A Table of Elliptic Integrals: Cubic Cases*

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Abstract. Forty-one integrands that are rational except for the square root of a cubic polynomial with known real zeros are integrated in terms of \( R \)-functions for which Fortran codes are available. In contrast to conventional tables the interval of integration is not required to begin or end at a singular point of the integrand. The table contains one elliptic integral of the first kind, 26 of the second kind, and 14 of the third kind. Only 10 of the integrals are treated in standard tables, which list a large number of special cases that are unified here.

1. Introduction. Two earlier installments [2], [3] of a new table of elliptic integrals deal primarily with "quartic cases" in which the integrand is rational except for the square root of a quartic polynomial with known real zeros. In this paper we consider "cubic cases" of the form

\[
[p] = [p_1, \ldots, p_4] = \int_y^x \prod_{i=1}^4 (a_i + b_i t)^{p_i/2} \, dt,
\]

where \( p_1, p_2, p_3 \) are odd integers, \( p_4 \) is an even integer (omitted if it is zero), and all quantities are real. Integrands containing complex conjugate factors will be considered in a later paper. Although a cubic case can usually be calculated numerically by choosing \( a_i = 1 \) and \( b_i = 0 \) for some value of \( i \) in a suitable quartic case, it is preferable to have an explicit formula, which is often tedious to obtain in a uniform notation from the quartic case.

The integral (1.1) is an elliptic integral of the third kind if \( p_4 \) is even and negative. Otherwise, it is second kind except for \([-1, -1, -1]\), which is first kind. Many integrals like

\[
\int (\alpha \cos^2 \theta + \beta \sin^2 \theta)^{p_1/2} d\theta \quad \text{and} \quad \int (\alpha + \beta z^2)^{p_1/2} (\gamma + \delta z^2)^{p_2/2} dz
\]

can be put in the form (1.1) by letting \( t = \sin^2 \theta \) or \( t = z^2 \).

The integrals in the table are expressed in terms of four \( R \)-functions:

\[
(1.2) \quad R_F(x, y, z) = \frac{1}{2} \int_0^\infty \frac{[(t + x)(t + y)(t + z)]^{-1/2}}{[\gamma + \delta z^2]} \, dt,
\]

\[
(1.3) \quad R_J(x, y, z, w) = \frac{3}{2} \int_0^\infty \frac{[(t + x)(t + y)(t + z)]^{-1/2}}{[(t + w)^{1/2}} \, dt,
\]

Received July 8, 1988.

1980 Mathematics Subject Classification (1985 Revision). Primary 33A25; Secondary 33A30.

*Part of this work was done at the University of Maryland, where the author was a visitor at the Institute for Physical Science and Technology, with the support of AROD contract DAAG 29-80-C-0032. The rest was done in the Ames Laboratory, which is operated for the U. S. Department of Energy by Iowa State University under contract no. W-7405-ENG-82. The work was supported by the Director of Energy Research, Office of Basic Energy Sciences.
and two special cases,

\[ R_C(x, y) = R_F(x, y, y) \quad \text{and} \quad R_D(x, y, z) = R_J(x, y, z, z). \]

The functions \( R_F \) and \( R_J \) are symmetric in \( x, y, z \); \( R_F \) and \( R_C \) are homogeneous of degree \(-1/2\); \( R_J \) and \( R_D \) are homogeneous of degree \(-3/2\); each of the four functions has the value unity when all its arguments are unity; and \( R_C \) and \( R_J \) are interpreted as Cauchy principal values when the last argument is negative. Fortran codes for numerical computation of all four functions, including Cauchy principal values, are listed in the Supplements to [2] and [3]. The functions \( R_F, R_D, \) and \( R_J \) respectively replace Legendre’s integrals of the first, second, and third kinds, while \( R_C \) includes the inverse circular and inverse hyperbolic functions.

To select integrals that are relatively simple, we arbitrarily require \( \sum |p_i| \leq 7 \) and \( \sum p_i \leq 3 \). Apart from permutation of subscripts in (1.1), there are just 40 cases of this kind. The table in Section 2 contains all 40 as well as \([-3, -3, -3]\), while only 10 of the 41 are included in [4] and nine in [5]. Each of the formulas for \([1, -1, -1], [1, 1, -1], [-1, -1, -3], \) and \([-1, -1, -5]\) unifies 18 formulas in [4], and that for \([1, -1, -3]\) unifies 36. Moreover, the table in Section 2 does not require the interval of integration to begin or end at a singular point of the integrand.

Derivation of the formulas is discussed in Section 3. All integral formulas have been checked by numerical integration, and some details of the checks are given in Section 4.

2. Table of Cubic Cases. We assume \( x > y \) and \( a_i + b_i t > 0, y < t < x, \) for \( i = 1, 2, 3 \). Assumptions about \( a_4 + b_4 t \) will be stated where necessary. We define

\[
(2.1) \quad d_{ij} = a_i b_j - a_j b_i, \quad r_{ij} = \frac{d_{ij}}{b_i b_j} = \frac{a_i}{b_i} - \frac{a_j}{b_j};
\]

\[
(2.2) \quad X_i = (a_i + b_i t)^{1/2}, \quad Y_i = (a_i + b_i y)^{1/2};
\]

\[
(2.3) \quad U_i = \frac{(X_i Y_j Y_k + Y_i X_j X_k)}{(x - y)},
\]

where \( i, j, k \) is any permutation of \( 1, 2, 3 \);

\[
(2.4) \quad W_2^2 = U_1^2 - b_4 d_{12} d_{13} / d_{14};
\]

\[
(2.5) \quad Q_2^2 = (X_4 Y_4 W_2 / X_1 Y_1)^2, \quad P_2^2 = Q_2^2 + b_4 d_{24} d_{34} / d_{14};
\]

\[
(2.6) \quad A(p_1, \ldots, p_n) = X_1^{p_1} \cdots X_n^{p_n} - Y_1^{p_1} \cdots Y_n^{p_n}.
\]

These definitions imply, if \( P_2 \) is chosen positive,

\[
(2.7) \quad P_2 = (X_1^{-1} X_2 X_3 Y_4^{-1} Y_1^{-1} Y_2 Y_3 X_4^2) / (x - y),
\]

\[
(2.8) \quad U_2^2 = U_1^2 - b_3 d_{12}, \quad U_3^2 = U_1^2 - b_2 d_{13} = U_2^2 - b_1 d_{23},
\]

\[
(2.9) \quad W_2^2 = U_2^2 + b_1 d_{12} d_{34} / d_{14} = U_3^2 + b_1 d_{13} d_{24} / d_{14}.
\]

If one limit of integration is infinite, (2.3) simplifies to

\[
(2.10) \quad U_i = (b_j b_k)^{1/2} Y_i, \quad x = +\infty,
\]

\[
U_i = (b_j b_k)^{1/2} X_i, \quad y = -\infty,
\]

where the square roots are nonnegative, while

\[
(2.11) \quad Q_2^2 = (b_4 / b_1) (Y_4 W_2 / Y_1)^2, \quad x = +\infty,
\]

\[
Q_2^2 = (b_4 / b_1) (X_4 W_2 / X_1)^2, \quad y = -\infty.
\]
Cubic cases will be expressed in terms of the quantities

\begin{align}
I_{1c} &= 2R_F(U_3^2, U_2^2, U_1^2), \\
I_{2c} &= \frac{2}{3}d_{12}d_{13}R_D(U_3^2, U_2^2, U_1^2) + 2X_1Y_1/U_1, \\
I_{3c} &= \frac{-2b_1d_{12}d_{13}}{3d_{14}}R_J(U_3^2, U_2^2, U_1^2, W_2^2) + 2R_C(P_2^2, Q_2^2),
\end{align}

which are integrals of the first, second, and third kinds, respectively. It will be seen from the tables that

\begin{align}
I_{1c} &= [1, -1, -1], \quad I_{2c} = [1, -1, -1], \quad I_{3c} = [1, -1, -1, -2].
\end{align}

Thus $I_{3c}$ reduces to $I_{2c}$ if $a_4 = 1$ and $b_4 = 0$. The extra subscript $c$ stands for "cubic," and the quantities defined here are obtained from those used in [3] for quartic cases. Specifically, if we put $a_4 = 1$ and $b_4 = 0$ and subsequently replace the subscript 5 by 4, then $(U_{12}, U_{13}, U_{14})$ reduces to $(U_3, U_2, U_1)$, $(W, P, Q)$ to $(W_2, P_2, Q_2)$, and $(I_1, I_2, I_3, I_4)$ to $(I_{1c}, I_{2c}, I_{3c}, I_{2c})$.

It is convenient to define also the quantities

\begin{align}
J_{1c} &= d_{12}d_{13}I_{1c} - 2b_1A(1, 1, 1) \\
&= 2d_{12}d_{13}R_F(U_3^2, U_2^2, U_1^2) - 2b_1A(1, 1, 1), \\
J_{2c} &= b_1I_{2c} - 2A(1, 1, -1) \\
&= \frac{2}{3}b_2d_{12}d_{13}R_D(U_3^2, U_2^2, U_1^2) + \frac{2d_{13}X_2Y_2}{X_3Y_3U_1}.
\end{align}

The first of these appears in the formula for $[1, 1, -1]$ and is transmitted by recurrence relations to a dozen others; likewise, $J_{2c}$ is transmitted from $[1, -1, -3]$. The second line of (2.17) follows from the first with the help of the identity

\begin{align}
(x - y)U_1X_3Y_3A(1, 1, -1) = X_2Y_2(X_1^2Y_3^2 - X_3^2Y_1^2) + X_1X_3Y_1Y_3(X_2^2 - Y_2^2) \\
= (x - y)(d_{31}X_2Y_2 + b_2X_1X_3Y_1Y_3).
\end{align}

It is important to use $I_{1c}$ and $J_{2c}$, not $J_{1c}$ or $I_{2c}$, to evaluate integrals with $\sum p_i < -2$, which converge when $x = +\infty$ or $y = -\infty$. Both $I_{1c}$ and $I_{2c}$ then become infinite while $I_{1c}$ and $I_{2c}$ are finite. The second term in the second line of (2.17) becomes

\begin{align}
2d_{13}X_2Y_2/X_3Y_3U_1 &= 2d_{13}Y_2/b_1Y_1Y_3, \quad x = +\infty, \\
2d_{13}X_2Y_2/X_3Y_3U_1 &= -2d_{13}X_2/b_3X_1X_3, \quad y = -\infty.
\end{align}

If one limit of integration is a branch point of the integrand, then $X_i$ or $Y_i$ is 0 for some value of $i \leq 3$, and one of the two terms on the right-hand side of (2.3) vanishes. If $X_1Y_1 = 0$ then $P_2$ and $Q_2$ are infinite, and the $R_C$ function in (2.14) vanishes by homogeneity. If both limits of integration are branch points, the elliptic integral is called complete, and $U_1U_2U_3 = 0$. It is not assumed that $b_i \neq 0$ nor that $d_{ij} \neq 0$ unless one of these quantities occurs in a denominator. The relation $d_{ij} = 0$ is equivalent to proportionality of $a_i + b_i t$ and $a_j + b_j t$.

We shall now list 41 cases of

\begin{align}
[p_1, \ldots, p_4] = \int_y^z (a_1 + b_1 t)^{p_1/2} \cdots (a_4 + b_4 t)^{p_4/2} dt,
\end{align}
18 with $p_4 = 0$, nine with $p_4 = 2$ or 4, and 14 with $p_4 = -2$ or $-4$. Only the last 14 are integrals of the third kind involving $I_{3c}$. We omit $p_4 = 0$ in the first 18:

\begin{align}
(2.21) \quad [-1, -1, -1] &= I_{1c}. \\
(2.22) \quad [1, -1, -1] &= I_{2c}. \\
(2.23) \quad [1, 1, -1] &= [(b_1 d_{23} + b_2 d_{13}) I_{2c} - J_{1c}] / 3b_1 b_3. \\
(2.24) \quad [1, 1, 1] &= [-2b_1 b_2 b_3 (r_{12} r_{13} + r_{23}^2) I_{2c} + (r_{12} + r_{13}) J_{1c} + 6A(3, 1, 1)] / 15b_1. \\
(2.25) \quad [1, -1, -3] &= (J_{2c} - d_{12} J_{1c}) / d_{23}. \\
(2.26) \quad [-1, -1, -3] &= (b_3 J_{2c} - b_2 d_{13} J_{1c}) / d_{13} d_{23}. \\
(2.27) \quad [1, 1, -3] &= [2b_2 I_{2c} - d_{12} J_{1c} - 2A(1, 1, -1)] / b_3. \\
(2.28) \quad [3, -1, -1] &= [2(b_2 d_{13} + b_3 d_{12}) I_{2c} - J_{1c}] / 3b_2 b_3. \\
(2.29) \quad [3, 1, -1] &= [b_1 b_3 b_2 (3r_{13}^2 + 7r_{13} r_{23} - 2r_{23}^2) I_{2c} - (3r_{13} + r_{23}) J_{1c} + 6A(3, 1, 1)] / 15b_3. \\
(2.30) \quad [3, -1, -3] &= [(b_1 d_{23} + b_2 d_{13}) I_{2c} - d_{12} d_{13} J_{1c} - 2d_{23} A(1, 1, -1)] / b_3 d_{23}. \\
(2.31) \quad [3, 1, -3] &= [(b_1 d_{23} + 7b_2 d_{13}) I_{2c} - 4d_{12} d_{13} J_{1c} + 2b_1 A(1, 1, 1) - 6d_{13} A(1, 1, -1)] / 3b_2^2. \\
(2.32) \quad [1, -3, -3] &= [2b_3 J_{2c} - (b_2 d_{13} + b_3 d_{12}) J_{1c} + 2d_{23} A(1, -1, -1)] / d_{23}^2. \\
(2.33) \quad [-1, -3, -3] &= [b_3 (b_2 d_{13} + b_3 d_{12}) J_{2c} - 2b_2 b_3 d_{12} d_{13} J_{1c} + 2b_2 d_{13} d_{23} A(1, -1, -1)] / d_{12} d_{13} d_{23}. \\
[-3, -3, -3] &= (b_3 / d_{12} d_{13} d_{23}) ((2 / r_{12} r_{13} r_{23}) (r_{12} r_{13} + r_{23}^2) J_{2c} - (b_1 b_2 / r_{23}) (r_{12} + r_{13}) J_{1c} + (2 / r_{12}) [b_1 r_{23} A(-1, 1, -1) + b_2 r_{13} A(1, -1, -1)]).
\end{align}

The next nine integrals have $p_4 = 2$ or 4. No restriction is placed on $a_4$ or $b_4$.

\begin{align}
(2.35) \quad [1, -1, -5] &= [-b_2 (1 + r_{12} / r_{13}) J_{2c} + 2b_2 d_{12} J_{1c} - 2d_{23} A(1, 1, -3)] / 3d_{23}. \\
(2.36) \quad [1, 1, -5] &= [(r_{13}^{-1} + r_{23}^{-1}) J_{2c} - d_{12} r_{23}^{-1} J_{1c} - 2b_3 A(1, 1, -3)] / 3b_2^2. \\
(2.37) \quad [-1, -1, -5] &= [-2(r_{13}^{-1} + r_{23}^{-1}) J_{2c} + b_1 b_2 (1 + 2r_{13} / r_{23}) J_{1c} - 2b_3 A(1, 1, -3)] / 3d_{13} d_{23}. \\
(2.38) \quad [5, -1, -1] &= [b_2^2 b_3 (8r_{12}^2 + 8r_{13}^2 + 7r_{12} r_{13}) J_{2c} - 4b_1 (r_{12} + r_{13}) J_{1c} + 6b_1 A(3, 1, 1)] / 15b_2 b_3.
\end{align}
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(2.43) \[ [1, -1, -3, 2] = \left( b_4 d_{23} - b_2 d_{34} \right) I_{2c} + d_{12} d_{34} I_{1c} + 2d_{34} A(1, 1, -1) / b_3 d_{23}. \]

(2.44) \[ [1, 1, -3, 2] = \left( b_4 / 3b_3 \right) \left[ b_2 (r_{13} + r_{23} - 6r_{34}) I_{2c} + d_{12} (3r_{34} - r_{13}) I_{1c} + (2/b_3) A(1, 1, 1) + 6r_{34} A(1, 1, -1) \right]. \]

(2.45) \[ [3, -1, -1, 2] = \left( b_4 / 15b_2 b_3 \right) \left[ b_1 b_2 b_3 \left[ 8r_{12}^2 + 8r_{13}^2 + 7r_{12} r_{13} \right] - 10r_{14} (r_{12} + r_{13}) I_{2c} - (4r_{12} + 4r_{13} - 5r_{14}) J_{1c} + 6A(3, 1, 1) \right]. \]

(2.46) \[ [-1, -1, -1, 4] = \left( b_4^2 / 3b_1 \right) \left[ -2(r_{14} + r_{24} + r_{34}) I_{2c} + 3b_1 r_{14}^2 I_{1c} - J_{1c} / b_1 b_2 b_3 \right]. \]

(2.47) \[ [1, -1, -1, 4] = \left( b_4^2 / 15b_1 b_2 b_3 \right) \left[ b_1 b_2 b_3 \left[ 8r_{23}^2 + 3r_{12} r_{13} - 5r_{14}^2 + 20r_{24} r_{34} \right] I_{2c} + 2(r_{14} + 2r_{24} + 2r_{34}) J_{1c} + 6A(3, 1, 1) \right]. \]

The final 14 integrals have \( p_4 = -2 \) or \(-4 \) and are integrals of the third kind. We assume either that \( a_4 + b_4 t \) is positive on the closed interval of integration or else, if \( p_4 = -2 \), that it changes sign in the open interval of integration. In the latter case the integral is interpreted as a Cauchy principal value (see [3, Section 6]).

(2.48) \[ [1, -1, -1, -2] = I_{3c}. \]

(2.49) \[ [-1, -1, -1, -2] = \left( b_4 I_{3c} - b_1 I_{1c} \right) / d_{14}. \]

(2.50) \[ [1, 1, -1, -2] = \left( d_{24} I_{3c} + b_2 I_{2c} \right) / b_4. \]

(2.51) \[ [1, 1, 1, -2] = \left[ 3b_2 r_{24} d_{34} I_{3c} + b_2 b_3 (r_{14} + r_{24} + r_{34}) I_{2c} - J_{1c} / b_1 \right] / 3b_4. \]

(2.52) \[ [1, 1, -3, -2] = \left( d_{24} I_{3c} - J_{2c} + d_{12} I_{1c} \right) / d_{34}. \]

(2.53) \[ [1, -1, -3, -2] = \left( b_4 d_{23} I_{3c} - b_3 J_{2c} + b_3 d_{12} I_{1c} \right) / d_{23} d_{34}. \]

(2.54) \[ [-1, -1, -3, -2] = \left[ (b_4^2 / d_{14}) I_{3c} - (b_3^2 / d_{13} d_{23}) J_{2c} + (r_{23}^{-1} - r_{14}^{-1}) I_{1c} \right] / d_{34}. \]

(2.55) \[ [3, -1, -1, -2] = \left( d_{14} I_{3c} + b_1 I_{2c} \right) / b_4. \]

(2.56) \[ [3, 1, -1, -2] = b_1 b_2 r_{14} r_{24} I_{3c} + (b_1 b_2 / 3b_4) (2r_{13} + 2r_{14} + r_{24}) I_{2c} - J_{1c} / 3b_3 b_4. \]

(2.57) \[ [3, 1, 1, -2] = \left( d_{14} d_{24} d_{34} / b_3^2 \right) I_{3c} + \left( b_1 b_2 b_3 \left[ 5r_{14} (r_{14} + r_{24} + r_{34}) - r_{12}^2 - r_{13}^2 - r_{23}^2 \right] I_{2c} - (3r_{14} + r_{24} + r_{34}) J_{1c} + 6A(3, 1, 1) / 15 b_4. \]

(2.58) \[ [1, 1, 1, -4] = \left[ b_2 b_3 (r_{24} + r_{34} + r_{24} r_{34} / r_{14}) I_{3c} + (3b_2 b_3 / b_4) I_{2c} - (d_{12} d_{14} / d_{14}) J_{1c} - 2A(1, 1, 1, -2) \right] / 2b_4. \]

In the next three formulas we use the abbreviation

(2.59) \[ K_{2c} = b_2 b_3 I_{2c} - 2b_4 A(1, 1, 1, -2) = b_3 J_{2c} - 2d_{34} A(1, 1, -1, -2). \]
The second equality, showing that \( K_{2c} \) is finite if \( x = +\infty \) or \( y = -\infty \), follows from (2.17) and [3, (4.8)].

(2.60) \[
[1, 1, -1, -4] = \left[ b_1 b_2 b_3 b_4 (r_{34}^2 - r_{13} r_{23}) I_{3c} + b_1 r_{14} K_{2c} - d_{12} d_{13} I_{1c} \right]/2d_{14} d_{34}.
\]

(2.61) \[
[1, -1, -1, -4] = (1/2b_4) (r_{14}^{-1} - r_{24}^{-1} - r_{34}^{-1}) I_{3c} + [K_{2c} - (b_4 d_{12} d_{13}/d_{14}) I_{1c}]/2d_{24} d_{34}.
\]

(2.62) \[
[-1, -1, -1, -4] = - (1/2d_{14}) (r_{14}^{-1} + r_{24}^{-1} + r_{34}^{-1}) I_{3c} + (b_4/2d_{14} d_{24} d_{34}) K_{2c} + (b_1/d_{14})^2 (1 - r_{12} r_{13}/2r_{24} r_{34}) I_{1c}.
\]

3. Derivation of the Formulas. Six of the 41 formulas are obtained by putting \( a_4 = 1 \) and \( b_4 = 0 \) (see the remarks following (2.15)) in suitable quartic cases in [3]. Thus \([-1, -1, -1], [1, -1, -1], [1, 1, -1], [1, -1, -3], [-1, -1, -3], \) and \([1, 1, -3, -3] \) come respectively from \([-1, -1, -1, -1], [1, -1, -1, -3], [1, 1, -1, -5], [1, -1, -3, -3], [-1, -1, -3, -3], \) and \([1, 1, -3, -3] \). Seven more are obtained by putting \( a_4 = 1 \) and \( b_4 = 0 \) and then replacing \( a_5 \) by \( a_4 \) and \( b_5 \) by \( b_4 \). Thus \([-1, -1, -1, 2], [1, -1, -1, -2], [3, -1, -1, -2], [1, 1, 1, -4], [1, 1, -1, -4], [1, -1, -1, -4], \) and \([-1, -1, -1, -4] \) come respectively from \([-1, -1, -1, 2], [1, -1, -1, -2], [3, -1, -1, -2], [1, 1, 1, -4], [1, 1, -1, -4], [1, -1, -1, -4], \) and \([-1, -1, -1, -4] \). The formulas are often simplified by using identities such as [3, (4.6) to (4.9)].

The remaining cases are then obtained by recurrence relations. Let \( e_i \) denote an \( n \)-tuple with 1 in the \( i \)th place and 0's elsewhere (for example, \([p + 2e_1] = [p_1 + 2, p_2, \ldots, p_n]\)). From [4, Section 4] we have

\[
(A_i) \quad (p_1 + \cdots + p_n + 2)b_i [p] = \sum_{j \neq i} p_j d_{ji} [p - 2e_j] + 2A(p + 2e_i),
\]

\[
(B_{ij}) \quad d_{ij} [p] = b_j [p + 2e_i] - b_i [p + 2e_j],
\]

\[
(C_{ij}) \quad b_j [p] = b_i [p - 2e_i + 2e_j] + d_{ij} [p - 2e_i].
\]

To get \([1, 1, 1] \) we use (A1) and evaluate \([1, -1, 1] \) by interchanging the subscripts 2 and 3 in \([1, 1, -1] \). Equations (2.28) to (2.31) then follow in order from (C12), (C13), (C13), and (C13). To get \([\pm 1, -3, -3] \) we use (B23) and evaluate \([\pm 1, -3, -1] \) by interchanging the subscripts 2 and 3 in \([\pm 1, -1, -3] \). Then \([-3, -3, -3] \) follows from putting \([p] = [-1, -3, -3] \) and \( i = 1 \) in [2, (5.5)] and using [3, (4.8)]. After \([1, -1, -5] \) has been obtained from (A2) with \([p] = [-1, -1, -3] \), Eqs. (2.36) and (2.37) follow respectively from (C23) and (B13). Equation (2.38) comes from (C12).

Equations (2.40) to (2.47) follow in order from (C42), (C43), (C43), (C43), (C42), (C41), and (C42). To get (2.49), (2.50), and (2.51), we use (B14), (C24), and (C34), respectively. Equations (2.52), (2.53), (2.54), (2.56), and (2.57) follow in order from (B34), (B34), (B34), (C24), and (C14).

4. Numerical Checks. The 41 formulas in Section 2 were checked numerically when \( x = 2.0, y = 0.5, a_i = 0.1 + 0.2i, b_i = 0.5 - 0.2i, 1 \leq i \leq 4 \). In each formula the integral on the left side, defined by (2.20), was integrated numerically by the SLATEC code QNG. On the right-hand side, \( I_{1c}, I_{2c}, \) and \( I_{3c} \) were calculated from
(2.12) to (2.14) by using the codes for R-functions in the Supplements to [2] and [3]. The remaining calculations, including those of \(J_{1c}, J_{2c},\) and \(K_{2c}\) by (2.16), (2.17), and (2.59), were done with a hand calculator. For each of the 41 formulas the values obtained for the two sides agreed to better than one part in a million.

Some of the intermediate values in these calculations are listed here:

\[
\begin{align*}
U_1^2 &= 0.41309998, & W_2^2 &= 0.38909998, \\
U_2^2 &= 0.40109998, & P_2^2 &= 0.24016665, \\
U_3^2 &= 0.43709998, & Q_2^2 &= 0.21616665, \\
R_C(P_2^2, Q_2^2) &= 2.1128946, & I_{1c} &= 3.0973715, \\
R_F(U_3^2, U_2^2, U_1^2) &= 1.5486858, & I_{2c} &= 2.0520132, \\
R_D(U_3^2, U_2^2, U_1^2) &= 3.7353179, & J_{3c} &= 4.2877248, \\
R_J(U_3^2, U_2^2, U_1^2, W_2^2) &= 3.8709720, & J_{1c} &= -0.00688951, \\
& & J_{2c} &= -0.80566308,
\end{align*}
\]

\[
\begin{align*}
A(1, 1, 1) &= 0.16015635, & K_{2c} &= 0.78110328, \\
A(1, 1, -1) &= 0.50543220, \\
A(1, -1, -1) &= 0.48163106, & A(3, 1, 1) &= 0.32463223, \\
A(-1, 1, -1) &= -0.12403646, & A(1, 1, 1, -2) &= 1.3360390, \\
A(1, 1, -3) &= 1.2956636, & A(1, 1, -1, -2) &= 2.9189040.
\end{align*}
\]

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