NONNEGATIVE AND SKEW-SYMMETRIC PERTURBATIONS OF A MATRIX WITH POSITIVE INVERSE

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Abstract. Let \( A \) be a nonsingular matrix with positive inverse and \( B \) a nonnegative matrix. Let the inverse of \( A + vB \) be positive for \( 0 \leq v < v^* < +\infty \) and at least one of its entries be equal to zero for \( v = v^* \); an algorithm to compute \( v^* \) is described in this paper. Furthermore, it is shown that if \( A + A^T \) is positive definite, then the inverse of \( A + v(B - B^T) \) is positive for \( 0 \leq v < v^* \).

1. Introduction

Let

\[
A + vB
\]

be an \( n \times n \) real matrix, where \( A \) is a nonsingular matrix with positive inverse ([5, 2, 1]), \( B \ (B \neq 0) \) a nonnegative matrix and \( v \) a nonnegative real parameter,

\[
A^{-1} > 0, \quad B \geq 0, \quad B \neq 0, \quad v \geq 0.
\]

The parameter \( v \) may be considered as a measure of the size of the nonnegative perturbation \( vB \) of the matrix \( A \). Let

\[
Z(v) = (A + vB)^{-1} = [z_{ij}(v)].
\]

For \( v = 0 \), we have \( Z(0) = A^{-1} > 0 \); thus, \( \det(A + vB) \neq 0 \) and \( Z(v) > 0 \) in a sufficiently small neighborhood of 0. This paper addresses the problem of finding the largest, possibly infinite, number \( v^* \) such that \( A + vB \) is nonsingular and \( Z(v) > 0 \) in \([0, v^*)\). We will describe an algorithm (the iterative process (6)) to compute \( v^* \) if \( v^* < +\infty \). In the case \( v^* = +\infty \), the successive approximations defined by (6) form a sequence diverging monotonically to \( +\infty \).

We shall consider also matrices of the type

\[
C(v) = A + v(B - B^T);
\]

here the matrix \( A \) is perturbed by a skew-symmetric matrix which may be written as \( B - B^T \) with \( B \geq 0 \). It will be shown that if \( A + A^T \) is positive definite, then \( C^{-1}(v) \geq Z(v) > 0 \) in \([0, v^*)\), where \( Z \) is defined by (3).

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Numerical calculations have been performed by using the matrix involved in the discrete analog of the integro-differential equation

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( p \frac{\partial u}{\partial x} \right) + q[u_0 - u] + v \int_0^1 K(x, x')[u_0(x') - u(x')] \, dx'
\]

with boundary conditions \( u(0) = u(1) = 0 \), where \( p(x) > 0 \), \( q(x) \geq 0 \), \( u_0(x) \geq 0 \), and \( K(x, x') \geq 0 \). Equation (5) is a model for a spatially distributed community whose migration has both a random and a special deterministic component; more complicated models (n-species communities, nonlinear) can be obtained including birth-death processes, competition and predator-prey interactions [4]. A direct finite difference approach to (5) provides a discrete approximation \( u \) of the steady state solution \( u \) satisfying an equation of the type \((A + vB)u = f \geq 0\), where \( A + vB \) is of type (2); the positivity of its inverse assures the positivity and the stability of \( u \).

2. The inverse of \( A + vB \)

Lemma 1. Assume (2) and let \( \det(A + vB) \neq 0 \) and \( Z(v) > 0 \), with \( Z(v) \) as defined in (3). Then \( Z'(v) < 0 \) and \( Z''(v) > 0 \).

Proof. From the identity \((A + vB)Z(v) = I\) we obtain

\[
Z' = -ZBZ, \quad Z'' = -2ZBZ' = 2ZBZBZ,
\]

where \( Z' = dZ/dv = [z'_ij] \) and \( Z'' = dZ'/dv = [z''ij] \). As \( B \geq 0 \), \( B \neq 0 \), and \( Z(v) > 0 \), there follows \( Z'(v) < 0 \) and \( Z''(v) > 0 \). □

Lemma 2. Under the assumptions of Lemma 1, let \( v_n \) be the largest number such that \( \det(A + vB) \neq 0 \) in the interval \([0, v_n)\). Then, either \( v_n = +\infty \), or an element of \( Z(v) \) must change sign in \([0, v_n)\).

Proof. As \( v \to v_n \), at least one entry of \( Z(v) \) must become infinite. Otherwise, in any interval \([0, v_B)\) where \( Z(v) > 0 \) we have \( Z'(v) < 0 \) (Lemma 1); therefore, \( Z(v) \) is bounded in \([0, v_B)\),

\[
0 < Z(v) \leq Z(0) = A^{-1}.
\]

It follows that \( v^* = \max v_B \leq v_n \), with strict inequality if \( v^* < +\infty \), because \( 0 \leq Z(v^*) \leq A^{-1} \). When \( v^* < +\infty \), the thesis follows from \( Z'(v^*) \leq 0 \) and Lemma 1 (note that the entries \( z_{ij}(v) \) cannot vanish identically). □

Theorem 1. Let \( v^* \) be the largest, possibly infinite, number such that \( Z(v) > 0 \) in \([0, v^*)\). Then \( v^* \) is the limit of the sequence \( \{v_k\} \) given by

\[
v_{k+1} = v_k + \min_{i,j:w_{kij} > 0} z_{kij}/w_{kij}, \quad k = 0, 1, 2, \ldots, n; \quad v_0 = 0,
\]

where \( Z_k = Z(v_k) = [z_{kij}] \), \( W_k = -Z'(v_k) = Z_kBZ_k = [w_{kij}] \).

Proof. Let \( v^*_{ij} \) be the smallest value of \( v \) for which \( z_{ij}(v) = 0 \), if such a value exists, or \( +\infty \) otherwise. We have \( v^* = \min_{i,j} v^*_{ij} \). In \([0, v^*)\), the matrix \( Z(v) \) does not have singularities (Lemma 2) and its entries are strictly
decreasing and convex functions of $v$ (Lemma 1). These regularity conditions on the entries $z_{ij}(v)$ allow us to obtain the sequence $\{v_k\}$, given by (6), as follows: we compute the Newton steps for the elements of the equation $Z(v) = 0$ and use the smallest of them to update $v$.

The first iteration, with starting value $v_0 = 0$, produces the equations $z_{ij}(v_0) + v z'_{ij}(v_0) = 0$, where $z_{ij}(0) > 0$ and $z'_{ij}(0) < 0$. The smallest solution of these equations is the first approximation $v_1$ in (6) and it is the largest value of $v$ for which

$$Z(0) + v Z'(0) = A^{-1} - v A^{-1} BA^{-1} \geq 0.$$  

As $Z(v) > Z(0) + v Z'(0)$ for $0 < v < v^*$, we have $v_1 < v_{ij}^*$, $i, j = 1, 2, \ldots, n$; therefore, $0 < v_1 < v^*$ and $Z_1 > 0$, $W_1 > 0$.

The successive approximations $v_k$ are defined as follows. Suppose we have computed the approximation $v_k$, for some $k > 0$, for which we have $0 < v_k < v^*$, $Z_k > 0$, $W_k > 0$. We compute the Newton steps starting from the value $v_k$, common to all the equations $z_{ij}(v) = 0$; this produces the equations $z_{ij}(v_k) + (v - v_k) z'_{ij}(v_k) = 0$. The approximation $v_{k+1}$ (the smallest solution of these equations) is the largest value of $v$ for which

$$Z(v_k) + (v - v_k) Z'(v_k) = Z_k - (v - v_k) W_k \geq 0$$

and it is given by (6). As $Z(v) > Z(v_k) + (v - v_k) Z'(v_k)$ for $v_k < v < v^*$, we have $v_{k+1} < v_{ij}^*$, $i, j = 1, 2, \ldots, n$; therefore $v_k < v_{k+1} < v^*$ and $Z_{k+1} > 0$, $W_{k+1} > 0$. We conclude that the sequence $\{v_k\}$ is increasing, bounded from above by $v^*$ if $v^* < +\infty$, and convergent to $v^*$ (note that $\{v_k\}$ cannot converge to a limit $v^* < v^*$ since this would imply $(v_{k+1} - v_k) \longrightarrow \min_{ij} z_{ij}(v_1)/\vert z'_{ij}(v_1) \vert > 0$).

When $v^* = +\infty$, all the entries $z_{ij}(v)$ are positive, strictly decreasing, and convex functions of $v \in [0, +\infty)$ (the only possible solution of each equation $z_{ij}(v) = 0$ is $v^* = +\infty$). If the sequence $\{v_k\}$ were bounded, then it would be convergent: $v_k \rightarrow v^* < +\infty$; as above, we would have $(v_{k+1} - v_k) \rightarrow \text{constant} > 0$. Thus, $\{v_k\}$ is not bounded and it is diverging monotonically to $+\infty$.

Remarks. (a) It is possible to show that the sequence $\{v'_{k}\}$ given by

$$v'_{k+1} = v'_{k} + \min_{i,j; w_{ij}>0} z_{kij}/w_{0ij}, \quad k = 0, 1, 2, \ldots ; v'_0 = 0,$$

is convergent to $v^*$, if $v^* < +\infty$, or divergent to $+\infty$ otherwise.

(b) Only for very small $n$ (the first few integers) we can obtain the analytic expressions of the entries $z_{ij}(v)$ $(i, j = 1, 2, \ldots, n)$ and find their zeros to evaluate $v^*$. The application of the iterative process (6) involves the numerical computation of the inverses $Z_k$, and each iteration requires $O(n^3)$ operations; however, the method has been applied successfully with $n$ equal to 30, 40, and 50 (for example, by using matrices from one-dimensional boundary value problems).
We can show the quadratic convergence \[3, p. 260\] of the process (6) when \(v^* < +\infty\) and \(Z'(v^*) > 0\). We introduce in (6) \(z_{kij}\) obtained from Taylor's formula

\[z_{ij}(v^*) = z_{kij} - (v^* - v_k)w_{kij} + \frac{1}{2}(v^* - v_k)^2z''_{ij}(v_{kij}),\]

where \(v_k \leq v_{kij} \leq v^*\). After some manipulations we have

\[v^* - v_{k+1} = \min_{i, j: w_{kij} > 0} \left[ \frac{\frac{1}{2}(v^* - v_k)^2z''_{ij}(v_{kij}) - z_{ij}(v^*)}{w_{kij}} \right];\]

thus, as \(z_{ij}(v^*) \geq 0\) and \(w_{kij} > 0\), it follows that

\[s_{k+1} \leq \max_{i, j: w_{kij} > 0} z''_{ij}(v_{kij})/w_{kij} - \max_{i, j} z''_{ij}(v^*)/|z'_{ij}(v^*)|,\]

where

\[(7) \quad s_{k+1} = (v^* - v_{k+1})/(v^* - v_k)^2.\]

3. THE INVERSE OF \(C(v) = A + v(B - B^T)\)

**Theorem 2.** Let the symmetric matrix \(A + A^T\) be positive definite. Then, in \([0, v^*)\) the spectral radius \(\rho\) of the nonnegative matrix

\[(8) \quad H(v) = vZ(v)B^T\]

is less than 1, and \(C^{-1}(v) \geq Z(v) > 0\).

**Proof.** The matrix \(C(v)\) given by (4) is now written as

\[C(v) = (A + vB)(I - H(v)),\]

where \(H(v)\) is given by (8). In \([0, v^*)\) we have \(Z(v) > 0\); it follows that \(C^{-1}(v) \geq Z(v) > 0\) if the spectral radius \(\rho(v) = r(H)\) of the nonnegative matrix \(H(v)\) is less than 1 [5, p. 83]. To the spectral radius \(\rho\) there corresponds an eigenvector \(u \geq 0\); from the eigenvalue equation \(vB^Tu = \rho(A + vB)u\) we obtain

\[\rho = uu^TBBu/(uu^TAu + vu^TBu).\]

We have \(uu^TAu = \frac{1}{2}[u^T(A + A^T)u] > 0\), because \(A + A^T\) is assumed positive definite. Thus, as \(v \geq 0, u \geq 0, B \geq 0\), it follows that \(\rho < 1\). \(\Box\)

**Remarks.** By means of simple examples it is possible to show that:

(a) The condition \(A + A^T\) positive definite is not necessary to have \(\rho(v) < 1, 0 \leq v < v^*\).

(b) The condition \(H(v) \geq 0, 0 \leq v < v^*\), is not sufficient by itself to have \(\rho(v) < 1\).

4. NUMERICAL RESULTS

As a sample problem we use the matrix \(A + vB\) obtained from a finite difference approximation to (5) using central differences and the trapezium rule.
Here we present the results obtained by assuming in (5) that \( p = 1 \), \( q = 0 \), and \( K(x, x') = \exp(-(x - x')^2) \) (sample problem 1). In this case, \( A \) is a Stieltjes matrix [5, p. 85] and \( B \) is a positive matrix. The inverses \( Z_k \) are computed by means of the routine LINV2F of the IMSL Library. The results (double-precision computation) are shown in Table 1. The quantities \( s_k \), given by (7), tend to a constant value confirming quadratic convergence. Values of \( v \) greater than \( v^* \), for which some computed entries of \( Z(v) \) are less than zero are reported in the row *.

**Table 1**

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<th>( m = 40 )</th>
<th>( m = 50 )</th>
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<td>( v_k )</td>
<td>( s_k )</td>
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<tr>
<td>*</td>
<td>8.85</td>
<td>0.0658</td>
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**Table 2**

<table>
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<th>( m = 50 )</th>
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<tr>
<td>( k )</td>
<td>( v_k )</td>
<td>( s_k )</td>
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<tr>
<td>**</td>
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<td>0.36</td>
</tr>
</tbody>
</table>

Now we consider the matrix \( C(v) = A + v(B - B^T) \) obtained by assuming in (5) that \( p = 1 \), \( q = 0 \), and \( K(x, x') = x - x' \) (sample problem 2). Here the matrix \( B \) is the nonnegative contribution due to \( K(x, x') \) for \( x \geq x' \). The
results are shown in Table 2. Values of $v$ greater than $v^{*}$, for which some computed entries of $Z(v)$ and of $C^{-1}(v)$ are less than zero are reported in the rows * and **, respectively. We note that $[0, v^{*})$ is a sufficiently good approximation of the interval in which $C^{-1}(v) > 0$.

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BIBLIOGRAPHY


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