|f(z)|}{|f'(z)|} \left( \frac{|f'(z)|}{|f'(z)|} \right)^{1/(j-1)} \right\}
\]

from Section 4.

Now we describe the algorithms.

**Algorithm A_k.** For a polynomial \( f \) and a complex number \( z_0 \in \mathbb{C} \), define iteratively,

\[
z_{n+1} = E_{k,h_n,f}(z_n), \quad \text{where } h_n = \text{Min} \left( 1, \frac{1}{48a_{f,z_n}} \right).
\]
For example, if $k = 1$ we have $z_{n+1} = z_n - h_n(f(z_n)/f'(z_n))$. □

**ALGORITHM B_k.** For a polynomial $f$ and $z_0 \in \mathbb{C}$, let $w_0 = f(z_0)$. Define iteratively

$$z_{n+1} = E_{k,1, g_n}(z_n),$$

where $g_n = f - w_{n+1}$, $w_{n+1} = (1 - h_n)w_n$, and $h_n = \text{Min}(1, 1/1800a_{f,z_n})$.

**Remark.** Note that

$$z_{n+1} = E_{k,1,g_n}(z_n) = E_{k,h,f}(z_n),$$

where $h = (f(z_n) - w_{n+1})/f(z_n)$ by Lemma 2.8(3). For example, if $k = 1$ we have $z_{n+1} = z_n - (f(z_n) - w_{n+1})/f'(z_n)$.

**THEOREM 5.A.** $z_n$ in Algorithm A_k always converges to a root or a critical point of $f$.

**Proof.** Note that once $a_{f,z_n} \leq 1/48$ (i.e., $h_n = 1$) then $z_n$ is an approximate zero of $f$ and converges to a root of $f$ by Theorem 4.4. We may assume that $a_{f,z_n} > 1/48$ and hence $h_n \equiv 1/48a_{f,z_n} < (1/8)H_{f,z_n}$ by Corollary 4.3. Applying Theorem 3.2 with $h_n$, we obtain $|f(z_{n+1})/f(z_n)| \leq 1 - h_n'$, where $|h_n' - h_n| \leq B_k(1/8)h_n \leq (3/4)h_n$ for all $k$. Inductively one has $f(z_N)/f(z_0) = \prod_N (1 - h_n')$, where $|h_n' - h_n| \leq (3/4)h_n$.

Notice that $|f(z_n)]/|f(z_0)|$ converges always since it is decreasing. We will show that $z_n$ converges to a critical point of $f$, if $|f(z_n)|/|f(z_0)|$ converges to a nonzero number. Recall from the theory of infinite products that this implies that $\sum b_n$ is bounded, where $1 - b_n = |1 - h_n'|$. Note that $\sum h_n$ and $\sum |h_n'|$ are also bounded since $b_n \geq h_n' - h_n' - h_n \geq 1/4 h_n \geq 1/16 |h_n|$ by (1). Again by the theory of infinite products we have $\prod (1 - h_n') \to w$, a nonzero complex number. This $w$ is a critical value of $f$ since $h_n \to 0$ and $|f'(z_n)| \to 0$ by the definitions of $h_n$ and $a_{f,z_n}$. Further, we claim that $|z_{n+1} - z_n| \to 0$ and $z_n$ converges to a critical point $\theta$. To see this, just note that

$$a_{f,z} = \max_{j=2, \ldots, d} \frac{|f(z)|}{f'(z)} \left| \frac{f^{(j)}(z)}{j!f'(z)} \right|^{1/(j-1)} \geq \left| \frac{f(z)}{f'(z)} \right| \left| \frac{1}{f'(z)} \right|^{1/(d-1)}.$$ 

Hence

$$|z_{n+1} - z_n| = h_n |f(z_n)|/|f'(z_n)| \leq 1/48 |f'(z_n)|^{1/(d-1)} \to 0.$$ 

Since there are finite preimages of $w$, we conclude that $z_n \to \theta$ where $w = f(\theta)$. □

**THEOREM 5.B.** For any polynomial $f$ and $z_0 \in \mathbb{C}$, $z_n$ in Algorithm B_k converges to a root or a critical point of $f$. Further, $z_n$ converges to a root unless there is a critical value of $f$ on the ray $(0, f(z_0)]$.

**Proof.** It is easy to see that once $h_n = 1$ (i.e., $a_{f,z_n} \leq 1/1800 \leq 1/48$) then $z_n$ is an approximate zero of $f$ and hence $z_n$ converges to a root of $f$ by Theorem 4.4. Note that if $H_n \geq 7200$ then $h_n = 1$, and $z_n$ is an approximate zero by Corollary 4.3. We will show inductively that either $z_n$ is an approximate zero or $z_n$ satisfies the bound

$$\frac{w_n}{f(z_n)} = 1 + \epsilon_n, \quad |\epsilon_n| \leq \frac{H_n}{14400} \leq \frac{1}{2}.$$
For simplicity, we denote \( f_n = f(z_n) \), \( R_n = R_{f,z_n} \), \( H_n = R_n/|f_n| \). We claim that (1) completes the proof: Recall that \( R_n = |f_n - f(\theta^*)| \) for some critical point \( \theta^* \) by Lemma 2.8(1) and that \( H_{f,z_n}/7200 \leq h_n \leq H_{f,z_n}/308 \) for \( h_n < 1 \) by Corollary 4.3. Now in the case \( h_n < 1 \),

\[
\frac{|w_n - f(\theta^*)|}{|w_n|} = \frac{|f_n|}{|w_n|} \left| \frac{w_n - f_n}{f_n} \right| + \left| \frac{f_n - f(\theta^*)}{f_n} \right| \leq \frac{1}{|1 + \varepsilon_n|} (|\varepsilon_n| + H_n) \\
\leq 2 \left( \frac{H_n}{14400} + H_n \right) \quad \text{since } |\varepsilon_n| \leq \frac{1}{2} \\
\leq 2.5 H_n \leq 20000 h_n.
\]

Using the same argument as in Theorem 5.A, \( w_n = \prod_{m=1}^{n} (1 - h_m) \) converges to a nonzero number only if \( h_n \to 0 \) and hence only if \( |w_n - f(\theta^*)| \to 0 \). Since there is no critical value on \((0, w_0]\), this is possible only if \( w_n \to f(\theta^*) = 0 \). Again using the same argument as in Theorem 5.A, we conclude that \( z_n \to \theta^* \) where \( f(\theta^*) = 0 \). Now we start an induction to show (1). Suppose \( f_n/w_n = 1 + \varepsilon_n \), \( |\varepsilon_n| \leq H_n/14400 \leq 1/2 \). Then we will show that either \( z_{n+1} \) is an approximate zero or it satisfies \( f_{n+1}/w_{n+1} = 1 + \varepsilon_{n+1} \), \( |\varepsilon_{n+1}| \leq H_{n+1}/14400 \leq 1/2 \). Recall that \( z_{n+1} = E_{k,h,f}(z_n) \), where \( h = (f_n - w_{n+1})/f_n \) and \( w_{n+1} = (1 - h) w_n \). Note that

\[
|h| = |f_n - w_{n+1}| = |f_n - (1 - h_n) w_n| = |f_n - (1 - h_n)(1 + \varepsilon_n) f_n| \\
= |1 - (1 - h_n)(1 + \varepsilon_n)| = |h_n - \varepsilon_n (1 - h_n)| \leq h_n + |\varepsilon_n|
\]

\[
\leq \frac{H_n}{308} + \frac{H_n}{14400} \leq \frac{H_n}{300}.
\]

Applying Theorem 3.2 to \( z_n \) with \( h \), we have

\[
\frac{f_{n+1}}{f_n} = 1 - h + h \delta, \quad \text{where } |\delta| \leq B_k \left( \frac{1}{300} \right) \leq \frac{1}{145} \text{ for all } k,
\]

\[
\frac{f_{n+1}}{w_{n+1}} = 1 + \frac{h \delta}{1 - h}, \quad \text{since } w_{n+1} = (1 - h) f_n,
\]

and

\[
\frac{w_{n+1}}{f_{n+1}} = 1 + \varepsilon_{n+1} = \frac{1}{1 + \mu}, \quad \mu = \frac{h \delta}{1 - h}.
\]

Note that

\[
H_{n+1} = \frac{R_{n+1}}{|f_{n+1}|} \geq \left| \frac{f_n}{f_{n+1}} \right| \frac{R_n - |f_n - f_{n+1}|}{|f_n|} \quad \text{by Lemma 2.8(2)}
\]

\[
\geq \frac{1}{|1 - h + h \delta|} |H_n - |h - h \delta||
\]

\[
\geq \frac{1}{|1 - h + h \delta|} \left( H_n - \frac{H_n}{300} \left( 1 + \frac{145}{145} \right) \right)
\]

\[
\geq \frac{296 H_n}{297 |1 - h + h \delta|}.
\]

Now

\[
|\mu| = \left| \frac{h \delta}{1 - h} \right| \leq \frac{H_n}{300} \frac{145}{|1 - h|} \leq \left| \frac{1 - h + h \delta}{1 - h} \right| \frac{H_{n+1} 297}{296 300 145} \frac{1}{43000}.
\]

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Note that if $|\mu| \geq \frac{1}{4}$, then

$$H_{n+1} \geq \frac{|\mu|}{|1 + \mu|} 43000 \geq 7200$$

and hence $z_{n+1}$ is an approximate zero. If $z_{n+1}$ is not an approximate zero then $H_{n+1} < 7200$ and $|\mu| < \frac{1}{4}$. Hence we have

$$|\varepsilon_{n+1}| \leq 2|\mu| \leq 2 \cdot \frac{5}{4} \cdot \frac{H_{n+1}}{43000} \leq \frac{H_{n+1}}{30000} \leq \frac{H_{n+1}}{14400} \leq \frac{1}{2}. \quad \Box$$

**Acknowledgment.** This work is partly extracted from the author’s thesis at CUNY and I would like to thank my advisor, Mike Shub, for stimulating discussions and encouragement throughout graduate school. I also would like to thank Professor Steve Smale for his interest in this work. I would like to express my very special thanks to the referee of this paper for his patience, for his careful reading, and for his helpful comments.

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