BOOLEAN METHODS FOR DOUBLE INTEGRATION

FRANZ-J. DELVOS

Abstract. This paper is concerned with numerical integration of continuous functions over the unit square $U^2$. The concept of the $r$th-order blending rectangle rule is introduced by carrying over the idea from Boolean interpolation. Error bounds are developed, and it is shown that $r$th-order blending rectangle rules are comparable with number-theoretic cubature rules. Moreover, $r$th-order blending midpoint rules are defined and compared with the $r$th-order blending rectangle rules.

1. BIVARIATE RECTANGLE RULES

The problem we consider is the numerical evaluation of integrals of the form

$$
\mathcal{J}(f) = \int_0^1 \int_0^1 f(x, y) \, dx \, dy,
$$

where $f$ is a continuous function on the unit square $U^2 = [0, 1]^2$. Moreover, we assume that $f$ satisfies the periodicity conditions

$$
f(x, 0) = f(x, 1), \quad f(0, y) = f(1, y) \quad (0 \leq x, y \leq 1).
$$

The inner product of $f, g \in L^2(U^2)$ is

$$
(f, g) = \int_0^1 \int_0^1 f(x, y) g(x, y) \, dx \, dy.
$$

We introduce the notations

$$
e_k(x) = \exp(i2\pi k x) \quad (k \in \mathbb{Z}),
$$

$$
e_{k,l}(x, y) = e_k(x) \cdot e_l(y) \quad (k, l \in \mathbb{Z}),
$$

where $x, y \in U$. The functions $e_{k,l} \quad (k, l \in \mathbb{Z})$ form an orthonormal basis of the Hilbert space $L^2(U^2)$. We denote by $A(U^2)$ the Wiener algebra of those functions $f \in L^2(U^2)$ with the property that the Fourier series of $f$ is absolutely convergent:

$$
\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |(f, e_{k,l})| < \infty.
$$

Let \( \mathcal{C}(U^2) \) denote the subspace of those functions \( f \in L^2(U^2) \) which are continuous on \( U^2 \). Moreover, \( \mathcal{C}_0(U^2) \) denotes the subspace of those functions \( f \in \mathcal{C}(U^2) \) which satisfy the periodicity conditions (1.2). It follows from relation (1.3) that
\[
A(U^2) \subseteq \mathcal{C}_0(U^2)
\]
and, for \( f \in A(U^2) \),
\[
(1.4) \quad f(x, y) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (f, e_{k,l}) \cdot e_{k,l}(x, y) \quad (x, y \in U).
\]

Let \( m \) and \( n \) be positive integers. The most obvious cubature formula is the bivariate rectangle rule:
\[
\mathcal{J}_{m,n}(f) = \frac{1}{m \cdot n} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f \left( \frac{j}{m}, \frac{k}{n} \right).
\]
The bivariate rectangle rule is not an efficient cubature formula in view of the large number of function evaluations. On the other hand, \( \mathcal{J}_{m,n}(f) \) is a basic tool in constructing a more sophisticated cubature formula, the \( r \)-th order blending rectangle rule. For this reason we will briefly derive a convenient remainder formula for \( \mathcal{J}_{m,n}(f) \).

**Proposition 1.** If \( f \in A(U^2) \), then
\[
(1.5) \quad \mathcal{J}_{m,n}(f) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} (f, e_{um, vn}).
\]

**Proof.** In view of (1.4), we have
\[
\mathcal{J}_{m,n}(f) = \frac{1}{m \cdot n} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} (f, e_{r,s}) \cdot \frac{1}{m \cdot n} \sum_{j=0}^{m-1} e_{r,j} \left( \frac{j}{m} \right) \sum_{k=0}^{n-1} e_{s,k} \left( \frac{k}{n} \right)
= \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} (f, e_{um, vn}). \quad \Box
\]

It is useful to define the series
\[
R_{m,\infty}(f) = \sum_{u \neq 0} (f, e_{um,0}), \quad R_{\infty,n}(f) = \sum_{v \neq 0} (f, e_{0,vn}),
\]
\[
R_{m,n}(f) = \sum_{u \neq 0} \sum_{v \neq 0} (f, e_{um,vn}).
\]

**Proposition 2.** If \( f \in A(U^2) \), then the error in the bivariate rectangle rule is
\[
(1.6) \quad \mathcal{J}_{m,n}(f) - \mathcal{J}(f) = R_{m,\infty}(f) + R_{\infty,n}(f) + R_{m,n}(f).
\]
Proof. It follows from relation (1.5) that

$$J_{m,n}(f) = (f, e_{0,0}) + R_{m,\infty}(f) + R_{\infty,n}(f) + R_{m,n}(f).$$

Since $J(f) = (f, e_{0,0})$, Proposition 2 is proved. □

Following Korobov, we define, for each $a \geq 1$, the linear space

$$E^a(U^2) = \{ f \in L^2(U^2) : (f, e_{m,n}) = O((m \cdot n)^{-a}) \ (m, n \to \infty) \},$$

where $m = \max\{1, |m|\}$ ($m \in \mathbb{Z}$). It is easily seen that

$$E^a(U^2) \subseteq A(U^2) \quad (a > 1).$$

We denote by $C^{p,p}(U^2)$ the linear subspace of $C(U^2)$ of those functions $f$ whose partial derivatives satisfy

$$D^k,l f \in C(U^2) \quad (0 \leq k, l \leq p).$$

Similarly, $C^{q,q}(U^2)$ is the linear subspace of $C_q(U^2)$ of functions $f$ with

$$D^k,l f \in C_q(U^2) \quad (0 \leq k, l \leq p).$$

It was shown in Baszenski and Delvos [1] that

$$C^{q-1,q-1}(U^2) \cap C^{q+1,q+1}(U^2) \subseteq E^{q+1}(U^2) \quad (q \in \mathbb{N}).$$

Proposition 3. If $f \in E^a(U^2)$ with $a > 1$, then the error in the bivariate rectangle rule satisfies

$$J_{m,n}(f) - J(f) = O((m^{-a} + n^{-a}) \ (m, n \to \infty)).$$

Proof. Since $f \in E^a(U^2)$, we have

$$R_{m,\infty}(f) = O(m^{-a}), \quad R_{\infty,n}(f) = O(n^{-a}),$$

$$R_{m,n}(f) = O(m^{-a} \cdot n^{-a}) \ (m, n \to \infty),$$

from which (1.9) follows by virtue of Proposition 2. □

Proposition 4. If $f \in C^{q-1,q-1}(U^2) \cap C^{q+1,q+1}(U^2)$ with $q \in \mathbb{N}$, then the error in the bivariate rectangle rule satisfies

$$J_{m,n}(f) - J(f) = O((m^{-q-1} + n^{-q-1}) \ (m, n \to \infty)).$$

Proof. Using (1.8), an application of Proposition 3 yields (1.11) □

2. rTH-ORDER BLENDING RECTANGLE RULES

We introduce the rth-order sum of bivariate rectangle rules

$$S_r^2(f) = \sum_{m=1}^{r} J_{2^m,2^{r+1-m}}(f) \quad (r \in \mathbb{Z}_+).$$

Then the rth-order blending rectangle rule $J_r^2(f)$ is

$$J_r^2(f) = S_r^2(f) - S_{r-1}^2(f).$$
where \( r \in \mathbb{N} \) and \( r > 1 \). The construction of the \( r \)th-order blending rectangle rule resembles the explicit formula of the interpolation projector of \( r \)th-order blending (Delvos and Posdorf [3] and Delvos [2]). The cubature points of \( \mathcal{J}^2_r(f) \) are mainly determined by the points occurring in \( \mathcal{S}^2_r(f) \):

\[
\bigcup_{m=1}^{r} \{(j \cdot 2^{-m}, k \cdot 2^{-r-1+m}) : 0 \leq j < 2^m, \ 0 \leq k < 2^{r+1-m}\}.
\]

Their number is given by

\[
n_r = (r + 1) \cdot 2^r.
\]

Next we will determine a remainder formula for the \( r \)th-order blending rectangle rule.

**Proposition 5.** If \( f \in A(U^2) \), then the error in the \( r \)th-order blending rectangle rule is

\[
\mathcal{J}^2_r(f) - \mathcal{J}(f) = R_{2^r, \infty}(f) + R_{\infty, 2^r}(f)
\]

\[
\quad + \sum_{m=1}^{r} R_{2^m, 2^{r+1-m}}(f) - \sum_{m=1}^{r-1} R_{2^m, 2^{r-m}}(f).
\]

**Proof.** Using (1.6), we get

\[
\mathcal{J}^2_r(f) - \mathcal{J}(f) = \sum_{m=1}^{r} (\mathcal{J}_{2^m, 2^{r+1-m}}(f) - \mathcal{J}(f)) - \sum_{m=1}^{r-1} (\mathcal{J}_{2^m, 2^{r-m}}(f) - \mathcal{J}(f))
\]

\[
= \sum_{m=1}^{r} (R_{2^m, 2^{r+1-m}}(f) + R_{2^m, \infty}(f) + R_{\infty, 2^m}(f))
\]

\[
- \sum_{m=1}^{r-1} (R_{2^m, 2^{r-m}}(f) + R_{2^m, \infty}(f) + R_{\infty, 2^m}(f))
\]

\[
= R_{2^r, \infty}(f) + R_{\infty, 2^r}(f)
\]

\[
\quad + \sum_{m=1}^{r} R_{2^m, 2^{r+1-m}}(f) - \sum_{m=1}^{r-1} R_{2^m, 2^{r-m}}(f). \quad \square
\]

**Proposition 6.** If \( f \in E^a(U^2) \) with \( a > 1 \), then the error in the \( r \)th-order blending rectangle rule is

\[
\mathcal{J}^2_r(f) - \mathcal{J}(f) = \mathcal{O}((r + 1) \cdot (2^r)^{-a}) \quad (r \to \infty).
\]

**Proof.** From (1.10) we have

\[
R_{2^r, \infty}(f) = \mathcal{O}((2^r)^{-a}), \quad R_{\infty, 2^r}(f) = \mathcal{O}((2^r)^{-a}) \quad (r \to \infty),
\]

\[
R_{2^m, 2^{r+1-m}}(f) = \mathcal{O}((2^{r+1})^{-a}) \quad (1 \leq m \leq r, \ r \to \infty),
\]

\[
R_{2^m, 2^{r-m}}(f) = \mathcal{O}((2^r)^{-a}) \quad (1 \leq m < r, \ r \to \infty).
\]

Now (2.6) follows from the remainder formula (2.5). \( \square \)
Remark 1. Recall that the number of cubature points of the $r$th-order blending rectangle rule $J^2_r(f)$ is bounded by

$$n_r = (r + 1)2^r.$$ 

It is easily seen that the error relation (2.6) of the $r$th-order blending rectangle rule obtains the form

$$J^2_r(f) - J(f) = O((\log(nr))^{(r+1)/(2r)}) \quad (r \to \infty),$$

where $f \in E^a(U^2)$ with $a > 1$. Thus, the $r$th-order blending rectangle rule is comparable with the bivariate number-theoretic "good-lattice" rules (see Sloan [5]). The attractive feature of the $r$th-order blending rectangle rule is its easy computation based on relations (2.1) and (2.2).

Proposition 7. If $f \in C_0^{q-1}(U^2) \cap C^{q+1}(U^2)$ with $q \in \mathbb{N}$, then the error in the $r$th-order blending rectangle rule satisfies

$$(2.7) \quad J^2_r(f) - J(f) = O((r + 1) \cdot (2^r)^{-q-1}) \quad (r \to \infty).$$

Proof. Use of (1.8) and an application of Proposition 6 yields (2.7). □

3. Bivariate midpoint rules

Let $m$ and $n$ be positive integers. A simple cubature formula closely related to the bivariate rectangle rule is the bivariate midpoint rule:

$$M_{m,n}(f) = \frac{1}{m \cdot n} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f \left( \frac{2j + 1}{2m}, \frac{2k + 1}{2n} \right).$$

Again, the bivariate midpoint rule is not an efficient cubature formula in view of the large number of function evaluations. However, $M_{m,n}(f)$ is a basic tool in constructing the more sophisticated cubature formula of the $r$th-order blending midpoint rule. For this reason we will briefly derive a convenient remainder formula for $M_{m,n}(f)$.

Proposition 8. If $f \in A(U^2)$, then

$$(3.1) \quad M_{m,n}(f) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} (f, e_{um,vn}) \cdot (-1)^{u+v}.$$

Proof. By (1.4), we have

$$M_{m,n}(f) = \frac{1}{m \cdot n} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f \left( \frac{2j + 1}{2m}, \frac{2k + 1}{2n} \right)$$

$$= \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} (f, e_{rs}) \frac{1}{m \cdot n} \sum_{j=0}^{m-1} e_r \left( \frac{2j + 1}{2m} \right) \sum_{k=0}^{n-1} e_s \left( \frac{2k + 1}{2n} \right)$$

$$= \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} (f, e_{um,vn}) \cdot (-1)^{u+v}. \, \Box$$
We define the series
\[ Q_{m, \infty}(f) = \sum_{u \neq 0} (f, e_{um, 0}) \cdot (-1)^u, \]
\[ Q_{\infty, n}(f) = \sum_{v \neq 0} (f, e_{0, vn}) \cdot (-1)^v, \]
\[ Q_{m, n}(f) = \sum_{u \neq 0} \sum_{v \neq 0} (f, e_{um, vn}) \cdot (-1)^{u+v}. \]

**Proposition 9.** If \( f \in A(U^2) \), then the error in the bivariate midpoint rule is

\[ \mathcal{M}_{m, n}(f) - \mathcal{I}(f) = Q_{m, \infty}(f) + Q_{\infty, n}(f) + Q_{m, n}(f). \]

**Proof.** From (3.1) we get
\[ \mathcal{M}_{m, n}(f) = (f, e_{0, 0}) + Q_{m, \infty}(f) + Q_{\infty, n}(f) + Q_{m, n}(f). \]
Since \( \mathcal{I}(f) = (f, e_{0, 0}) \), Proposition 9 follows. \( \square \)

**Proposition 10.** If \( f \in E^a(U^2) \) with \( a > 1 \), then the error in the bivariate midpoint rule satisfies

\[ \mathcal{M}_{m, n}(f) - \mathcal{I}(f) = \mathcal{O}(m^{-a} + n^{-a}) \quad (m, n \to \infty). \]

**Proof.** Since \( f \in E^a(U^2) \), we have
\[ Q_{m, \infty}(f) = \mathcal{O}(m^{-a}), \quad Q_{\infty, n}(f) = \mathcal{O}(n^{-a}), \]
\[ Q_{m, n}(f) = \mathcal{O}(m^{-a} \cdot n^{-a}) \quad (m, n \to \infty), \]
and (3.3) follows from Proposition 9. \( \square \)

**Proposition 11.** If \( f \in E_0^{q-1, q-1}(U^2) \cap E^{q+1, q+1}(U^2) \) with \( q \in \mathbb{N} \), then the error in the bivariate midpoint rule satisfies

\[ \mathcal{M}_{m, n}(f) - \mathcal{I}(f) = \mathcal{O}(m^{-q-1} + n^{-q-1}) \quad (m, n \to \infty). \]

**Proof.** The proof of Proposition 11 is similar to that of Proposition 4. \( \square \)

4. **RTh-ORDER BLENDING MIDPOINT RULES**

We introduce the \( r \)-th order sum of bivariate midpoint rules

\[ T_r^2(f) = \sum_{m=0}^{r-1} \mathcal{M}_{2m, 2r-1-n}(f) \quad (r \in \mathbb{N}). \]

Then the \( r \)-th order blending midpoint rule \( \mathcal{M}_r^2(f) \) is

\[ \mathcal{M}_r^2(f) = T_r^2(f) - T_{r-1}^2(f), \]
where \( r \in \mathbb{N} \) and \( r > 1 \). The construction of the \( r \)-th order blending midpoint rule is analogous to the construction of the \( r \)-th order blending rectangular rule. While the latter may be interpreted as an interpolatory cubature formula based
on Boolean periodic spline interpolation, no such interpolatory characterization holds for the \( r \)th-order blending midpoint rule.

The cubature points of \( M_r^2(f) \) are mainly determined by the points occurring in \( T_{r-1}^2(f) \):

\[
\bigcup_{m=0}^{r-1} \{(2j+1) \cdot 2^{-m-1}, (2k+1) \cdot 2^{-r+m} \colon 0 \leq j < 2^m, 0 \leq k < 2^{r-1-m}\}.
\]

Their number is given by

\[
m_r = r \cdot 2^{r-1}.
\]

Next we will determine a remainder formula for the \( r \)th-order blending midpoint rule.

**Proposition 12.** If \( f \in A(U^2) \), then the error in the \( r \)th-order blending midpoint rule is

\[
M_r^2(f) - J(f) = Q_{2^{r-1}, \infty}(f) + Q_{\infty, 2^{r-1}}(f) + \sum_{m=0}^{r-2} Q_{2^m, 2^{r-1-m}}(f) - \sum_{m=0}^{r-2} Q_{2^m, 2^{r-2-m}}(f).
\]

**Proof.** In view of relations (3.2), (3.4), (4.1), and (4.3), the proof of (4.5) is similar to that of (2.5). \( \square \)

**Proposition 13.** If \( f \in E^a(U^2) \) with \( a > 1 \), then the error in the \( r \)th-order blending midpoint rule is

\[
M_r^2(f) - J(f) = \Theta(r \cdot (2^{r-1})^{-a}) \quad (r \to \infty).
\]

**Proof.** In view of relations (3.4) and (4.5), the proof of (4.6) is similar to that of (2.6). \( \square \)

**Remark 2.** Recall that the number of cubature points of the \( r \)th-order blending midpoint rule \( M_r^2(f) \) is mainly determined by \( m_r = r \cdot 2^{r-1} \). It is easily seen that the error relation (4.6) of the \( r \)th-order blending midpoint rule obtains the form

\[
M_r^2(f) - J(f) = \Theta((\log(m_r))^{a+1} \cdot (m_r)^{-a}) \quad (r \to \infty),
\]

where \( f \in E^a(U^2) \) with \( a > 1 \). Thus, the \( r \)th-order blending midpoint rule is comparable with the bivariate number-theoretic “good lattice” rules (see Sloan [5]). Again, the attractive feature of the \( r \)th-order blending midpoint rule is its easy computation based on relations (4.1) and (4.2).

**Proposition 14.** If \( f \in C_{0}^{q-1}, q-1(U^2) \cap C_{q+1}, q+1(U^2) \) with \( q \in \mathbb{N} \), then the error in the \( r \)th-order blending midpoint rule satisfies

\[
M_r^2(f) - J(f) = \Theta(r \cdot (2^{r-1})^{-q-1}) \quad (r \to \infty).
\]

**Proof.** The proof of Proposition 14 is similar to that of Proposition 7. \( \square \)
5. A NUMERICAL EXAMPLE

We consider the double integral

$$\mathcal{I}(f) = \int_0^1 \int_0^1 f(x, y) \, dx \, dy$$

with the function

$$f(x, y) = \frac{x + y}{1 + x \cdot y} \quad (x, y \in U).$$

The function $f$ is an element of the Korobov space $E^1(U^2)$. Following Hua and Wang [4, p. 122] we introduce the function

$$g(x, y) = \frac{1}{4}(f(x, y) + f(x, 1 - y) + f(1 - x, y) + f(1 - x, 1 - y)).$$

It is easily seen that

$$\mathcal{I}(g) = \mathcal{I}(f) = 2 \cdot (\log(4) - 1)$$

and

$$g \in \mathcal{E}_0^{0, 0}(U^2) \cap \mathcal{E}_2^{2, 2}(U^2).$$

It follows from relation (1.8) that Propositions 4 and 7 are applicable to $g$ with $q = 1$. The errors and the number of cubature points for the blending rectangle rule and the ordinary rectangle rule are shown in Table 1.

**Table 1**

*Errors in blending and ordinary rectangle rules*

<table>
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<tr>
<th>$r$</th>
<th>$(r + 1) \cdot 2^r$</th>
<th>$\mathcal{I}_r^2(g) - \mathcal{I}(g)$</th>
<th>$2^{2r}$</th>
<th>$\mathcal{I}_{r, 2^r}(g) - \mathcal{I}(g)$</th>
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</table>

Similarly, it follows from relation (1.8) that Propositions 11 and 14 are applicable to $g$ with $q = 1$. Table 2 shows the errors and the number of cubature points for the blending midpoint rule and the ordinary midpoint rule. In Figure 1 we exhibit the distribution of cubature points in $r$th-order sum of midpoint rules.

**Remark 3.** It follows from (2.3) and (4.3) that the cubature points of $T_r^2(f)$ form a subset of the cubature points of $S_r^2(f)$ which are not contained in the set of cubature points of $S_{r-1}^2(f)$.

**Remark 4.** The Boolean methods for double integration can be extended to arbitrary dimensions by using the method of $d$-variate Boolean interpolation developed in [2]. This is the topic of a forthcoming paper.
Table 2

Errors in blending and ordinary midpoint rules

<table>
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<tr>
<th>$r$</th>
<th>$r \cdot 2^{r-1}$</th>
<th>$m^2_r(g) - J(g)$</th>
<th>$2^{2r-2}$</th>
<th>$m_{2r-1,2r-1}(g) - J(g)$</th>
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</table>

Figure 1

Points of $r$th-order sum of midpoint rules

Bibliography


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