LINEAR COMBINATIONS OF ORTHOGONAL POLYNOMIALS
GENERATING POSITIVE QUADRATURE FORMULAS

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Abstract. Let \( p_k(x) = x^k + \cdots, k \in \mathbb{N}_0 \), be the polynomials orthogonal on \([-1, +1]\) with respect to the positive measure \( d\sigma \). We give sufficient conditions on the real numbers \( \mu_j, j = 0, \ldots, m \), such that the linear combination of orthogonal polynomials \( \sum_{j=0}^{m} \mu_j p_{n-j} \) has \( n \) simple zeros in \((-1, +1)\) and that the interpolatory quadrature formula whose nodes are the zeros of \( \sum_{j=0}^{m} \mu_j p_{n-j} \) has positive weights.

1. Introduction

Let \( \sigma \) be a positive measure on \([-1, 1]\) such that the support of \( d\sigma \) contains an infinite set of points. In this paper we consider interpolatory quadrature formulas with positive weights, i.e., quadrature formulas of the form

\[
\int_{-1}^{+1} f(x) \, d\sigma(x) = \sum_{j=1}^{n} c_j f(x_j) + R_n(f),
\]

where \(-1 < x_1 < x_2 < \cdots < x_n < 1\), \( c_j > 0 \) for \( j = 1, \ldots, n \), and \( R_n(f) = 0 \) for \( f \in \mathbb{P}_{2n-1-m} \), \( 0 \leq m \leq n \) (\( \mathbb{P}_n \) denotes as usual the set of polynomials of degree at most \( n \)). As in [6], such a quadrature formula is called a positive \((2n-1-m, n, d\sigma)\) quadrature formula. If \( \sigma \) is absolutely continuous on \([-1, 1]\), with \( \sigma'(x) = w(x) \), we write also \((2n-1-m, n, w)\) instead of \((2n-1-m, n, d\sigma)\). Furthermore, we say that a polynomial \( t_n \in \mathbb{P}_n \) generates a positive \((2n-1-m, n, d\sigma)\) quadrature formula if \( t_n \) has \( n \) simple zeros \( x_1 < x_2 < \cdots < x_n \) in \((-1, +1)\) and the interpolatory quadrature formula based on the nodes \( x_j \) is a positive \((2n-1-m, n, d\sigma)\) quadrature formula. Since the degree of exactness is \( 2n-1-m \), we get with the help of (1.1) the well-known fact that such a polynomial \( t_n \) is orthogonal to \( \mathbb{P}_{n-1-m} \) with respect to \( d\sigma \), and hence is of the form

\[
t_n(x) = \sum_{j=0}^{m} \mu_j p_{n-j}(x),
\]
where \( \mu_j \in \mathbb{R} \) and \( p_k(x) = x^k + \cdots \), \( k \in \mathbb{N}_0 \), denotes the polynomial of degree \( k \) orthogonal with respect to \( d\sigma \). For that reason we are interested in conditions on the numbers \( \mu_j \) such that \( t_n \) generates a positive \((2n-1-m, n, d\sigma)\) quadrature formula. For small \( m \), \( m = 1, 2, 3 \), necessary and sufficient conditions on the numbers \( \mu_j \) can be obtained from the general characterizations of positive quadrature formulas given by the author in [7, 8] (see in particular [8, Theorem 2(b)]), by Sottas and Wanner [10] (note that the conditions given there do not imply that the nodes are in \((-1, +1)\)), and recently by H. J. Schmid [9]. But for larger \( m \) the computational work increases rapidly, and the conditions become very complex (see the examples given in [9, 10]). Thus, the problem arises to find “simple and applicable” sufficient conditions on the numbers \( \mu_j \) such that \( \sum_{j=0}^m \mu_j p_{n-j} \) generates a positive \((2n-1-m, n, d\sigma)\) quadrature formula. This problem is studied and partly solved in this paper by giving first a general sufficient condition on the \( \mu_j \)'s, from which simpler conditions are derived.

2. Preliminary results

In order to state our results, we need some known facts on polynomials orthogonal on \([-1, 1]\), resp. orthogonal on the circumference of the unit circle \(|z| = 1\). Let us recall that the polynomials \( p_n = x^n + \cdots \), \( n \in \mathbb{N} \), orthogonal with respect to \( d\sigma \) on \([-1, +1]\) satisfy a recurrence relation of the form

\[
p_n(x) = (x - \alpha_n) p_{n-1}(x) - \lambda_n p_{n-2}(x) \quad \text{for } n \in \mathbb{N},
\]

where \( p_{-1} = 0 \), \( p_0 = 1 \), \( \alpha_n \in (-1, +1) \) for \( n \in \mathbb{N} \), and \( \lambda_n > 0 \) for \( n \geq 2 \). \( p_n^{(1)}, n \in \mathbb{N}_0 \), denotes the so-called associated polynomial, defined by

\[
p_n^{(1)}(x) = \frac{1}{d_0} \int_{-1}^{+1} \frac{p_{n+1}(x) - p_{n+1}(t)}{x - t} \, d\sigma(t),
\]

where \( d_0 = \int_{-1}^{+1} d\sigma(t) \). Note that the \( p_n^{(1)} \)'s are polynomials of degree \( n \) with leading coefficient one, which satisfy the following recurrence relation (see e.g. [2, Chapter 3, §4])

\[
p_n^{(1)}(x) = (x - \alpha_{n+1}) p_{n-1}^{(1)}(x) - \lambda_{n+1} p_{n-2}^{(1)}(x) \quad \text{for } n \in \mathbb{N},
\]

where the \( \alpha_n \)'s and \( \lambda_n \)'s are determined by (2.1).

We are now ready to state the first simple characterization of positive quadrature formulas.

Lemma 1. Let \( n, m \in \mathbb{N}_0 \), \( n \geq m \), and let \( \mu_j \in \mathbb{R} \) for \( j = 0, \ldots, m \), \( \mu_0 \neq 0 \). Then \( \sum_{j=0}^m \mu_j p_{n-j} \) generates a positive \((2n-1-m, n, d\sigma)\) quadrature formula if and only if \( \sum_{j=0}^m \mu_j p_{n-j} \) has \( n \) simple zeros in \((-1, +1)\) and the zeros of \( \sum_{j=0}^m \mu_j p_{n-j} \) and \( \sum_{j=0}^m \mu_j p_{n-1-j}^{(1)} \) separate each other.

Proof. Setting

\[
t_n = \sum_{j=0}^m \mu_j p_{n-j} \quad \text{and} \quad t_n^{(1)} = \sum_{j=0}^m \mu_j p_{n-1-j}^{(1)},
\]
we get for the weights \( c_j \), using relation (2.2),

\[
c_j = \int_{-1}^{+1} \frac{t_n(x)}{(x-x_j)t_n'(x_j)} d\sigma(x) = d_0 \frac{t_n^{(1)}(x_j)}{t_n'(x_j)} \quad \text{for} \quad j = 1, \ldots, n.
\]

Hence the conditions \( c_j > 0 \) for \( j = 1, \ldots, n \) are equivalent to the interlacing property of the zeros of \( t_n \) and \( t_n^{(1)} \). \( \square \)

Next, denote by \( P_n(z) = z^n + \cdots, n \in \mathbb{N}_0 \), the polynomial orthogonal on \([0, 2\pi]\) with respect to the positive measure

\[
\psi(\phi) = \begin{cases} 
-\sigma(\cos \phi) & \text{for } \phi \in [0, \pi], \\
\sigma(\cos \phi) & \text{for } \phi \in (\pi, 2\pi],
\end{cases}
\]

i.e.,

\[
\int_0^{2\pi} e^{-ik\phi} P_n(e^{i\phi}) d\psi(\phi) = 0 \quad \text{for} \quad k = 0, \ldots, n - 1.
\]

Note if \( \sigma \) is absolutely continuous on \([-1, +1]\) and \( \sigma'(x) = w(x) \), then \( \psi \) is absolutely continuous with \( \psi'(\phi) = w(\cos \phi)|\sin \phi| \) for \( \phi \in [0, 2\pi] \). It is well known (polynomials orthogonal on the unit circle are studied extensively in [3]) that the \( P_n \)'s satisfy a recurrence relation of the type

\[
P_n(z) = zP_{n-1}(z) - a_{n-1}P_{n-1}^*(z) \quad \text{for} \quad n \in \mathbb{N},
\]

where \( a_n \in (-1, +1) \) for \( n \in \mathbb{N}_0 \), and where \( P_{n}^*(z) = z^nP_{n}(z^{-1}) \) denotes the reciprocal polynomial of \( P_n \). The reason that the parameters \( a_n \) are real and have absolute value less than one consists in the facts that \( \psi \) is odd with respect to \( \pi \) and that \( \psi \) has an infinite set of points of increase (see [3, p. 5]). Furthermore, let \( \Omega_n(z) = z^n + \cdots \) be defined by the recurrence relation

\[
\Omega_n(z) = z\Omega_{n-1}(z) + a_{n-1}\Omega_{n-1}^*(z) \quad \text{for} \quad n \in \mathbb{N}.
\]

\( \Omega_n \) is called the associated polynomial of \( P_n \). It is well known that both polynomials \( P_n \) and \( \Omega_n \), \( n \geq 1 \), have all their zeros in the open unit disk \(|z| < 1\). The following relations hold between polynomials \( p_n \) orthogonal on \([-1, 1]\) with respect to \( d\sigma \) and polynomials \( P_n \):

\[
p_n(x) = 2^{-n+1} \text{Re}\{z^{-n+1}P_{2n-1}(z)\},
\]

\[
p_{n-1}^{(1)}(x) = 2^{-n+1} \text{Im}\{z^{-n+1}\Omega_{2n-1}(z)\}/\sin \phi,
\]

where \( x = \frac{1}{2}(z + z^{-1}), \ z = e^{i\phi}, \ \phi \in [0, \pi]. \) The parameters \( (a_n) \) are given by [3, (31.4)]

\[
a_{2n-1} = 1 - (u_n + v_n) \quad \text{and} \quad a_{2n} = \frac{v_n - u_n}{v_n + u_n},
\]

where

\[
u_n = \frac{P_{n+1}(1)}{P_n(1)} \quad \text{and} \quad v_n = \frac{P_{n+1}(-1)}{P_n(-1)}.
\]
Moreover,
\[(2.10) \quad a_{2n} = 0 \quad \text{for} \quad n \in \mathbb{N}_0, \quad \text{if} \quad \sigma(x) = -\sigma(-x) \text{ a.e. on } [-1, 1].\]

For example, we obtain for the Jacobi polynomials \( P_{n}^{(\alpha, \beta)}(x) = x^n + \cdots \) which are orthogonal on \([-1, 1]\) with respect to the weight function \( w^{(\alpha, \beta)}(x) = (1-x)^\alpha(1+x)^\beta, \quad \alpha, \beta > -1, \) that the corresponding parameters \( a_{2n}^{(\alpha, \beta)} \) appearing in the recurrence relation of \( P_{n}^{(\alpha, \beta)}(z) = z^n + \cdots \) are given by
\[(2.11) \quad a_{2n+1}^{(\alpha, \beta)} = -\frac{\alpha + \beta + 1}{\alpha + \beta + 2n + 3}, \quad a_{2n}^{(\alpha, \beta)} = \frac{\beta - \alpha}{\alpha + \beta + n + 2} \quad \text{for} \quad n \in \mathbb{N}_0.\]

Hence we get for the ultraspherical case \( P_{n}^{(\lambda)}(x) := P_{n}^{(\lambda-1/2, \lambda-1/2)}(x) \) and \( w^{(\lambda)}(x) = (1-x^2)^{\lambda-1/2} \) that
\[(2.12) \quad a_{n}^{(\lambda)} = \frac{\lambda}{n + 1 + \lambda} \quad \text{and} \quad a_{2n}^{(\lambda)} = 0 \quad \text{for} \quad n \in \mathbb{N}_0,\]
and in particular for the Chebyshev case, i.e., for the case where \( \lambda = 0 \) and \( w(x) = (1-x^2)^{-1/2}, \) that
\[(2.13) \quad a_{n} = 0 \quad \text{for} \quad n \in \mathbb{N}_0, \quad \Omega_{n}(z) = P_{n}(z) = z^n \quad \text{for} \quad n \in \mathbb{N}_0.\]

Finally, we shall need

**Lemma 2.** Let \( n \in \mathbb{N} \) and \( l \in \mathbb{Z} \) with \( 2|l| \leq n. \) Assume that the real polynomial \( t_{n}(z) = z^n + \cdots \) has all its zeros in the open unit disk \( |z| < 1. \) Then the cosine-polynomial \( \text{Re}\{z^{-l}t_{n}(z)\} \), resp. the sine-polynomial \( \text{Im}\{z^{-l}t_{n}(z)\}, \quad z = e^{i\phi}, \phi \in [0, \pi], \) has \( n - l \) zeros \( \phi_j \) in \( (0, \pi), \) resp. \( n - l - 1 \) zeros \( \psi_j \) in \( (0, \pi), \) and their zeros separate each other, i.e., \( 0 < \phi_1 < \psi_1 < \phi_2 < \cdots < \psi_{n-l-1} < \phi_{n-l} < \pi. \)

**Proof.** Since \( \text{Re}\{z^{-l}t_{n}(z)\} \) (respectively \( \text{Im}\{z^{-l}t_{n}(z)\} \)) is zero at \( z = e^{i\phi}, \phi \in (0, 2\pi), \) if and only if
\[ z^{-2l}t_{n}(z) = -1 \quad \text{(respectively} + 1), \]
which is equivalent to
\[ \text{arg} \ z^{n-2l} + \text{arg} \ \frac{t_{n}(z)}{t_{l}(z)} = (2k - 1)\pi \quad \text{(respectively} 2k\pi), \]
\( k \in \mathbb{N}_0, \) we get, taking into consideration the fact that \( \text{arg} \ t_{n}(e^{i\phi})/t_{l}(e^{i\phi}) \) increases from 0 to \( 2n\pi \) if \( \phi \) varies from 0 to \( 2\pi, \) that both \( \text{Re}\{z^{-l}t_{n}(z)\} \) and \( \text{Im}\{z^{-l}t_{n}(z)\} \) have \( 2(n-l) \) zeros in \( [0, 2\pi) \) and that their zeros separate each other. Observing that \( \text{Im}\{z^{-l}t_{n}(z)\} \) has a zero at \( \phi = 0 \) and \( \phi = \pi, \) the assertion follows by the symmetry of trigonometric polynomials. \( \square \)

## 3. Main results

First, let us introduce the following polynomials, which play a crucial role in this paper.
Definition. For given \( n \in \mathbb{N} \) let the polynomials \( Q_{\nu, 2n-1}(z) = z^\nu + \cdots \), \( \nu \in \{0, \ldots, 2n-1\} \), be defined by the recurrence relation

\[
Q_{\nu, 2n-1}(z) = zQ_{\nu-1, 2n-1}(z) - a_{2n-1-\nu}Q_{\nu-1, 2n-1}
\]

for \( \nu = 1, \ldots, 2n-1 \),

where \( Q_{0, 2n-1} = 1 \) and the \( a_{2n-1-\nu} \)'s are the parameters appearing in the recurrence relation (2.5) of the \( P_n \)'s.

The polynomials \( Q_{\nu, 2n-1} \) have the following important properties.

Lemma 3. Let \( n \in \mathbb{N} \). The following propositions hold:

(a) \( \prod_{k=0}^{n-1} (1 - \sqrt{a_{2n-2-k}})^{n} \leq |Q_{\nu, 2n-1}(z)| \leq \prod_{k=0}^{n-1} (1 + \sqrt{a_{2n-2-k}})^{n} \) for \( |z| \leq 1 \), where \( \nu \in \{0, \ldots, 2n-1\} \). Moreover, \( Q_{\nu, 2n-1} \) has all zeros in \( |z| < 1 \).

(b) Let \( \nu \in \{0, \ldots, n-1\} \); then \( \tau = e^{i\phi}, \chi = \cos \phi, \phi \in [0, \pi] \)

\[
p_n(x) = 2^{-n+1} \text{Re}\{z^{-n+1} Q_{\nu, 2n-1}(z)Q_{\nu, (n-\nu)-1}(z)\}
\]

and

\[
p_{n-1}^{(1)}(x) = 2^{-n+1} \text{Im}\{z^{-n+1} Q_{\nu, 2n-1}(z)Q_{\nu, (n-\nu)-1}(z)\}/\sin \phi.
\]

Proof. (a) follows immediately from (3.1) and [3, (26.6)].

(b) We first note that the recurrence relations (2.5), resp. (3.1), imply (see [3, (3.6)]) that

\[
(2.5') \quad P_n^*(z) = P_{n-1}^*(z) - a_{n-1} z P_{n-1}(z) \quad \text{for } n \in \mathbb{N},
\]

and

\[
(3.1') \quad Q_{\nu, 2n-1}^*(z) = Q_{\nu-1, 2n-1}^*(z) - a_{2n-1-\nu} z Q_{\nu-1, 2n-1}(z)
\]

for \( \nu = 1, \ldots, 2n-1 \).

With the help of all these recurrence relations it follows by induction arguments that

\[
z P_{2n-1}(z) + P_{2n-1}^*(z) = z Q_{\nu, 2n-1}(z) P_{2n-1}(z) + Q_{\nu, 2n-1}^*(z) P_{2n-1}(z),
\]

which, in view of (2.7) and taking into consideration the fact that for \( z = e^{i\phi} \)

\[
2 \text{Re}\{z^{-n+1} P_{2n-1}(z)\} = z^{-n}(z P_{2n-1}(z) + P_{2n-1}(z)),
\]

gives the first relation.

Analogously as above, one demonstrates that

\[
z \Omega_{2n-1}(z) - \Omega_{2n-1}^*(z) = z Q_{\nu, 2n-1}(z) \Omega_{2n-1}(z) - Q_{\nu, 2n-1}^*(z) \Omega_{2n-1}(z),
\]

which in conjunction with (2.8) gives the second relation. \( \square \)
The main result is now the following

**Theorem 1.** Let \( n, m \in \mathbb{N}_0 \), \( m \leq n \), \( \mu_0, \ldots, \mu_m \in \mathbb{R} \) and \( \mu_0 \neq 0 \). Then \( \sum_{j=0}^m \mu_j p_{n-j} \) generates a positive \((2n - 1 - m, n, d\sigma)\) quadrature formula if \( \sum_{j=0}^m \mu_j z^j Q_{2m-2j, 2(n-j)-1}(z) \), where \( \mu_j = 2^j \mu_j \), has all its zeros in the open unit disk \(|z| < 1\).

**Proof.** Putting

\[
t_n(x) = \sum_{j=0}^m \mu_j p_{n-j}(x) \quad \text{and} \quad t_{n-1}^{(1)}(x) = \sum_{j=0}^m \mu_j p_{n-1-j}^{(1)}(x),
\]

we get with the help of Lemma 3(b) that \((z = e^{i\phi}, x = \cos \phi, \phi \in [0, \pi])\)

\[
t_n(x) = 2^{-n+1} \text{Re}\{z^{-m}q_{2m}(z)z^{-(n-m)+1}P_{2(n-m)-1}(z)\}
\]

and

\[
t_{n-1}^{(1)}(x) = 2^{-n+1} \text{Im}\{z^{-m}q_{2m}(z)z^{-(n-m)+1}\Omega_{2(n-m)-1}(z)\}/\sin \phi,
\]

where

\[
q_{2m}(z) = \sum_{j=0}^m \mu_j z^j Q_{2m-2j, 2(n-j)-1}(z).
\]

Assume now that \( q_{2m} \) has all its zeros in \(|z| < 1\). Since the same is true for \( P_{2(n-m)-1} \), it follows from Lemma 2 that \( t_n \) has \( n \) simple zeros in \((-1, +1)\).

Thus, by Lemma 1, it remains to demonstrate that the zeros of \( t_n \) and \( t_{n-1}^{(1)} \) separate each other.

Using the relation

\[
\text{Re} a \text{Re} b + \text{Im} a \text{Im} b = \text{Re}\{ab\},
\]

where \( a, b \in \mathbb{C} \), we get for \( z = e^{i\phi} \)

\[
\text{Re}\{z^{-(n-1)}q_{2m}(z)P_{2(n-m)-1}(z)\} \text{Re}\{z^{-(n-1)}q_{2m}(z)\Omega_{2(n-m)-1}(z)\}
\]

\[
+ \text{Im}\{z^{-(n-1)}q_{2m}(z)P_{2(n-m)-1}(z)\} \text{Im}\{z^{-(n-1)}q_{2m}(z)\Omega_{2(n-m)-1}(z)\}
\]

\[
= |q_{2m}(z)|^2 \text{Re}\{P_{2(n-m)-1}(z)\Omega_{2(n-m)-1}(z)\} \text{Im}\{P_{2(n-m)-1}(z)\Omega_{2(n-m)-1}(z)\}
\]

\[
= c|q_{2m}(z)|^2, \quad c \in \mathbb{R}^+,
\]

where the last equality follows from the known relation \([3, (5.6)]\)

\[
P_{2(n-m)-1}(z)\Omega_{2(n-m)-1}(z) + \Omega_{2(n-m)-1}(z)P_{2(n-m)-1}(z)
\]

\[
= \hat{c} z^{2n-2m-1}, \quad \text{where} \ \hat{c} \in \mathbb{R}^+.
\]

Considering relation (3.4) at the zeros \( x_j \), \(-1 < x_1 < x_2 < \cdots < x_n < 1\), of \( t_n(x) \) and taking into account that by Lemma 2 the zeros of \( t_n(x) \) and \( r_{n-1}(x) := \text{Im}\{z^{-(n-1)}q_{2m}(z)P_{2(n-m)-1}(z)\}/\sin \phi \), \( x = \frac{1}{2}(z + 1/z) \), \( z = e^{i\phi} \),
\[ \phi \in [0, \pi], \] separate each other, we obtain
\[ (-1)^{n-j} t^{(1)}_{n-1}(x) > 0 \quad \text{for } j = 1, \ldots, n, \]
which proves the interlacing property of \( t_n \) and \( t^{(1)}_{n-1} \) and thus the theorem.

**Remark 1.** From the general characterization of positive quadrature formulas given by the author in [7, Theorem 2] it follows with the help of relation (3.2) that the sufficient condition of Theorem 1 is also necessary if \( 2m \leq n \).

From Theorem 1 we obtain, using some ideas of Cauchy and Kojima on the location of the zeros of polynomials (see [4, §30, in particular Exercise 6]), the following sufficient conditions which are easy to verify.

**Corollary 1.** Let \( n, m \in \mathbb{N}_0, m \leq n, \mu_0, \ldots, \mu_m \in \mathbb{R} \) and \( \mu_0 \neq 0 \). Put
\[ A_0 = |\mu_0|, \]

\[ A_j = 2^j |\mu_j| \frac{\prod_{k=0}^{2m-1-2j} (1 + |a_{2(n-j-1)-k}|)}{\prod_{k=0}^{2m-1-1} (1 - |a_{2(n-1)-k}|)} \quad \text{for } j = 1, \ldots, m, \]

and let \( j_\nu \in \{0, 1, \ldots, m\} \), \( j_0 := 0 < j_1 < \cdots < j_m \) be those indices for which \( A_{j_\nu} \neq 0 \) for \( \nu = 1, \ldots, m^* \) and \( A_j = 0 \) for \( j \in \{1, \ldots, m\} \setminus \{j_0, j_1, \ldots, j_m\} \).

Then each of the following two conditions is sufficient that \( \sum_{j=0}^m \mu_j p_{n-j} \) generates a positive \((2n-1-m, n, d\sigma)\) quadrature formula:

1. \( \sum_{j=1}^{m^*} A_{j_\nu} < A_0 \).
2. \( A_{j_\nu} \geq 2 A_{j_{\nu+1}} \) for \( \nu = 0, \ldots, m^* - 2 \) and \( A_{m^*} > A_{m^*} \).

**Proof.** First let us note that condition (2) implies condition (1). In fact, applying successively the inequalities given in (2), we obtain
\[ A_{j_0} \geq A_{j_1} + A_{j_1} \geq A_{j_2} + \cdots + A_{j_{m^*}} \]
which is condition (1).

Next we show that condition (1) implies that
\[ q^*_m(z) := \sum_{j=0}^m \tilde{\mu}_j z^j Q^*_m,2(n-j)-1(z), \quad \tilde{\mu}_j = 2^j \mu_j, \]
has all zeros in \(|z| > 1\), which is equivalent to the fact that
\[ \sum_{j=0}^m \tilde{\mu}_j z^j Q^*_m,2(n-j)-1(z) \]
has all zeros in \(|z| < 1\) and proves the corollary. Assume, to the contrary, that \( q^*_m \) has a zero \( \zeta \) in \(|z| \leq 1\). Then it follows, using from Lemma 3 the fact that \( Q^*_m,2n-1 \) has no zero in \(|z| \leq 1\), that
\[ (3.6) \quad |\mu_0| = \left| \sum_{j=1}^m \tilde{\mu}_j \zeta^j \frac{Q^*_m,2(n-j)-1(\zeta)}{Q^*_m,2n-1(\zeta)} \right| \leq \sum_{j=1}^m A_j |\zeta|^j \leq \sum_{j=1}^m A_j, \]
where the first inequality follows with the help of Lemma 3, which is a contra-
diction to (1). □

Let us give an illustrative

**Example.** Let \( n, m \in \mathbb{N}_0, n > m \), and suppose that the parameters \( a_\nu \) satisfy

\[
0 < 1 / \gamma \leq 1 - |a_\nu| \quad \text{for } \nu = 2(n - m) - 1, \ldots, 2n - 2.
\]

Then we get by Corollary 1 that

\[
p_n - \mu_m p_{n-m}, \quad |\mu_m| < (2\gamma^{2})^{-m},
\]

generates a positive \((2n - 1 - m, n, d\sigma)\) quadrature formula, where because of (2.10) the condition on \(|\mu_m|\) can be replaced by \(|\mu_m| < (2\gamma^{2})^{-m}\) if \(\sigma\) is odd. In particular, we obtain for the Jacobi weight by a rough estimate of the parameters \(a_n^{(\alpha, \beta)}\) from (2.11) that

\[
p_n^{(\alpha, \beta)} - \mu_m p_{n-m}^{(\alpha, \beta)}, \quad |\mu_m| < 2^{-m},
\]

generates a positive \((2n - 1 - m, n, (1 - x)\alpha(1 + x)\beta)\) quadrature formula for each \(n \geq m + \max\{2, \alpha + \beta + 1 + 2|\beta - \alpha|\}\). In the ultraspherical case \(\alpha = \beta = \lambda - 1/2, \lambda \in (-1/2, \infty)\), the conditions on \(|\mu_m|\), resp. \(n\), can be replaced by \(|\mu_m| < 2^{-2m}\) and \(n \geq m + \max\{\lambda, -3\lambda\}\).

Let us note in this connection that the conditions of Corollary 1 are in general too rough to get the known results (see [1]) on the positivity of \((n - 1, n, (1 - x)^\alpha(1 + x)^\beta)\) quadrature formulas generated by \(p_n^{(a, b)}\), \(a, b > -1\). But this is not surprising because the proof of such results requires very special properties of Jacobi polynomials.

In order to weaken the sufficient conditions of Corollary 1, a better estimate for \(\max_{0 \leq \phi \leq 2\pi}|Q_{2m-2j, 2(n-j)-1}(e^{i\phi})|/Q_{2m, 2n-1}(e^{i\phi})|\) than that one used in (3.6) would be needed.

In the following, let \(T_n\), resp. \(U_n\), denote the Chebyshev polynomial of the first, resp. second, kind of degree \(n\) and \(\hat{T}_n(x) = 2^{-n+1}T_n(x) = x^n + \cdots\), resp. \(\hat{U}_n(x) = 2^{-n}U_n(x) = x^n + \cdots\). For the case of the Chebyshev distribution \(d\sigma(x) = (1 - x^2)^{-1/2} \, dx\) we get in view of (2.13) particularly simple conditions, which hold also for the distribution \(d\sigma(x) = (1 - x^2)^{1/2} \, dx\).

**Corollary 2.** Let \( n, m \in \mathbb{N}_0, m \leq n, \mu_0, \ldots, \mu_m \in \mathbb{R}, \mu_0 \neq 0, \) and put \(\hat{\mu}_j = 2^j \mu_j\) for \(j = 0, \ldots, m\). Then the following propositions hold:

(a) \(\sum_{j=0}^{m} \hat{\mu}_j T_{n-j}\) generates a positive \((2n - 1 - m, n, (1 - x^2)^{-1/2})\) quadrature formula if \(\sum_{j=0}^{m} \hat{\mu}_j z^{m-j}\) has all its zeros in the open unit disk \(|z| < 1\). In particular (besides conditions (1) and (2) of Corollary 1), the condition

\[
(3) \hat{\mu}_0 > \hat{\mu}_1 > \cdots > \hat{\mu}_m > 0
\]

is sufficient that \(\sum_{j=0}^{m} \mu_j T_{n-j}\) generates a positive \((2n - 1 - m, n, (1 - x^2)^{-1/2})\) quadrature formula.
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(b) The sufficient conditions given in (a) (including conditions (1) and (2) of Corollary 1 with \( a_n = 0 \) for \( n \in \mathbb{N}_0 \)) are also sufficient for \( \sum_{j=0}^{m} \mu_j z^{m-j} \) to generate a positive \( (2n - 1 - m, n, (1 - x^2)^{1/2}) \) quadrature formula.

Proof. (a) The first statement follows immediately from Theorem 1. Since by the Kakeya-Eneström Theorem (see, e.g., [4]) condition (3) implies that \( \sum_{j=0}^{m} \mu_j z^{m-j} \) has all zeros in \( |z| < 1 \), part (a) is proved.

(b) We shall demonstrate, independently from Theorem 1, that \( \sum_{j=0}^{m} \mu_j \hat{U}_{n-j} \) generates a positive \( (2n - 1 - m, n, (1 - x^2)^{1/2}) \) quadrature formula if \( \sum_{j=0}^{m} \mu_j \cdot z^{m-j} \) has all zeros in \( |z| < 1 \), which also implies all other statements of (b). Setting

\[
r_n(z) = z^{n-m} \sum_{j=0}^{m} \mu_j z^j
\]

and

\[
2^n t_n(x) = \sum_{j=0}^{m} \mu_j U_{n-j}(x) = \text{Im}\{zr_n(z)\}/\sin \phi,
\]

we obtain, since, as is well known, the associated polynomial of \( \hat{U}_k \) is \( \hat{U}_{k-1} \), \( k \in \mathbb{N}_0 \), that the associated polynomial \( t_{n-1}^{(1)} \) of \( t_n \) with respect to \( (1 - x^2)^{1/2} \) is of the form

\[
2^{n-1} t_{n-1}^{(1)}(x) = \sum_{j=0}^{m} \mu_j U_{n-1-j}(x) = \text{Im}\{r_n(z)\}/\sin \phi.
\]

Observing that

\[
(3.7) \quad \text{Re}\{r_n(z)\}\frac{\text{Im}\{zr_n(z)\}}{\sin \phi} - \text{Re}\{zr_n(z)\}\frac{\text{Im}\{r_n(z)\}}{\sin \phi} = |r_n(z)|^2,
\]

we deduce with the help of Lemma 2, by considering relation (3.7) at the \( n \) zeros of \( t_n \), that \( t_n \) and \( t_{n-1}^{(1)} \) have interlacing zeros. In view of Lemma 1 the assertion is proved. ☐

The sufficiency of condition (3) for the Chebyshev weight \( (1 - x^2)^{-1/2} \) is due to C. A. Micchelli [5], who derived this result in order to demonstrate that the ultraspherical polynomials \( p_n^{(\lambda)} \), \( 0 \leq \lambda < 1 \), generate a positive \( (n - 1, n, (1 - x^2)^{-1/2}) \) quadrature formula. Let us mention in this connection (for a different approach see [5]) that for \(-1/2 < \lambda \leq 0\) the positivity can be demonstrated with the help of condition (1), using the simple fact that \( T_k(1) = 1 \) for \( k \in \mathbb{N}_0 \). Proceeding similarly as in the proof of Corollary 2(b), it could also be demonstrated that Corollary 2(b) holds for the more general weight \( (1 - x)^{\alpha}(1 + x)^{\beta} \), \( \alpha, \beta \in \{-1/2, 1/2\} \), a result which has been given by the author in [8, Corollary 2], using different methods.

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Using the fact that the sufficient condition of Theorem 1 is also necessary if \(2m \leq n\) (see Remark 1), we get

**Corollary 3.** Let \(n, m \in \mathbb{N}_0\), \(2m \leq n\), \(\mu_0, \ldots, \mu_m \in \mathbb{R}\) and \(\mu_0, \mu_m \in \mathbb{R}\backslash\{0\}\). For \(k \in \{0, \ldots, m\}\) put \(A^{(k)}_k = 2^k |\mu_k|\) and

\[
A_j^{(k)} = 2^j |\mu_j| \prod_{k=0}^{2m-1-2j} \frac{1 + |a_{2(n-j-1)-k}|}{1 - |a_{2(n-k-1)-k}|} \quad \text{for } j = 0, \ldots, m, j \neq k.
\]

If there is a \(k \in \{1, \ldots, m\}\) such that \(A_k^{(k)} > \sum_{j=0, j \neq k}^{m} A_j^{(k)}\), then \(\sum_{j=0}^{m} \mu_j \cdot p_{n-j}\) does not generate a positive \((2n - 1 - m, n, d\sigma)\) quadrature formula.

**Proof.** In view of Remark 1 it is sufficient to demonstrate that

\[
q_{2m}^*(z) := \sum_{j=0}^{m} \hat{\mu}_j z^j Q_{2m-2j, 2(n-j)-1}^*(z), \quad \hat{\mu}_j = 2^j \mu_j,
\]

has at least one zero in \(|z| < 1\). With the help of Lemma 3 we get on the circumference \(|z| = 1\)

\[
|\hat{\mu}_k z^k Q_{2m-2k, 2(n-k)-1}^*(z)| \geq |\hat{\mu}_k| \prod_{k=0}^{2m-1-2k} (1 - |a_{2(n-k-1)-k}|)
\]

\[
> \sum_{j=0}^{m} |\hat{\mu}_j| \prod_{k=0}^{2m-1-2j} (1 + |a_{2(n-j-1)-k}|)
\]

\[
\geq \left| \sum_{j=0, j \neq k}^{m} \hat{\mu}_j z^j Q_{2m-2j, 2(n-j)-1}^*(z) \right|.
\]

Using the fact that \(Q_{2m-2k, 2(n-k)-1}^*\) has no zero in \(|z| < 1\), this implies by Rouché’s Theorem that \(q_{2m}^*\) has \(k\) zeros in \(|z| < 1\), which proves the assertion. \(\square\)

If one is interested only in such linear combinations of orthogonal polynomials whose zeros are simple and are in \((-1, +1)\), conditions (1) and (2) can be weakened in the following way.

**Theorem 2.** Let \(n, m \in \mathbb{N}_0\), \(m \leq n\), \(\mu_0, \ldots, \mu_m \in \mathbb{R}\) and \(\mu_0 \neq 0\). Put \(|B_0| = \mu_0\),

\[
B_j = 2^j |\mu_j| \prod_{k=0}^{2n-2j} (1 - |a_k|) \quad \text{for } j = 1, \ldots, m,
\]

and let \(j_0, j_1, \ldots, j_m^* \in \{0, 1, \ldots, m\}\) be those indices for which \(B_j \neq 0\) for \(j = 1, \ldots, m^*\) and \(B_j = 0\) for \(j \in \{1, \ldots, m\} \backslash \{j_0, j_1, \ldots, j_m^*\}\). Then each of the following two conditions is sufficient that \(\sum_{j=0}^{m^*} \mu_j p_{n-j}\) has \(n\)
simple zeros in \((-1, 1)\):

\[(1') \sum_{\nu=1}^{m^*} B_{\nu} < B_0.\]

\[(2') B_{\nu} \geq 2B_{\nu + 1} \text{ for } \nu = 0, \ldots, m^* - 1 \text{ and } B_{m^* - 1} > B_{m^*}.\]

**Proof.** Since by (2.7)

\[
\sum_{j=0}^{m} \mu_j p_{n-j}(x) = 2^{-n+1} \Re \left\{ \sum_{j=0}^{m} \hat{\mu}_j z^j P_{2(n-j)-1}(z) \right\},
\]

where \(\hat{\mu}_j = 2^j \mu_j\), we deduce with the help of Lemma 2 that \(\sum_{j=0}^{m} \mu_j p_{n-j}(x)\) has \(n\) simple zeros in \((-1, +1)\) if \(\sum_{j=0}^{m} \mu_j P_{2(n-j)-1}^*\) has all zeros in \(|z| > 1\).

Observing that by relation (26.5) of [3]

\[
\max_{|z| \leq 1} \left| \frac{P_{2(n-j)-1}^*(z)}{P_{2n-1}^*(z)} \right| \leq \frac{1}{\prod_{\kappa=2(n-j)-1}^{2n-2} (1 - |a_\kappa|)} \quad \text{for } j = 1, \ldots, m,
\]

the assertion can be proved in the same way as Corollary 1. \(\square\)

**Bibliography**


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