THE LOCAL HURWITZ CONSTANT AND
DIOPHANTINE APPROXIMATION ON HECKE GROUPS

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Abstract. Define the Hecke group by

\[ G_q = \left\langle \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle, \]

\[ \lambda_q = 2 \cos \frac{\pi}{q}, \ q = 3, 4, \ldots \] We call \( G_q(\infty) \) the \( G_q \)-rationals, and \( \mathbb{R} - G_q(\infty) \) the \( G_q \)-irrationals. The problem we treat here is the approximation of \( G_q \)-irrationals by \( G_q \)-rationals. Let \( M(\alpha) \) be the upper bound of numbers \( c \) for which \( |\alpha - k/m| < 1/cm^2 \) for all \( G_q \)-irrationals and infinitely many \( k/m \in G_q(\infty) \). Set \( h'_q = \inf \alpha M(\alpha) \). We call \( h'_q \) the Hurwitz constant for \( G_q \). It is known that \( h'_{2q} = 2, q \) even; \( h'_{2q} = 2(1 + (1 - \lambda_q/2)^2)^{1/2}, q \) odd. In this paper we prove this result by using \( \lambda_q \)-continued fractions, as developed previously by D. Rosen. Write

\[ \alpha - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}{m_{n-1}(\alpha) Q_{n-1}^2}, \]

where \( \varepsilon_i = \pm 1 \) and \( P_i/Q_i \) are the convergents of the \( \lambda_q \)-continued fraction for \( \alpha \). Then \( M(\alpha) = \lim_{n} m_{n}(\alpha) \). We call \( m_{n}(\alpha) \) the local Hurwitz constant.

In the final section we prove some results on the local Hurwitz constant. For example (Theorem 4), it is shown that if \( q \) is odd and \( \varepsilon_{n+1} = \varepsilon_{n+2} = +1 \), then \( m_i \geq (\lambda_q^2 + 4)^{1/2} > h'_q \) for at least one of \( i = n-1, n, n+1 \).

1. Introduction

Let the Hecke group

\[ G_q = \left\langle \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad \lambda_q = 2 \cos \frac{\pi}{q}, \ q \geq 3, \]

act on the upper half-plane \( \text{Im} \ z > 0 \) by Möbius transformations \( z \rightarrow (kz + l)/(mz + n), \ (k \ m, l \ n) \in G_q \). \( G_q \) is a horocyclic group with cusp set \( G_q(\infty) \), which are called \( G_q \)-rationals. The points of \( \mathbb{R} - G_q(\infty) \) are the \( G_q \)-irrationals. In [4] we considered the problem of approximating a \( G_q \)-irrational by \( G_q \)-rationals.

When \( q = 3 \), \( G_3 \) becomes the classical modular group \( \text{PSL}(2, \mathbb{Z}) \) and we are considering classical Diophantine approximation of rationals by irrationals.
A. Hurwitz showed that when $\alpha$ is irrational, there exist infinitely many reduced fractions $k/m$ for which
\[ \left| \frac{\alpha - k}{m} \right| < \frac{1}{\sqrt{5}m^2}, \]
where $\sqrt{5}$ is the best constant possible. From now on we consider only $q \geq 4$.

Let $\alpha$ be $G_q$-irrational and suppose
\[ (1.1) \quad \left| \frac{\alpha - k}{m} \right| < \frac{1}{cm^2}, \quad \frac{k}{m} \in G_q(\infty), \quad m > 0. \]

We denote by $M(\alpha)$ the upper bound of numbers $c$ for which (1.1) holds for infinitely many $k/m$ and put
\[ (1.2) \quad h'_q = \inf_\alpha M(\alpha), \quad \alpha \text{ } G_q\text{-irrational.} \]

We call $h'_q$ the Hurwitz constant for $G_q$. In [4] we proved that $h'_q = 2$ when $q$ is even and gave bounds for $h'_q$ when $q$ is odd. In [3] A. Haas and C. Series found the exact value of $h'_q$. So we now know that $h'_q = h_q$, where $h_q$ is defined by
\[ (1.3) \quad h_q = \begin{cases} 2, & q \text{ even, } \geq 4, \\ 2(1 + (1 - \lambda_q/2)^2)^{1/2}, & q \text{ odd.} \end{cases} \]

(Note that the notation of [3] differs from ours—their $h_q$ is the reciprocal of ours—and the methods of the two papers are quite different.)

From now on we write $G$ for $G_q$, and $\lambda$ for $\lambda_q$. In [4] we made use of a type of continued fraction expansion of the limit set of $G_q$, i.e., of $\mathbb{R}$, developed by D. Rosen [5]. (This limit set was also studied by Thea Pignataro in her Princeton thesis (1984, unpublished).) This expansion is called a (reduced) $\lambda$-fraction and represents every real number $\alpha$ uniquely:
\[ (1.4) \quad \alpha \equiv \alpha_0 = r_0, r_1, \ldots = \left[ r_0, \frac{e_1}{r_1}, \frac{e_2}{r_2}, \ldots \right]. \]

Here $e_i = \pm 1$, $r_0 = r_0(\alpha_0)$ is an integer, $r_i = r_i(\alpha_0)$, $i \geq 1$, are positive integers, and certain conditions are placed on the $e_i$ and $r_i$. The above expansion, referred to as $\lambda$CF $\alpha_0$, is finite if and only if $\alpha_0$ is $G$-rational. Denote the convergents of (1.4) by
\[ (1.5) \quad \frac{P_n}{Q_n} = \left[ r_0, \ldots, \frac{e_n}{r_n}, \ldots \right], \quad Q_0 = 1. \]

Our general plan of attack follows Hurwitz and was described in [4] at the beginning of §3. Hurwitz first shows that if (1.1) is satisfied by any rational number $P/Q$ in lowest terms, then $P/Q$ must be a convergent in the expansion of $\alpha$ as a regular continued fraction. The problem is thus reduced to studying the approximation of $\alpha$ by its convergents.

Here we follow a similar plan. By a preliminary theorem [4, Theorem 3] the approximation of a $G$-irrational $\alpha_0$ by $G$-rationals was reduced to the
approximation of $\alpha_0$ by the convergents $P_n/Q_n$ of $\lambda$-CF $\alpha_0$. Thus the inequality (1.1) was replaced by an inequality derived from

$$\alpha_0 - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1} e_1 e_2 \cdots e_n}{m_{n-1} Q_{n-1}^2}, \quad m_{n-1} = m_{n-1}(\alpha_0),$$

and the object of study was $m_{n-1}(\alpha_0)$. Clearly,

$$M(\alpha_0) = \lim_{n \to \infty} m_{n-1}(\alpha_0), \quad h_q = \inf_{\alpha_0} M(\alpha_0).$$

We call $m_n(\alpha_0)$ a local Hurwitz constant.

Two $\lambda$-CF $\alpha$ and $\beta$ are said to be equivalent, and we write $\alpha \sim \beta$, if their expansions agree from a certain point on. It is easy to check that $\alpha \sim \beta$ if and only if $\alpha = \pm V \beta$ for a $V \in G$. It is clear that

$$\alpha \sim \beta \Rightarrow M(\alpha) = M(\beta).$$

The object of the present paper is to provide inequalities for the local Hurwitz constants. First, however, we shall prove that the Hurwitz constant $h_q^2$ has the value $h_q$ in (1.3), using the method of $\lambda$-fractions. The result follows from

Theorem 1. Let $\alpha_0$ be a $G$-irrational given by (1.4). When $q$ is odd,

$$M(\alpha_0) \geq 2(1 + (1 - \lambda/2)^2)^{1/2},$$

with equality if and only if

$$\alpha_0 \sim 1 - \lambda/2 + (1 + (1 - \lambda/2)^2)^{1/2}.$$

When $q$ is even, $M(\alpha_0) \geq 2$, with equality if and only if $\alpha_0 \sim 1$.

Of course, knowledge of the value of $h_q$, $q$ odd, given in [3], was of the greatest value in constructing the proof.

The local Hurwitz constants are also discussed. Let $m_{n-1} = m_{n-1}(\alpha)$ be defined by (1.6).

Theorem 2. If $e_{n+1} = 1$, then $m_{n-1} > 2$, $m_n < 2$, or $m_{n-1} < 2$, $m_n > 2$.

Theorem 3. Let $q$ be odd. If $r_n \geq 2$ and $e_{n-1} = 1$, then $m_{n-1} \geq h_q$.

Theorem 4. Let $q$ be odd. If $e_{n+1} = e_{n+2} = 1$, then $m_i \geq (\lambda^2 + 4)^{1/2} > h_q$ for at least one of $i = n - 1, n, n + 1$.

2. Definitions and basic lemmas

In this section we gather together definitions and theorems needed in the sequel; most of these can be found in [5 and 4]. Let $q \geq 4$. With the notations of (1.4), (1.5) we have

$$P_n = r_n \lambda P_{n-1} + e_n P_{n-2}, \quad Q_n = r_n \lambda Q_{n-1} + e_n Q_{n-2}, \quad n \geq 1,$$
where
\begin{equation}
\begin{aligned}
P_{-1} &= 1, & P_0 &= r_0^\lambda, & Q_{-1} &= 0, & Q_0 &= 1, \\
P_n Q_{n-1} - P_{n-1} Q_n &= (-1)^{n-1} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n, & n &\geq 1,
\end{aligned}
\end{equation}

\begin{equation}
\alpha - \frac{P_{n-1}}{Q_{n-1}} = (-1)^{n-1} \frac{\varepsilon_1 \cdots \varepsilon_n}{m_{n-1}(\alpha) Q_{n-1}^2}.
\end{equation}

Here,
\begin{equation}
m_{n-1}(\alpha) = m_{n-1} = \alpha_n + \varepsilon_n / \alpha_n', & n &\geq 3,
\end{equation}

\begin{equation}
\alpha_n = \left[ r_n^\lambda, \frac{\varepsilon_{n+1}}{r_{n+1}^\lambda}, \ldots \right], & n &\geq 0;
\end{equation}

\begin{equation}
\alpha_n' = \left[ r_{n-1}^\lambda, \frac{\varepsilon_{n-1}}{r_{n-1}^\lambda}, \ldots, \frac{\varepsilon_2}{r_1^\lambda} \right].
\end{equation}

As we shall see later, \( Q_n \geq 1 \) and \( m_{n-1}(\alpha) > 0 \). Note that \( P_n / Q_n \) is a strictly decreasing sequence when all \( \varepsilon_i = -1 \). The periodic \( \lambda \) CF of period \( p \),

\begin{equation}
\alpha = \left[ r_0^\lambda, \frac{\varepsilon_1}{r_1^\lambda}, \ldots, \frac{\varepsilon_{p-1}}{r_{p-1}^\lambda}, \frac{\varepsilon_p}{r_0^\lambda}, \frac{\varepsilon_1}{r_1^\lambda}, \ldots \right]
\end{equation}

can be written as

\begin{equation}
\alpha = \left[ r_0^\lambda, \frac{\varepsilon_1}{r_1^\lambda}, \ldots, \frac{\varepsilon_{p-1}}{r_{p-1}^\lambda}, \frac{\varepsilon_p}{r_0^\lambda} \right],
\end{equation}

or as

\begin{equation}
\alpha = \left[ r_0^\lambda, \frac{\varepsilon_1}{r_1^\lambda}, \ldots, \frac{\varepsilon_{p-1}}{r_{p-1}^\lambda}, \alpha \right].
\end{equation}

The following lemma is slightly more general than [5, p. 556].

**Lemma 1.** Let

\begin{equation}
\alpha_{nv} = [b_n, \varepsilon_{n+1} / b_{n+1}, \ldots, \varepsilon_v / b_v]
\end{equation}

and

\begin{equation}
\alpha'_{nv} = [b'_n, -1 / b'_{n+1}, \ldots, -1 / b'_v]
\end{equation}

have \( b_n, b'_n > 0 \), \( 0 \leq n \leq \mu < \nu \). If \( b_{\mu} \geq b'_{\mu} \), then \( \alpha_{nv} \geq \alpha'_{nv} \), and \( \alpha_{nv} > \alpha'_{nv} \) if some \( b_{\mu} > b'_{\mu} \). If

\begin{equation}
\alpha_n = [b_n, \varepsilon_{n+1} / b_{n+1}, \ldots]
\end{equation}

and

\begin{equation}
\alpha'_n = [b'_n, -1 / b'_{n+1}, \ldots]
\end{equation}

are convergent fractions, and \( b_{\mu} \geq b'_{\mu} \), \( \mu \geq n \), then \( \alpha_n \geq \alpha'_n \).

For \( q \) odd, write \( q = 2l - 1 \), \( l \geq 3 \); for \( q \) even, write \( q = 2l \), \( l \geq 2 \). Let

\begin{equation}
s = [(q - 3) / 2] = l - 2, & l &\geq 2.
\end{equation}
The notation $(-1/r\lambda)^n$ means a block of $n$ consecutive terms $-1/r\lambda$. We shall frequently need the ACF

$$B(n) = [\lambda, (-1/\lambda)^{n-1}], \quad n \geq 2, \quad B(1) = \lambda,$$

with $n$ partial quotients. Thus [5, p. 556],

$$B(n + 1) = \lambda - 1/B(n), \quad 1 \leq n \leq s + 1,$$

$B(n)$ is strictly decreasing,

$$B(s) = 1/(\lambda - 1), \quad B(s + 1) = 1, \quad q \text{ odd},$$

$$B(s) = \lambda/(\lambda^2 - 2), \quad B(s + 1) = 2/\lambda, \quad q \text{ even}.\quad (2.6)$$

Also let

$$C(n) = [2\lambda, (-1/2\lambda)^{n-1}], \quad n \geq 2, \quad C(1) = 2\lambda.$$

Then

$$C(n + 1) = 2\lambda - 1/C(n), \quad n \geq 1,$$

$C(n)$ is strictly decreasing,

$$\lim_{n \to \infty} C(n) = \lambda + (\lambda^2 - 1)^{1/2}.\quad (2.7)$$

We have

$$\left[ C(n + 1), -\frac{1}{T} \right] > \left[ C(n), -\frac{1}{T} \right], \quad n \geq 1, \quad 0 < T < \lambda + (\lambda^2 - 1)^{1/2}.\quad (2.8)$$

Indeed, by Lemma 1,

$$\left[ C(n + 1), -\frac{1}{T} \right] = \left[ C(n), -\frac{1}{2\lambda - 1/T} \right] > \left[ C(n), -\frac{1}{T} \right],$$

since $T + 1/T < \lambda + (\lambda^2 - 1)^{1/2} + \lambda - (\lambda^2 - 1)^{1/2} = 2\lambda$. Similarly,

$$\left[ B(k), -\frac{1}{T} \right] > \left[ B(k + 1), -\frac{1}{T} \right], \quad k \leq s, \quad T > 0.\quad (2.9)$$

In fact,

$$\left[ B(k), -\frac{1}{T} \right] > \left[ B(k), -\frac{1}{\lambda - 1/T} \right] = \left[ B(k) - \frac{1}{\lambda}, -\frac{1}{T} \right] = \left[ B(k + 1), -\frac{1}{T} \right],$$

since $T + 1/T \geq 2 > \lambda$.

When ACF $\alpha$ is reduced (see §§3 and 5 for the definition), we have

$$\alpha_{nv} \geq 2/\lambda, \quad \nu \geq n; \quad \alpha_n \geq 2/\lambda \quad \text{if } r_0 \geq 1 \text{ [5, Lemma 2]},$$

where

$$\alpha_{nv} = \left[ r_n \lambda, \frac{e_{n+1}}{r_{n+1} \lambda}, \ldots, \frac{e_\nu}{r_\nu \lambda} \right], \quad \nu > n; \quad \alpha_n = r_n \lambda,\quad (2.10)$$

$$\alpha_n = \left[ r_n \lambda, \frac{e_{n+1}}{r_{n+1} \lambda}, \ldots \right],$$

$$Q_n \geq Q_{n-1}, \quad n \geq 1 \text{ [5, Theorem 3]}.\quad (2.11)$$
Using these inequalities in (2.4) and (2.1), we get
\[ m_{n-1}(\alpha) \geq \frac{2}{\lambda} - 1 > 0, \quad n \geq 3; \quad Q_n \geq 1, \quad n \geq 0, \]
as stated earlier.

3. Evaluation of the Hurwitz constant

In this section our object is to prove Theorem 1. The result for even \( q \) having been established in [4, Theorem 1], we now assume \( q \) odd.

A \( \lambda \)CF \( \alpha_0 = [r_0, \epsilon_1/r_1, \ldots] \) is said to be reduced [5, p. 555] if

The inequality \( r_i \lambda + \epsilon_{i+1} < 1 \) (i.e., \( r_i = 1, \epsilon_{i+1} = -1 \)) is satisfied
\[ (3.1) \]
for no more than \( s \) consecutive values \( i = j, j + 1, \ldots, j + s - 1, j \geq 1 \). Here \( s \) is defined in (2.5a).

If \( r_j \lambda + \epsilon_{j+1} < 1 \) is satisfied for \( s \) consecutive values \( i = j, \ldots, j + s - 1 \), then \( r_{j+s} \geq 2 \).
\[ (3.2) \]

(3.3) If \([B(s), -1/2 \lambda, -1/B(s)]\) occurs, the succeeding \( \epsilon \) is +1.

(3.4) If \( \lambda \)CF terminates with \( \epsilon /B(s + 1) \), then \( \epsilon = +1 \).

A reduced \( \lambda \)CF has the following properties, in addition to (2.9) and (2.10):

(3.5) An infinite reduced \( \lambda \)CF converges.

(3.6) Every real number \( \alpha \) can be expanded uniquely by the “nearest integer algorithm” in a reduced \( \lambda \)CF. If the fraction is infinite, it converges to \( \alpha \).

(3.7) \[ Q_n \to \infty, \quad n \to \infty. \]

From now on, \( \lambda \)CF shall mean reduced \( \lambda \)CF. Bear in mind that at this point we are interested in \( \lim_{n \to \infty} m_{n-1}(\alpha_0) \) rather than \( m_{n-1}(\alpha_0) \) itself, because of (1.7).

We first consider the \( \lambda \)CF \( \alpha_0 = [r_0, \epsilon, -1/r_1, \ldots] \), with all \( \epsilon, = -1 \). In \( \alpha_0 \), some terms \(-1/r_\lambda, r \geq 2\), must occur by (3.1); in fact, there is at least one such term in every block of length \( s + 1 \). We shall make a series of transformations in \( \lambda \)CF \( \alpha_0 \), each having the effect of decreasing \( \alpha_0 \) while leaving it reduced. The first transformation is to replace each \( r_\nu > 2 \) by \( r_\nu = 2 \), which by Lemma 1 decreases \( \alpha_0 \). For convenience let \( r_0 = 2 \), so that now

(3.8) \[ \alpha_0 = [C(t_1), -1/B(u_1), -1/C(t_2), \ldots], \quad t_i \geq 1, \quad 1 \leq u_i \leq s, \]

by (3.1). By (2.8) we can assume further that \( t = 1 \) or 2.

The case \( q = 5 \) is simpler to treat than the higher values of \( q \). Let \( \lambda = \lambda_5 \); then \( s = 1 \), so \( u_i = 1 \). Moreover, \( t_i \geq 2 \) for all \( i \geq 2 \), otherwise (3.3) is
violated. Thus, we decrease $\alpha_0$ by assuming $t_i = 2$, and we shall temporarily assume $t_1 = 1$. Hence,

\begin{equation}
(3.9) \quad \alpha_0 \geq \left[ 2\lambda, -\frac{1}{\lambda}, -\frac{1}{2\lambda}, -\frac{1}{2\lambda}; -\frac{1}{\lambda} \right] =: \tau_0 = \tau_{3n},
\end{equation}

a periodic $\lambda$CF of period 3. If any $t_i$ is greater than 2, we have strict inequality.

The reverse $\alpha'_{3n-1}$ can be extended to a periodic fraction with a decrease in value. This fraction, still denoted by $\alpha'_{3n-1}$, obviously satisfies

\begin{equation}
(3.10) \quad \alpha'_{3n-1} \geq \left[ 2\lambda, -\frac{1}{\lambda}, \left( -\frac{1}{2\lambda} \right)^2, \ldots \right] = \tau_0.
\end{equation}

Therefore,

\begin{equation}
(3.11) \quad m_{3n-1} \geq \tau_0 - 1/\tau_0.
\end{equation}

By similar calculations we can show that

\begin{equation}
(3.12) \quad m_{3n-2} \geq \tau_2 - \frac{1}{\tau_{3n+1}} = 2\lambda - \frac{1}{\tau_0} + \tau_0 - 2\lambda = \tau_0 - \frac{1}{\tau_0},
\end{equation}

where we used $\tau'_{3n+1} = \tau_{3n+1} = \tau_1$, $-1/\tau_1 = \tau_0 - 2\lambda$. Thus, $m_{3n-1}$ and $m_{3n-2}$ are both bounded below by $\tau_0 - 1/\tau_0$. On the other hand,

\begin{equation}
(3.13) \quad m_3 = \tau_{3n+1} - 1/\tau'_{3n} < \tau_{3n+1} - \tau_1 = \lambda - \ldots < \lambda.
\end{equation}

It remains to evaluate $\tau_0 - 1/\tau_0$. Now, $\tau_0 = 2\lambda - 1/\tau_1$, and it was shown in [4, p. 126] that $\tau_1$ satisfies

\begin{equation}
(3.14) \quad \tau_1^2 - \lambda \tau_1 + \frac{2\lambda - 1}{5} = 0,
\end{equation}

where we used $\lambda^2 - \lambda - 1 = 0$. From this we calculate that

\begin{equation}
(3.15) \quad \tau_1^2 + (2 - 3\lambda)\tau_1 + 1 = 0,
\end{equation}

or

\begin{equation}
(3.16) \quad \tau_0 \approx 1 - \lambda/2 + (1 + (1 - \lambda/2)^2)^{1/2}.
\end{equation}

Let $\tau_0^*$ be the other root, $\tau_0\tau_0^* = 1$. Then,

\begin{equation}
(3.17) \quad m_{3n-1}, m_{3n-2} \geq \tau_0 - \frac{1}{\tau_0} = \tau_0 - \tau_0^* = (9\lambda^2 - 12\lambda)^{1/2}
\end{equation}

\begin{equation}
(3.18) \quad = (9 - 3\lambda)^{1/2} = 2 \left( 1 + \left( -\frac{\lambda}{2} \right)^2 \right)^{1/2} = h_5.
\end{equation}

From (3.10), (3.12), and (3.13) it follows that $M(\tau_0) = h_5$ when $\tau_0$ satisfies (3.12), and this is the only case of equality. Theorem 1 is now proved for $q = 5$. 

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We next assume \( q \geq 7 \). The case \( t_i = 2 \) for some \( i \) in (3.8) is not difficult. Suppose \( B(u_1), -1/2\lambda, -1/2\lambda, -1/B(u_1) \) occurs. Setting \([2\lambda, -1/B(u_2), \ldots] = [r_n\lambda, \ldots]\), we have \( \alpha_n \geq 2\lambda - \lambda/2 = 3\lambda/2 \) by (2.10). Also, \( \alpha_{n-1}' = [2\lambda, -1/B(u_1), \ldots] \geq 3\lambda/2 \), since \( \alpha_{n-1}' \) is reduced. Hence,

\[
(3.14) \quad m_{n-1} \geq 3\lambda/2 - 2/3\lambda > h_q + 0.3, \quad q \geq 7,
\]
as a calculation shows. It follows that

\[
(3.15) \quad M(\alpha_0) \geq h_q + 0.3, \quad q \geq 7,
\]
for \( \alpha_0 \) in this class.

We may now assume all \( t_i = 1 \). Define two periodic \( \lambda \)CF of period \( p = 2s + 1 \):

\[
(3.16) \quad \beta_0 = [2\lambda, -1/B(s), -1/2\lambda, -1/B(s - 1), \beta_0] = \beta_p,
\]

\[
(3.17) \quad \gamma_0 = [2\lambda, -1/B(s - 1), -1/2\lambda, -1/B(s), -1/\gamma_0] = \gamma_p.
\]

Note that \( \gamma_0 = \beta_{s+1} \), so that \( \beta_0 \sim \gamma_0 \).

Let

\[
\beta_0 = \lambda + \delta_0, \quad \delta_0 = \left[ \lambda, -\frac{1}{B(s)}, -\frac{1}{2\lambda}, -\frac{1}{B(s - 1)}, -\frac{1}{\beta_0} \right],
\]

and let \( P_i/Q_i, \ i \geq 0 \), be the convergents of \( \delta_0 \). \( P_i \) and \( Q_i \) satisfy the recurrence (2.1), and we calculate certain convergents explicitly. Recall \( q = 2l - 1 \). When \( 2 \leq j \leq s \), the recurrence (2.1) has constant coefficients and we solve for

\[
Q_j = A\zeta^j + B\zeta^{-j}, \quad \text{where} \quad \zeta = 2^{-1}(\lambda + (\lambda^2 - 4)^{1/2}) = e^{\pi i/q}.
\]

Hence,

\[
Q_0 = A + B = 1, \quad Q_1 = A\zeta + B\zeta^{-1} = \lambda,
\]
yielding

\[
(3.18) \quad A = -\zeta/(\zeta^{-1} - \zeta), \quad B = \zeta^{-1}(\zeta^{-1} - \zeta),
\]

\[
(\zeta^{-1} - \zeta)Q_j = -\zeta^{j+1} + \zeta^{-j-1} = -2i \sin \pi (j + 1)/q.
\]

In particular, put \( j = s - 2 = l - 4 \):

\[
(\zeta^{-1} - \zeta)Q_{s-2} = -2i \sin \pi \frac{l - 3}{2l - 1} = -2i \cos \frac{5\pi}{2q}.
\]

Let \( \omega = e^{\pi i/2q} \); note \( \zeta = \omega^2 \), \( \omega + \omega^{-1} = 2 \cos \pi/2q \), \( \omega^2 + \omega^{-2} = \lambda \), \( \omega^4 + \omega^{-4} = \lambda^2 - 2 \). Hence,

\[
(3.19) \quad 2 \cos 5\pi/q = \omega^5 + \omega^{-5} = (\omega + \omega^{-1})(\omega^4 - \omega^2 + 1 - \omega^{-2} + \omega^{-4})
\]

\[
= (\omega + \omega^{-1})(\lambda^2 - \lambda - 1).
\]

Also, \( \zeta^{-1} - \zeta = -2i \sin \pi/q \). Therefore,

\[
(3.20) \quad Q_{s-2} = (\lambda^2 - \lambda - 1)\Omega,
\]

where
with the abbreviation
\[ \Omega = \frac{\omega + \omega^{-1}}{2 \sin \pi/q} = \frac{1}{2 \sin \pi/2q}. \]

This illustrates the calculation. Similarly we find

\begin{equation}
Q_{s-1} = (\lambda - 1)\Omega,
\end{equation}

and by applying the recurrence (2.1) we derive further

\begin{equation}
Q_s = \Omega, \quad Q_{s+1} = (\lambda + 1)\Omega, \quad Q_{s+2} = (\lambda^2 + \lambda - 1)\Omega,
Q_{s+3} = (\lambda^3 + \lambda^2 - 2\lambda - 1)\Omega.
\end{equation}

Next we consider \( Q_j \) for \( s + 2 \leq j \leq 2s \). Write \( Q'_j = Q_{s+j+2} \), so now \( 0 \leq j \leq s - 2 \). \( Q'_j \) satisfies the same recurrence as \( Q_j \) with initial values \( Q'_0 = Q_{s+2}, Q'_1 = Q_{s+3} \). Solving, we find

\begin{equation}
(\zeta^{-1} - \zeta)Q'_j = -Q_{s+3}(\zeta'^{j-1} - \zeta^{-j-1}) + Q_{s+2}(\zeta'^{-j-1} - \zeta^{-j+1}),
\end{equation}

\( 0 \leq j \leq s - 2 \). For example, set \( j = s - 2 = l - 4 \). Then, \( \zeta^{s-2} - \zeta^{-s+2} = 2i \cos \gamma \pi/2q \) and \( \zeta^{s-3} - \zeta^{-s+3} = 2i \cos 9 \pi/2q \). The values of the cosines are calculated as in (3.19). Using \( \zeta^{-1} - \zeta = -2i \sin \pi/q \) and the values (3.22), we get

\begin{equation}
Q_{2s} = Q'_{s-1} = \Omega(\omega + \omega^{-1}) \left( (\lambda^3 + \lambda^2 - 2\lambda - 1)(\lambda^3 - \lambda^2 - 2\lambda + 1) \right.
\end{equation}

\begin{equation}
- (\lambda^2 + \lambda - 1)(\lambda^4 - \lambda^3 - 3\lambda^2 + 2\lambda + 1))
\end{equation}

\begin{equation}
= \frac{(\omega + \omega^{-1})^2}{4 \sin^2 \pi/q} \frac{\lambda}{4 - \lambda^2} = \frac{\lambda}{2 - \lambda}.
\end{equation}

In this same way one can derive \( Q_{2s-1} = (\lambda^2 - 2)/(2 - \lambda) \).

To calculate \( P_j \), we note that

\[ P_i = Q_{i+1}, \quad 0 \leq i \leq s - 1. \]

\( P_s, \ldots, P_{s+3} \) are calculated by the recurrence (2.1). We now use the analogues of (3.23), (3.24) to get \( P_{2s}, P_{2s-1} \).

In summary, we now have

\begin{equation}
Q_{s-2} = (\lambda^2 - \lambda - 1)\Omega, \quad Q_{s-1} = (\lambda - 1)\Omega, \quad Q_{s} = \Omega,
\end{equation}

\begin{equation}
P_{s-2} = Q_{s-1}, \quad P_{s-1} = \Omega, \quad P_{s} = \Omega, \quad \text{where } \Omega = 1/(2 \sin(\pi/2q)) \end{equation}

\begin{equation}
P_{2s-1} = (\lambda^3 - 2\lambda^2 + \lambda)\omega, \quad P_{2s} = (\lambda^2 - 2\lambda + 2)\omega,
\end{equation}

\begin{equation}
Q_{2s-1} = (\lambda^2 - 2)\omega, \quad Q_{2s} = \lambda\omega, \quad \text{where } \omega = 1/(2 - \lambda).\end{equation}
This gives

\[ \beta_0 - \lambda = \delta_0 = \frac{P_{2s+1}}{Q_{2s+1}} = \frac{\beta_0 P_{2s} - P_{2s-1}}{\beta_0 Q_{2s} - Q_{2s-1}}, \]

\[ (3.26) \]

\[ \beta_0^2 - (3\lambda - 2)\beta_0 + 2\lambda^2 - 2\lambda - 1 = 0, \]

\[ \beta_0 = \frac{3\lambda}{2} - 1 + \left(1 + \left(1 - \frac{\lambda}{2}\right)^2\right)^{1/2}; \]

we take the plus sign for the square root, since \( \beta_0 \geq 2/\lambda > 1 \) from (3.9).

The evaluation of \( \gamma_0 \) is similar:

\[ (3.27) \gamma_0 - \lambda = \left[ B(s), -\frac{1}{\beta_0} \right] = \frac{\beta_0 - (\lambda - 1)}{\beta_0(\lambda - 1) - (\lambda^2 - \lambda - 1)}. \]

At this point, it is convenient to introduce

\[ (3.28) \rho, \rho^* = 1 - \frac{\lambda}{2} \pm \left(1 + \left(1 - \frac{\lambda}{2}\right)^2\right)^{1/2}, \]

so that \( \rho \rho^* = -1 \). Then,

\[ \beta_0 = \lambda - \rho^*. \]

Substituting in (3.27),

\[ \gamma_0 = \lambda + \frac{\rho^* - 1}{\rho^*(\lambda - 1) - 1} = \lambda + \rho. \]

The reverse \( \beta'_{p-1} \) can be extended to a periodic \( \lambda \)CF of period \( p \) with a decrease in value. We denote this fraction by \( \beta'_{p-1} \) also. Hence

\[ (3.29) \beta'_{tp-1} = \left[B(s-1) - \frac{1}{2\lambda}, -\frac{1}{B(s), -\frac{1}{2\lambda}}; -\frac{1}{B(s-1)} \right] = \frac{1}{2\lambda - \gamma_0}, \]

\[ \gamma'_{tp-1} = \frac{1}{2\lambda - \beta_0}. \]

These values enable us to calculate (see (2.4))

\[ m_{tp-1}(\beta_0) = \beta_{tp} - \frac{1}{\beta'_{tp-1}} = \beta_0 - (2\lambda - \gamma_0) \]

\[ = \lambda - \rho^* + \lambda + \rho - 2\lambda \]

\[ = 2 \left(1 + \left(1 - \frac{\lambda}{2}\right)^2\right)^{1/2} = h_q, \]

\[ (3.30) \]

\[ m_{tp-1}(\gamma_0) = \gamma_0 - (2\lambda - \beta_0) = h_q. \]

On the other hand, if \( v \neq 0 \pmod{p} \), \( m_{v-1}(\beta_0) < \beta_v = \lambda - \cdots < \lambda < h_q \). So,

\[ \lim_{n \to \infty} m_{n-1}(\beta_0) = h_q = \lim_{n \to \infty} m_{n-1}(\gamma_0), \]
that is,

\[(3.32) \quad M(\beta_0) = M(\gamma_0) = h.\]

Next, we wish to show that \(\beta_0\) and \(\gamma_0\) are unique up to \(G\)-equivalence. Recall that \(q \geq 7\), so \(s \geq 2\). Define

\[(3.33) \quad \gamma_0^* = \left[2\lambda, -\frac{1}{B(s-1)}, -\frac{1}{T}\right], \quad T = \left[2\lambda, -\frac{1}{B(l_1)}, \ldots\right],\]

\[(3.34) \quad \beta_0^* = \left[2\lambda, -\frac{1}{B(s)}, -\frac{1}{T}\right], \quad T = \left[2\lambda, -\frac{1}{B(k_1)}, \ldots\right].\]

We shall show that every \(\alpha_0 \neq \beta_0\) can be replaced by \(\beta_0^*\) or \(\gamma_0^*\) with a decrease in \(M(\alpha_0)\).

Consider \(\gamma_0^*\). Since it is reduced, we have \(l_j \leq s\), \(l_j + l_{j+1} \leq 2s - 1\), \(j \geq 1\), by conditions (3.1) and (3.5). Replace \(l_j \leq s - 1\) by \(l_j = s - 1\); this decreases \(\gamma_0^*\). We say the sequence \(\{l_j\}\) is alternating if the entries \(s - 1\) and \(s\) occur in succession. Clearly, if \(\lambda\)CF \(\gamma_0^*\) ends in an infinite alternating sequence, then \(\gamma_0^* \sim \gamma_0\).

Suppose, on the contrary, that for some odd \(t\) the sequence \(l_1 = s, l_2, \ldots, l_{t+1} = s - 1\) is alternating, but \((l_{t+2}, l_{t+3}, \ldots, l_{t+k+3}) = (s, s - 1, \ldots, s - 1, s)\). There are \(k\) entries \(s - 1\). If \(k\) is odd, we can replace every other \(s - 1\) by \(s\) to obtain an alternating sequence. Suppose \(k\) is even, \(k = 4\), say. Then \((l_{t+3}, \ldots, l_{t+6})\) can be replaced by \((s - 1, s, s - 1, s - 1)\). Thus, the sequence we must treat is \((s, s - 1, s - 1, s)\), and we wish to replace it by \((s, s - 1, s, s - 1)\). This applies to any even \(k\).

What we must prove is that

\[(*) \quad \left[B(s-1), -\frac{1}{2\lambda}, -\frac{1}{B(s)}, -\frac{1}{U}\right] > \left[B(s), -\frac{1}{2\lambda}, -\frac{1}{B(s-1)}, -\frac{1}{U}\right],\]

where \(U = [2\lambda, -1/B(l_1), -1/V] = \lambda + [\lambda, \ldots] \geq \lambda + 2/\lambda > \lambda + 1\), \(U < 2\lambda\).

By writing the left member as \([\lambda, -1/B(s-2), \ldots]\), and similarly for the right member, and repeating the process, we eventually bring (*) to the form

\[(**) \quad \left[\lambda, -\frac{1}{B(s)}, -\frac{1}{U}\right] > \left[0, -\frac{1}{2\lambda}, -\frac{1}{B(s-1)}, -\frac{1}{U}\right].\]

Since \(U > \lambda + 1\),

\[\left[B(s), -\frac{1}{U}\right] > \left[B(s), -\frac{1}{\lambda + 1}\right] > \frac{2}{\lambda},\]

from which it follows that the left member of (**) is positive. But

\[2\lambda, -1/B(s-1), -1/U] > \lambda + [B(s), -1/U] > 0,\]

so the right member of (**) is negative. This establishes (*). We have shown that \(\gamma_0^* \geq \gamma_0\). Similarly, \(\beta_0^* \geq \beta_0\).
If $n$ is an index for which $\gamma^*_n = [2\lambda, -1/B(s - 1), \ldots]$, then by the previous reasoning

$$\gamma^*_n \geq \gamma_0 = \lambda + \rho, \quad \beta^*_n \geq \beta_0 = \lambda - \rho^*.$$ 

Now,

$$(\gamma^*_{n-1})' = \begin{bmatrix} B(l_k), & -\frac{1}{2\lambda}, & -\frac{1}{B(l_{k-1}')}, & -\frac{1}{W} \end{bmatrix} \geq \begin{bmatrix} B(s), & -\frac{1}{2\lambda}, & -\frac{1}{B(s' - 1)'}, & -\frac{1}{W} \end{bmatrix},$$

where $W = [2\lambda, -1/B(l_{k-2}), \ldots]$, with a finite alternating sequence $l_{k-2}$, $l_{k-3}, \ldots, l_1$. This can be extended to an infinite alternating sequence with a decrease in the value of $W$. Hence,

$$(\gamma^*_{k-1})' \geq \beta^*_1 = \frac{1}{2\lambda - \beta_0^*}.$$ 

It follows that

$$m_{n-1}(\gamma^*_n) = \gamma^*_n \frac{1}{(\gamma^*_{n-1})'} \geq \gamma_0 - (2\lambda - \beta_0^*)$$

$$\geq \gamma_0 + \beta_0 - 2\lambda = h_q;$$ 

see (3.30). Similarly,

$$m_{n-1}(\beta^*_n) \geq h_q.$$ 

On the other hand, if $r_n = 1$, we have $m_{n-1}(\beta_n) < \lambda - \ldots < h_q$. We have proved

(3.37) $$M(\beta_0^*) = \lim_{\nu \to \infty} m_{\nu}(\beta_0^*) \geq M(\beta_0) = h_q,$$

equality occurring if and only if $\beta_0^*, \gamma_0^* \sim \beta_0$.

Putting (3.13), (3.15), (3.35), and (3.36) together, we get

**Lemma 2.** If $\alpha_0$ has all $e_\nu = -1$ and $r_n = 2$, then

(3.38) $$m_{n-1}(\alpha_0) \geq h_q.$$ 

Hence,

$$M(\alpha_0) \geq h_q, \quad q \geq 5,$$

with equality if and only if $\alpha_0 \sim \rho$.

The last statement follows since $r_n = 2$ must occur infinitely often.

To complete the proof of Theorem 1, we proceed as follows. If $e_\mu = 1$ occurs in $\alpha_0$ only a finite number of times, we may assume it never occurs; then by

(3.2), $r_\mu \geq 2$ infinitely often. Hence $M(\alpha_0) \geq h_q$ by Lemma 2, with the cases of equality mentioned there. So we now assume $e_\mu = 1$ occurs infinitely
often but not always. We look for the largest block of terms with \( \varepsilon = -1 \), i.e., bounded by \( \varepsilon = +1 \) at both ends. Denote this block by

\[
\alpha_{\mu \nu} = \begin{bmatrix}
\frac{1}{r_{\mu}}, \ldots, \frac{1}{r_{\mu+1}}, \ldots, \frac{1}{r_{\nu}}
\end{bmatrix}, \quad \varepsilon_\mu = \varepsilon_{\nu+1} = 1.
\]

The terms with \( r_i = 1 \) yield only \( m_{t-1} < \lambda < h_q, \mu + 1 \leq t \leq \nu \). So let \( r_n = 2 \) for an \( n \) with \( \mu + 1 \leq n \leq \nu \). If \( \alpha_{\mu \nu} \) does not end in \( B(s), -1/2\lambda, -1/B(s) \), we can adjoin \( U \) with all \( \varepsilon = -1 \) so that \([\alpha_{\mu \nu}, -1/U]\) is reduced: for example, we could take a periodic \( U = [2\lambda, -1/2\lambda, \ldots] \). Then by (2.2), \( \alpha_n > \alpha_{n \nu} > [\alpha_{n \nu}, -1/U] \). Similarly, \( \alpha_{n-1} > \alpha_{n-1, \mu} > [\alpha_{n-1, \mu}, -1/V, \mu, -1/V] \), where \( V = [r_{\mu-1}, -1/r_{\mu-2}, \ldots, -1/r_1] \) is chosen so that \([\alpha_{n-1, \mu}, -1/V] \) is reduced. Then, \( \delta_0 := [V', -1/\alpha_{\mu \nu}, -1/U] \) has all \( \varepsilon = -1 \) and is reduced. By Lemma 2,

\[
m_{n-1}(\alpha_0) \geq m_{n-1}(\delta_0) \geq h_q, \quad r_n \geq 2.
\]

It follows that

\[
M(\alpha_0) \geq M(\delta_0) \geq h_q.
\]

When \( \alpha_{\mu \nu} \) ends with \( B(s), -1/2\lambda, -1/B(s) \), there is no \( U \) satisfying the required conditions because of (3.3). We derive successively, using the values (3.25):

\[
\alpha_0 = [\ldots, -1/B(s), 1/T],
\]

\[
\left[ B(s), \frac{1}{T} \right] = \frac{TP_{s-1} + P_{s-2}}{TQ_{s-1} + Q_{s-2}} > \frac{1}{\lambda - 1}, \quad \left[ 2\lambda, -\frac{1}{B(s)}, \frac{1}{T} \right] = \lambda + 1,
\]

\[
\left[ B(s), -\frac{1}{2\lambda}, -\frac{1}{B(s)}, \frac{1}{T} \right] > \left[ B(s), -\frac{1}{\lambda + 1} \right] = \frac{2}{\lambda}, \quad \left[ 2\lambda, -\frac{1}{B(s)}, \ldots, \frac{1}{T} \right] > \left[ 2\lambda, -\frac{\lambda}{2} \right] = \frac{3\lambda}{2},
\]

and finally

\[
\alpha_n \geq \eta_n := \left[ 2\lambda, -\frac{1}{B(l_1)}, \ldots, -\frac{1}{B(l_k)}, -\frac{1}{3\lambda/2} \right].
\]

We assign \( l_j = s \) or \( s - 1 \) in alternation, so that \( \eta_n \) is of the form \( \beta_0^* \) in (3.34) or \( \gamma_0^* \) in (3.33); then from (3.37), (3.36) we again get (3.39), (4.40).

The final case is: all \( \varepsilon = +1 \). If \( 1/r_n \lambda \) occurs with \( r_n \geq 2 \), then \( m_{n-1} \geq \alpha_n > 2\lambda > h_q \). When \( 1/r_n \lambda \) occurs infinitely often, we get

\[
M(\alpha) = \lim_{n \to \infty} m_{n-1}(\alpha) \geq 2\lambda > h_q.
\]

Otherwise, we may assume \( 1/r \lambda, r \geq 2 \), never occurs and

\[
\alpha_n = \left[ \lambda, \frac{1}{\alpha_n} \right] = \left[ \lambda, \frac{1}{\alpha_n} \right] = \frac{1}{2}(\lambda + (\lambda^2 + 4)^{1/2}) =: \mu.
\]
So,

$$(3.41) \quad m_{n-1} = \mu + \frac{1}{\mu} = (\lambda^2 + 4)^{1/2} > h_q,$$

as a small calculation shows, and this implies

$$M(\alpha_0) > h_q.$$ 

In all cases, then, $M(\alpha_0)$ is bounded below by $h_q$, with the cases of equality stated in (3.12), (3.37), (3.38). This completes the proof of Theorem 1.

4. THE LOCAL HURWITZ CONSTANT

In this section we shall consider the local Hurwitz constant, i.e., $m_i(\alpha_0)$. Our object is to compare $m_i$ with $h_q$.

We first use a geometric method. Let $\alpha$ be $G$-irrational. The Ford circle $C_n$ is defined by

$$C_n: |z - \left( \frac{P_n}{Q_n} + \frac{i}{2Q_n^2} \right)| = \frac{1}{2Q_n^2},$$

where $P_n/Q_n$ are the convergents of $\alpha$. Different $C_n$ do not overlap; $C_n$ and $C_m$ are tangent externally if and only if $m = n + 1$ or $n - 1$. These assertions follow easily from the determinant condition (2.2). Also from (2.3), (2.12), with $\alpha = [r_0, r_1, r_2, \ldots]$, we have

$$\text{sgn} \left( \alpha - \frac{P_{n-1}}{Q_{n-1}} \right) = \pm \text{sgn} \left( \alpha - \frac{P_n}{Q_n} \right)$$

according as $\varepsilon_{n+1} = -1$ or $+1$.

Suppose $\varepsilon_{n+1} = -1$. Then $\alpha$ is on the same side of both $P_n/Q_n$ and $P_{n-1}/Q_{n-1}$. It follows that

$$(4.1) \quad |\alpha - \frac{P_i}{Q_i}| > \frac{1}{2Q_i^2} \text{ for } i = n - 1 \text{ or } n.$$ 

Equality is impossible because $P_i/Q_i$ is $G$-rational, but $\alpha$ is $G$-irrational.

Next suppose $\varepsilon_{n+1} = 1$. Then $\alpha$ lies between $P_n/Q_n$ and $P_{n-1}/Q_{n-1}$. Let $(i, j)$ be a permutation of $(n - 1, n)$. Then,

$$(4.2) \quad \left| \alpha - \frac{P_i}{Q_i} \right| < \frac{1}{2Q_i^2}, \quad \left| \alpha - \frac{P_j}{Q_j} \right| > \frac{1}{2Q_j^2}.$$ 

Equality can occur only if $\alpha$ coincides with the real projection of the point of tangency of the Ford circles, which is impossible because $\alpha$ is $G$-irrational. Hence,

**Theorem 2.** If $\varepsilon_{n+1} = 1$, we have $m_{n-1} > 2$, $m_n < 2$, or $m_{n-1} < 2$, $m_n > 2$.

An elegant algebraic proof of this theorem in the rational case ($q = 3$) was given by K. Th. Vahlen [6].

Theorem 2 holds for all $q \geq 4$, even or odd. Since $h_q = 2$ when $q$ is even, it provides an estimate of the desired type for even $q$. We now concentrate on odd $q$. 
Theorem 3. Let \( q \) be odd. If \( r_n \geq 2 \) and \( \varepsilon_{n-1} = 1 \), then \( m_{n-1} \geq h_q \).

Theorem 3 is a special case of (3.39).

If we drop the assumption \( r_n \geq 2 \), we can have two consecutive \( m_i < h_q \), as we see from the following example: let

\[
\lambda = \lambda_q = 1.80, \ldots, r_{n-1} = 4,
\]

\[
\alpha_0 = [\ldots, -1/4\lambda, 1/\lambda, -1/\lambda, -1/\lambda, \ldots],
\]

for which \( m_n < \lambda, m_{n-1} < 1.97 < 2 \). We make further assumptions on the \( \varepsilon_i \).

Theorem 4. Let \( q \) be odd. If \( \varepsilon_{n+1} = \varepsilon_{n+2} = 1 \), then \( m_i \geq (\lambda^2 + 4)^{1/2} > h_q \) for at least one of \( i = n-1, n, n+1 \).

The proof is modelled after one by M. Fujiwara [2]; see also F. Bagemihl and J. R. McLaughlin [1]. In contradiction to the conclusion

\[
m_i(\alpha) \geq (\lambda^2 + 4)^{1/2},
\]

we can assert that

\[
\left| \alpha - \frac{P_j}{Q_j} \right| > \frac{1}{(\lambda^2 + 4)^{1/2}Q_j}, \quad n-1 \leq j \leq n+1.
\]

We observe from (2.3) that \( \alpha - P_{n-1}/Q_{n-1} \) and \( \alpha - P_n/Q_n \) have opposite signs, in view of \( \varepsilon_{n+1} = 1 \). Hence,

\[
\frac{1}{(\lambda^2 + 4)^{1/2}} \left( \frac{1}{Q_{n-1}^2} + \frac{1}{Q_n^2} \right) < \left| \alpha - \frac{P_{n-1}}{Q_{n-1}} \right| + \left| \alpha - \frac{P_n}{Q_n} \right| = \frac{1}{Q_nQ_{n-1}}.
\]

Write

\[
(\lambda^2 + 4)^{1/2} = u + \frac{1}{u}, \quad u > \lambda;
\]

then

\[
\frac{Q_n^2}{Q_{n-1}^2} - (\lambda^2 + 4)^{1/2} \frac{Q_n}{Q_{n-1}} + 1 = \left( \frac{Q_n}{Q_{n-1}} - \frac{1}{u} \right) \left( \frac{Q_n}{Q_{n-1}} - u \right) < 0.
\]

Now \( Q_n/Q_{n-1} - 1/u > 1 - 1 = 0 \), so \( Q_n/Q_{n-1} - u < 0 \), that is,

\[
\frac{Q_n}{Q_{n-1}} < u.
\]

Hence,

\[
\frac{Q_{n-1}}{Q_n} > \frac{1}{u}.
\]

Replacing \( n \) by \( n + 1 \) in (4.6)—recall \( \varepsilon_{n+2} = 1 \)—we get

\[
\frac{Q_{n+1}}{Q_n} < u.
\]
Therefore,
\[ u > \frac{Q_{n+1}}{Q_n} = r_{n+1} + \varepsilon_{n+1} \frac{Q_{n-1}}{Q_n} \geq \frac{Q_{n-1}}{Q_n}, \]
yielding
\[ \frac{Q_{n-1}}{Q_n} < u - \lambda. \]
But
\[ \left( u - \frac{1}{u} \right)^2 = \left( u + \frac{1}{u} \right)^2 - 4 = \lambda^2, \]
and so
\[ \frac{Q_{n-1}}{Q_n} < \frac{1}{u}, \]
contradicting (4.7). This completes the proof of Theorem 4.

Note added in proof. In a recent letter Thomas A. Schmidt has pointed out an error in [4] that carries over to the present paper. It can be corrected as follows.

We now consider the approximation of a \( G \)-irrational \( \alpha_0 \) by the convergents \( P_n/Q_n \) of its \( \lambda \)CF (1.4). Note that in (1.1) the fraction \( k/m \in G(\infty) \) determines \( k \) and \( m \) uniquely up to sign, since
\[
\begin{pmatrix} k \\ m \end{pmatrix}^{-1} \begin{pmatrix} k_1 \\ m_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
when \( k/m = k_1/m_1 \). Thus we write
\[
(1.6) \quad \alpha_0 - \frac{P_{n-1}}{Q_{n-1}} = (-1)^{n-1} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n \frac{m_{n-1}}{Q_{n-1}^2}, \quad m_{n-1} = m_{n-1}(\alpha_0),
\]
and study \( m_{n-1}(\alpha_0) \). Clearly,
\[
(1.7) \quad M(\alpha_0) = \lim_{n \to \infty} m_{n-1}(\alpha_0), \quad h'_{\alpha_0} = \inf_{\alpha_0} M(\alpha_0).
\]
We call \( m_n(\alpha_0) \) a local Hurwitz constant.

Similar changes are required in [4]. In particular, Theorem 3 should be eliminated.

Bibliography


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