A NOTE ON DISCRETE SOLUTIONS OF THE PLATEAU PROBLEM

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Abstract. In this paper we prove theorems for convergence of discrete solutions of the Plateau problem under the assumption that the contour is rectifiable.

1. Introduction

In [7] the discrete solutions of the Plateau problem were defined, and some theorems for its convergence were proved under a very restrictive condition. The purpose of this paper is to show that we can obtain the same conclusions if the contour is rectifiable.

It is well known [2, pp. 107–118] that the Plateau problem can be defined as the following variational problem:

Let \( D = \{(u, v) \in \mathbb{R}^2| u^2 + v^2 < 1\} \) be the unit disk with boundary \( \partial D \) and let \( \Gamma \) be a Jordan curve in \( n \)-dimensional Euclidean space \( \mathbb{R}^n, n \geq 2 \). Let \( C(\overline{D}; \mathbb{R}^n) \) be the space of continuous maps from \( \overline{D} \) into \( \mathbb{R}^n \), and let \( H^1(D; \mathbb{R}^n) \) be the ordinary Sobolev space (for the exact definitions, see [7]).

We define the class of maps by

\[
X_\Gamma = \{ f \in C(\overline{D}; \mathbb{R}^n) \cap H^1(D; \mathbb{R}^n) | f(\partial D) = \Gamma, \ f|_{\partial D} \text{ is monotone} \},
\]

where monotone means that, for each \( p \in \Gamma, \ (f|_{\partial D})^{-1}(p) \subset \partial D \) is connected. \( X_\Gamma \) may be empty [4, p. 58], but if \( \Gamma \) is rectifiable, then \( X_\Gamma \neq \emptyset \) [2, pp. 129–131]. We choose six arbitrary distinct points \( z_1, z_2, z_3 \in \partial D \) and \( \zeta_1, \zeta_2, \zeta_3 \in \Gamma \), and we define the subset of \( X_\Gamma \) by

\[
X_{\Gamma}^{\text{tp}} = \{ f \in X_\Gamma | f(z_i) = \zeta_i, \ i = 1, 2, 3 \},
\]

where the superscript “tp” stands for “three-point condition”. The Plateau problem is to find stationary points of the energy functional

\[
E(f) = \frac{1}{2} \int_D \left( |f_u|^2 + |f_v|^2 \right) \, du \, dv
\]

in \( X_{\Gamma}^{\text{tp}} \), where \( f_u = (\partial f_1/\partial u, \ldots, \partial f_n/\partial u) \) and \( f_v = (\partial f_1/\partial v, \ldots, \partial f_n/\partial v) \).

A solution of the Plateau problem is called a minimal surface spanned in \( \Gamma \) even if it is not a minimal point of the energy functional. For the existence of
the minimal surfaces the following theorem is known [2, pp. 101–105; 4, p. 71]:

**Theorem A (Douglas-Rado).** Let \( e_\Gamma = \inf\{E(f) : f \in X^{1p}_\Gamma\} \). If \( X^{1p}_\Gamma \neq \emptyset \), then there exists a map \( x \in X^{1p}_\Gamma \) such that \( E(x) = e_\Gamma \).

An \( x \) as in Theorem A is called a **Douglas solution**. Evidently, a Douglas solution is a minimal surface.

In §2 we define a (stable) discrete minimal surface using the simplest finite element scheme. In §3 we prove the relative compactness of bounded subsets of discrete maps when the Jordan curve is rectifiable. In [7] a very restrictive condition was assumed to prove the relative compactness, so §3 is the main part of this paper.

### 2. Definition of the discrete minimal surface

Let \( \Omega \subset D \) be a regular triangulation of \( D \) with \( \overline{\Omega} = \bigcup K_i \), where \( K_i \) are triangles. With the triangulation \( \Omega \) we associate the mesh size of \( \Omega \) defined by

\[
|\Omega| = \max_i \text{diam}(K_i).
\]

We assume that there exists a positive constant \( \omega \) which is independent of the triangulation \( \Omega \) such that the following inequality holds for each triangle \( K_i \subset \Omega \):

\[
\text{(H1)} \quad \text{diam}(K_i)/\rho(K_i) \leq \omega,
\]

where \( \rho(K_i) = \sup\{\text{diam}(S) ; K_i \subset S : \text{ball}\} \).

Let \( S_\Omega \) be the set of functions which are continuous on \( \overline{\Omega} \) and linear on each triangle \( K_i \). Let \( S_\Omega \) be the set of maps from \( \overline{\Omega} \) into \( \mathbb{R}^n \) such that each component function belongs to \( S_\Omega \). Let \( N_\Omega = \{b_i\}_{i=1}^{N+N'} \) be the set of nodal points of \( \Omega \) where \( b_i \in \Omega^\circ \), the interior of \( \Omega \), for \( 1 \leq i \leq N \), and \( b_i \in \partial \Omega \) for \( N + 1 \leq i \leq N + N' \). We number \( \{b_{N+1}, \ldots, b_{N+N'}\} = N_\Omega \cap \partial D \) in counter-clockwise order. We assume that

\[
\text{(H2)} \quad \Omega \text{ is of nonnegative type}.
\]

For the definition of the term “nonnegative type”, see [1, 7]. This assumption is for the **discrete maximum principle** [7, Lemma 3]. We introduce the admissible class of triangulations of \( D \) defined by

\[
\Delta^{1p} = \{\Omega | z_1, z_2, z_3 \in N_\Omega, \Omega \text{ satisfies (H1), (H2)}\}.
\]

When \( \Omega \) is given, we define

\[
X_{\Gamma,\Omega} = \{f \in S_\Omega|f(N_\Omega \cap \partial D) \subset \Gamma, f|_{\partial D} \text{ is } d\text{-monotone}\},
\]

where **d-monotone** means that the order of nodal points on \( \Gamma \) is the same as the order of nodal points on \( \partial D \). Let

\[
X^{1p}_{\Gamma,\Omega} = \{f \in X_{\Gamma,\Omega}|f(z_i) = \zeta_i, \quad i = 1, 2, 3\},
\]

and let \( E_\Omega(f) \) be the energy functional on \( \Omega \) defined by

\[
E_\Omega(f) = \frac{1}{2} \int \int_\Omega (|f_u|^2 + |f_v|^2) \, du \, dv.
\]
We extend \( f \in S_\Omega \) to \( D - \Omega \) as follows:

If \( p \in \partial \Omega \) and \( p \notin N_\Omega \), there exists an exterior normal half-line \( L_p \) of \( \partial \Omega \) on \( p \). For arbitrary \( q \in L_p \cap (D - \Omega) \), we define \( f(q) = f(p) \). Then the following estimate is valid:

\[
E_\Omega(f) \leq E(f) \leq (1 + C |\Omega|)E_\Omega(f) \quad \text{for any } f \in S_\Omega,
\]

where \( C \) is a constant which is independent of \( \Omega \) and \( f \).

**Definition 1.** Let \( \Omega \in \Delta_{ip} \).

(D1) \( f \in X_{\Gamma, \Omega}^{ip} \) is a stable \( d \)-minimal surface if there exists a positive constant \( \delta \) such that \( \|f - g\|_{C(\Omega; \mathbb{R}^n)} < \delta \) implies \( E_\Omega(f) \leq E_\Omega(g) \) for \( g \in X_{\Gamma, \Omega}^{ip} \).

(D2) \( f \in X_{\Gamma, \Omega}^{ip} \) is the \( d \)-Douglasolution if \( E_\Omega(f) = \inf \{ E_\Omega(g) : g \in X_{\Gamma, \Omega}^{ip} \} \).

**3. Relative compactness**

First, we recall a useful lemma [2, pp. 101–102; 4, pp. 67–68]. For any \( z \in \mathbb{R}^2 \) and any \( r > 0 \) we define

\[
C_{r, z} = \overline{B} \cap \{ w \in \mathbb{R}^2 : |w - z| = r \}.
\]

For \( f \in X_{\Gamma, \Omega}^{ip} \) we denote by \( l(f, C_{r, z}) \) the length of the image \( f(C_{r, z}) \). Let \( M \) be a constant with \( \varepsilon_r < M \).

**Lemma 2.** For arbitrary \( \delta, 0 < \delta < 1 \), and \( f \in X_{\Gamma, \Omega}^{ip} \) with \( E(f) \leq M \), there exists \( \rho, \delta \leq \rho \leq \delta^{1/2} \), depending on \( f \) and \( z \) such that

\[
l(f, C_{\rho, z})^2 \leq \lambda(\delta),
\]

where \( \lambda(\delta) = 8\pi M / \log(1/\delta) \).

For \( \Omega \in \Delta_{ip} \) and \( f \in X_{\Gamma, \Omega}^{ip} \) we define

\[
L(\Omega, f) = \max \{|f(b_i) - f(b_{i+1})| : b_i \in N_\Omega \cap \partial D, \ i = N + 1, \ldots, N + N'\},
\]

where \( b_{N+N'+1} = b_{N+1} \). The following lemma is valid.

**Lemma 3.** Let \( \Delta_{ip} \supset \{ \Omega_n \}_{n=1}^\infty \) be such that \( \lim_{n \to \infty} |\Omega_n| = 0 \), and let \( f_n \in X_{\Gamma, \Omega_n}^{ip} \). Suppose that \( \Gamma \) is rectifiable and \( E(f_n) \leq M \) for any \( n \). Then \( \lim_{n \to \infty} L(\Omega_n, f_n) = 0 \).

**Proof.** The proof is by contradiction. Assume that \( \limsup_{n \to \infty} L(\Omega_n, f_n) > 0 \). Then there exists a positive constant \( \varepsilon_0 \) such that, for any \( \xi > 0 \), there exist a positive integer \( m \) and \( b_i \in N_{\Omega_m} \cap \partial D \) such that

\[
|\Omega_m| < \xi \quad \text{and} \quad |f_m(b_i) - f_m(b_{i+1})| \geq \varepsilon_0.
\]

For \( b_i \in N_{\Omega_m} \cap \partial D \) and \( f_m \in X_{\Gamma, \Omega_m}^{ip} \) as in (3.2), a pair \( (\alpha_1, \alpha_2) \ (\alpha_i \in f_m(N_{\Omega_m} \cap \partial D), \ i = 1, 2) \) is said to be admissible if it satisfies the following properties: \( \Gamma_1 \), one of the two connected components of \( \Gamma - \{ \alpha_1, \alpha_2 \} \), contains at least two of \( \{ \zeta_1, \zeta_2, \zeta_3 \} \), and the other connected component \( \Gamma_2 \) contains
If \( \{\zeta_1, \zeta_2, \zeta_3\} \cap \{f_m(b_i), f_m(b_{i+1})\} \neq \emptyset \), for example in the case of \( \zeta_1 = f_m(b_i) \), a pair \((\alpha_1, \alpha_2)\) such that \( \alpha_1 = \zeta_1 = f_m(b_i) \) and \( \Gamma_1 \) contains at least one of \( \{\zeta_1, \zeta_2, \zeta_3\} \) and \( \Gamma_2 \) contains \( f_m(b_{i+1}) \) is also said to be admissible.

By a topological argument we can show that there exists a positive constant \( \eta \) depending on \( \{\Gamma, \zeta_1, \zeta_2, \zeta_3\} \) and \( \epsilon_0 \) such that \( |\alpha_1 - \alpha_2| \geq \eta \) for any admissible pair \((\alpha_1, \alpha_2)\) on \( \Gamma \).

Let \( b_k, b_h \in N_{\Omega_m} \cap \partial D \) be such that all of \((f_m(b_k+p), f_m(b_h+q))\) \((p, q = 0, 1)\) are admissible pairs. For \( b_j \in N_{\Omega_m} \cap \partial D \), we denote by \( \text{seg}(b_j) \) the segment which connects \( f_m(b_j) \) and \( f_m(b_{j+1}) \).

**Lemma 4.** Assume that there exist \( \beta_1 \in \text{seg}(b_k) \) and \( \beta_2 \in \text{seg}(b_h) \) such that \( |\beta_1 - \beta_2| < \eta/2 \). Then we have

\[
|f_m(b_k) - f_m(b_{k+1})| + |f_m(b_h) - f_m(b_{h+1})| > \eta.
\]

**Proof.** Since \( |f_m(b_k+p) - f_m(b_h+q)| > \eta \) \((p, q = 0, 1)\), we obtain (3.3) easily. \( \square \)

Let \( l(\Gamma) \) be the length of \( \Gamma \). Let \( A \) be the least integer that satisfies

\[
\left( \frac{l(\Gamma) - \epsilon_0}{\eta} \right) \leq A.
\]

We take sufficiently small \( \delta, 0 < \delta < 1 \), such that

\[
(2A - 1)(\lambda(\delta))^{1/2} < \eta/2,
\]

\[
2(A\delta^{1/2} + \gamma(\delta)) < \min\{|z_i - z_j| : i \neq j\},
\]

where \( \gamma(\delta) = \delta^{2^{t-1}} \). We set \( \xi = \gamma(\delta) \) in (3.2), and we choose and fix a positive integer \( m \) and \( b_i \in N_{\Omega_m} \cap \partial D \) as in (3.2). Then we have

\[
|\Omega_m| < \delta^{2^{t-1}}.
\]

Let \( z \in \partial D \) be the center of the shorter arc \( \widehat{b_i b_{i+1}} \). By Lemma 2 there exists a positive constant \( \rho, \delta \leq \rho \leq \delta^{1/2} \), such that \( l(f_m, C_{\rho, z}) \leq \lambda(\delta)^{1/2} \). Let \( l_i \) and \( r_1 \) be the left and right endpoint of \( C_{\rho, z} \) on \( \partial D \), respectively. Suppose that \( l_i \in \widehat{b_{k_i} b_{k_i+1}} \) and \( r_1 \in \widehat{b_{h_i} b_{h_i+1}} \), where \( b_{k_i}, b_{h_i} \in N_{\Omega_m} \cap \partial D \). Note that the pair \((f_m(b_{k_i}), f_m(b_{h_i}))\) is not admissible in the extraordinary case like Figure 1.

However, in such a case we can obtain a contradiction and prove this lemma immediately. Hence we may assume without loss of generality that all of the pairs \((f_m(b_{k_i+p}), f_m(b_{h_i+q}))\) \((p, q = 0, 1)\) are admissible because of (3.6). Note that, by (3.7), \( b_{k_i}, b_{h_i} \) and \( b_i \) are distinct. From (3.1) and (3.5) we have

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\[
|f_m(l_k) - f_m(r_k)| < \left( \frac{\eta}{2} \right) / (2A - 1) \leq \eta / 2. \quad \text{Thus, from Lemma 4, we obtain}
\]

\[
|f_m(b_k) - f_m(b_{k+1})| + |f_m(b_{k+1}) - f_m(b_{k+1+1})| > \eta.
\]

By Lemma 2 there exist positive constants \( \theta_1, \rho^2 \leq \theta_1 \leq \rho \), and \( \mu_1, \rho^2 \leq \mu_1 \leq \rho \) (\( \delta^2 \leq \theta_1, \mu_1 \leq \delta / 2 \)), such that

\[
l(f_m, C\theta_1, \rho^2) \lambda(\rho^2)^{1/2} \leq \lambda(\delta)^{1/2} < \frac{\eta}{2(2A - 1)}, \quad l(f_m, C_{\mu_1, \mu_1^*}) < \frac{\eta}{2(2A - 1)}.
\]

Let \( l_2 \) be the left endpoint of \( C\theta_1, \rho^2 \) and \( r_2 \) the right endpoint of \( C_{\mu_1, \rho^2} \). Let \( b_{k_2}, b_{h_2} \in N_{\Omega_n} \cap \partial D \) be nodal points such that \( l_2 \) and \( r_2 \) are on the arcs \( b_{k_2} \rightarrow b_{k_2+1} \) and \( b_{h_2} \rightarrow b_{h_2+1} \), respectively. Again, we may assume that all pairs \( (f_m(b_{k_2+p}), f_m(b_{h_2+q})) \) \( (p, q = 0, 1) \) are admissible because of (3.6). By (3.7), \( b_{k_1}, b_{h_1} \) \( (j = 1, 2) \) and \( b_i \) are distinct. From (3.1) and (3.5) we have \( |f_m(l_2) - f_m(r_2)| < 3(\eta/2) / (2A - 1) \leq \eta / 2 \). Thus, by Lemma 4 and (3.8), we obtain

\[
\sum_{j=1}^{2} (|f_m(b_{k_j}) - f_m(b_{k_{j+1}})| + |f_m(b_{h_j}) - f_m(b_{h_{j+1}})|) > 2\eta.
\]

Repeating this procedure \( A \) times, we conclude that there exist \( 2A \) distinct nodal points on \( \partial D \) such that

\[
\sum_{j=1}^{A} (|f_m(b_{k_j}) - f_m(b_{k_{j+1}})| + |f_m(b_{h_j}) - f_m(b_{h_{j+1}})|) > A\eta.
\]
By (3.4) the right side of (3.9) is greater than \( l(\Gamma) - \varepsilon_0 \). Thus we obtain

\[
\begin{align*}
 l(\Gamma) &\geq \sum_{i=N+1}^{N+N'} |f_m(b_i) - f_m(b_{i+1})| \\
&\geq \sum_{j=1}^{A} (|f_m(b_{k_j}) - f_m(b_{k_{j+1}})| + |f_m(b_{k_j}) - f_m(b_{k_{j+1}})|) + \varepsilon_0 > l(\Gamma) .
\end{align*}
\]

This is a contradiction, hence Lemma 3 is proved. \( \square \)

**Corollary 5.** Let \( \Delta^\text{tp} \supset \{\Omega_n\}_{n=1}^\infty \) be such that \( \lim_{n \to \infty} |\Omega_n| = 0 \), and let \( f_n \in X^\text{tp}_{\Gamma, \Omega} \). Suppose that \( \Gamma \) is rectifiable and \( E(f_n) \leq M \) for any \( n \). Then, for any \( \varepsilon > 0 \), there exist \( \delta > 0 \) and positive integer \( n_1 \) such that

\[
|s - t| < \delta \quad \text{implies} \quad |f_n(s) - f_n(t)| < \varepsilon ,
\]

for any \( s, t \in \partial D \) and \( n \geq n_1 \).

**Proof.** By a topological argument we can show that, for any \( \varepsilon \) with \( 0 < \varepsilon < \min \{|\alpha_i - \alpha_j| : i \neq j\} \), there exists \( \tau > 0 \) such that, if \( |\alpha_i - \alpha_j| > \tau \), \( \alpha, \alpha_2 \in \Gamma \), then the diameter of the smaller connected component of \( \Gamma - \{\alpha, \alpha_2\} \) is less than \( \varepsilon \).

Suppose that \( \varepsilon > 0 \) is given and \( \tau > 0 \) is chosen in the above manner. By Lemma 3 there exists a positive integer \( n_1 \) such that \( L(\Omega_n, f_n) < \tau/3 \) for all \( n \geq n_1 \). We choose \( \delta > 0 \) such that \( \lambda(\delta)^{1/2} < \tau/3 \) and \( 2\delta^{1/2} < \min \{|z_i - z_j| : i \neq j\} \). By Lemma 2, for any \( s \in \partial D \), there exists \( \rho, \delta \leq \rho \leq \delta^{1/2} \), depending on \( s, \delta \) and \( f_n \), such that \( l(f_n, C, s) < \tau/3 \). Let \( l, r \in \partial D \) be the left and right endpoints of \( C \), and let \( b_i, b_j \in \Omega_n \cap \partial D \) be such that \( l \) and \( r \) are on the arcs \( b_i b_{i+1} \) and \( b_{j-1} b_j \), respectively. Since \( L(\Omega_n, f_n) < \tau/3 \), we obtain

\[
|f_n(b_i) - f_n(b_j)| \leq |f_n(b_i) - f_n(l)| + |f_n(l) - f_n(r)| + |f_n(r) - f_n(b_j)| < \tau .
\]

Thus we conclude that the diameter of \( \Gamma_1 \), the smaller connected component of \( \Gamma - \{f_n(b_i), f_n(b_j)\} \), is less than \( \varepsilon \), and, for any \( t \in \partial D \) with \( |s - t| < \delta \), \( f_n(s) \) and \( f_n(t) \) are in the convex hull of \( \Gamma_1 \). Hence we obtain \( |f_n(s) - f_n(t)| < \varepsilon \). \( \square \)

**Lemma 6.** Let \( \Delta^\text{tp} \supset \{\Omega_n\}_{n=1}^\infty \) be such that \( \lim_{n \to \infty} |\Omega_n| = 0 \), and let \( f_n \in X^\text{tp}_{\Gamma, \Omega} \). Suppose that \( \Gamma \) is rectifiable and \( E(f_n) \) are uniformly bounded. Then there exists a subsequence \( \{f'_n\} \) such that \( f_n|_{\partial D} \) converges uniformly to a continuous map \( \varphi \in C(\partial D) \) on \( \partial D \). Moreover, \( \varphi(\partial D) = \Gamma \) and \( \varphi \) is monotone.

**Proof.** The proof is similar to that of the Ascoli-Arzelà theorem. Let \( \psi_n = \{(\cos(2\pi i/n), \sin(2\pi i/n)) : i = 0, \ldots, n - 1\} \), and let \( \Psi = \bigcup_{n=1}^\infty \psi_n \). Since \( \Psi \) is countable, we can number \( \Psi \) as \( \Psi = \{\gamma_1, \gamma_2, \ldots\} \). By the diagonal method we choose a subsequence \( \{f'_n\} \) such that, for each \( j \), \( f'_n(\gamma_j) \) converges as \( n_j \to \infty \).

Suppose that an arbitrary \( \varepsilon > 0 \) is given. For this \( \varepsilon \) we choose \( \delta > 0 \) and a positive integer \( n_1 \) as in Corollary 5. Let \( K \) be a positive integer such that
the length of an edge of the regular $K$-gon inscribed $\partial D$ is less than $\delta$, that is, $2 \sin (\pi/K) < \delta$. Let $\mathcal{V}_K = \{\xi_1, \ldots, \xi_K\}$, and let $n_2$ be a positive integer such that $|f_{n_i}(\xi_k) - f_{n_j}(\xi_k)| < \epsilon$, for $n_i, n_j \geq n_2$ and $k = 1, \ldots, K$. For arbitrary $s \in \partial D$ there exists $\xi_k \in \mathcal{V}_K$ such that $|s - \xi_k| < \delta$. Thus, by Corollary 5, we obtain

$$|f_{n_i}(s) - f_{n_j}(s)| \leq |f_{n_i}(s) - f_{n_i}(\xi_k)| + |f_{n_i}(\xi_k) - f_{n_j}(\xi_k)| + |f_{n_j}(\xi_k) - f_{n_j}(s)| < 3\epsilon,$$

for $n_i, n_j \geq n_0 = \max\{n_1, n_2\}$. Since $n_0$ is independent of $s$, $\{f_{n_i}\}$ converges uniformly on $\partial D$. The last part of the lemma is obvious. $\square$

4. Theorems

Using Lemma 6, we obtain the following theorems. The proofs of the theorems are quite similar to those of the theorems in [7].

**Theorem 7.** Suppose that $\Gamma$ is rectifiable. Let $\Delta^\Gamma \supset \{\Omega_n\}_{n=1}^\infty$ be such that $\lim_{n \to \infty} |\Omega_n| = 0$, and let $\{x_n \in X^\Gamma_{\Omega_n}\}_{n=1}^\infty$ be a sequence of the $d$-Douglas solutions.

Then there exists a subsequence $\{x_{n_i}\}$ which converges to one of the Douglas solutions $x \in X^\Gamma_{\Omega}$ in the following sense:

$$\lim_{n_i \to \infty} \|x - x_{n_i}\|_{H^1(D; \mathbb{R}^n)} = 0,$$

and if $x \in W^{1,p}(D; \mathbb{R}^n)$, $p > 2$, then

$$\lim_{n_i \to \infty} \|x - x_{n_i}\|_{C(\overline{D}; \mathbb{R}^n)} = 0.$$

If the Douglas solution is unique, then $x_{n_i}$ converges in the sense of (4.1) and (4.2).

A harmonic map $x \in X^\Gamma_{\Omega}$ is said to be an isolated stable minimal surface if there exists a constant $\delta$ such that

$$0 < \|x - y\|_{C(\overline{D}; \mathbb{R}^n)} < \delta \quad \text{implies} \quad E(x) < E(y) \quad \text{for} \quad y \in X^\Gamma_{\Omega}.$$

**Theorem 8.** Suppose that $\Gamma$ is rectifiable. Let $\Delta^\Gamma \supset \{\Omega_n\}_{n=1}^\infty$ be such that $\lim_{n \to \infty} |\Omega_n| = 0$, and let $x \in X^\Gamma_{\Omega}$ be an isolated stable minimal surface. Then there exists a sequence $\{x_n \in X^\Gamma_{\Omega_n}\}_{n=1}^\infty$ of stable $d$-minimal surfaces which converges to $x$ in the sense of (4.1) and (4.2).

Acknowledgment

The author would like to thank the referee for suggesting a number of improvements and corrections to the original version of this paper.
BIBLIOGRAPHY


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